Introduction to Accelerator Physics
2011 Mexican Particle Accelerator School

Lecture 1-2/7: Intro, Relativity, E&M, Weak Focusing, Betatrons, Transport Matrices

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Wednesday, September 28, 2011

About These Lectures

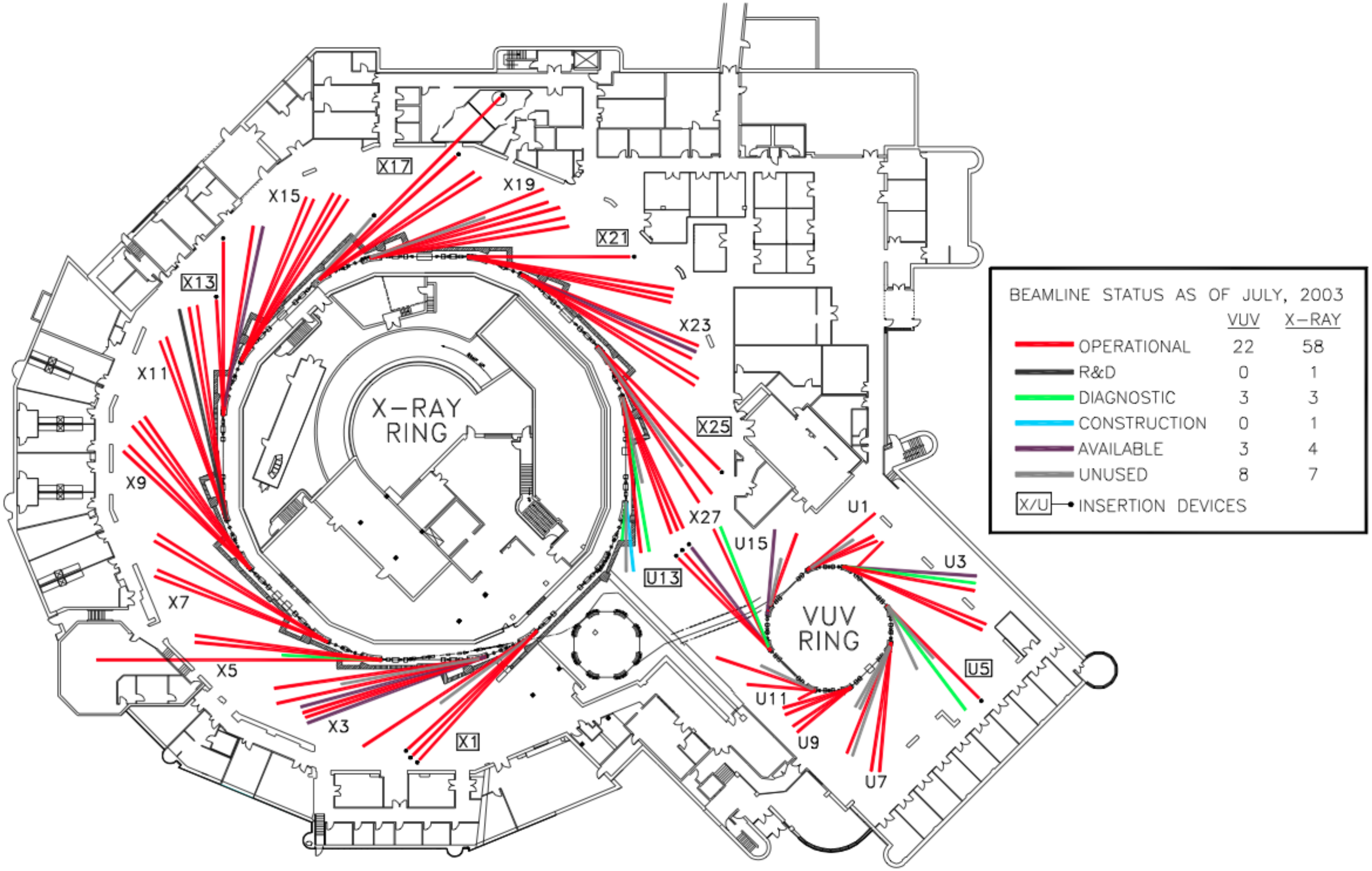
- **Objective:** Comfort with synchrotron transverse optics
  - Optics principles found in synchrotrons, light sources
  - Accelerator physics language/terminology
    - Ties many other concepts in the field together
  - Mostly “single particle” dynamics
    - John Byrd will teach instabilities next week

- Recommended Reading
  - Conte/MacKay: “An Introduction to the Physics of Particle Accelerators”, 2nd Edition
  - Edwards/Syphers: “An Introduction to the Physics of High Energy Accelerators”

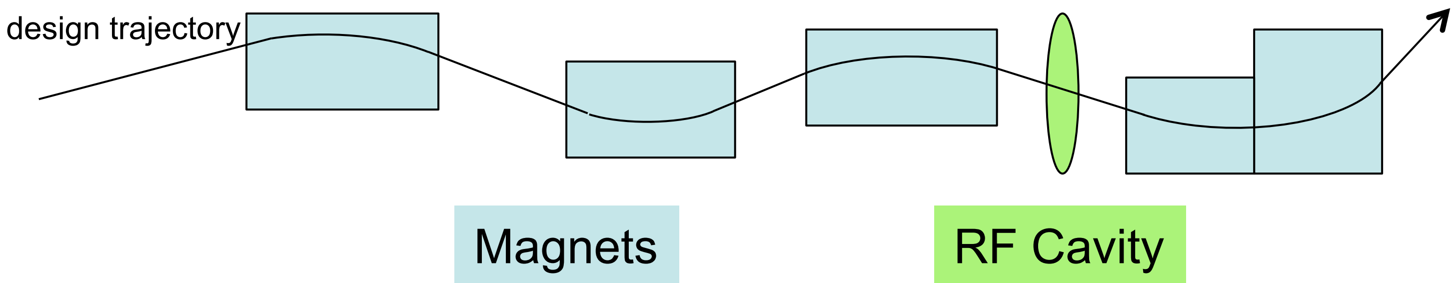
MePAS Accelerator Physics Syllabus

- 1-2: Wednesday
  - Relativity/EM review, coordinates, cyclotrons
  - Weak focusing, transport matrices, dipole magnets, dispersion
- 3: Thursday
  - Edge focusing, quadrupoles, accelerator lattices, start FODO
- 4: Friday
  - Periodic lattices, FODO optics, emittance, phase space
- 5: Saturday
  - Insertions, beta functions, tunes, dispersion, chromaticity
- 6: Monday
  - Dispersion suppression, light source optics (DBA, TBA, TME)
- 7: Tuesday
  - (Nonlinear dynamics), Putting it all together

NSLS (Brookhaven Lab, New York)



Simplified Particle Motion



- Design trajectory
  - Particle motion will be perturbatively expanded around a **design trajectory** or **orbit**
  - This orbit can be over 10^{10} km in a storage ring
- Separation of fields: Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
  - Magnetic fields from static or slowly-changing magnets
    - transverse to design trajectory $\hat{x}, \hat{y}$
  - Electric fields from high-frequency RF cavities
    - in direction of design trajectory $\hat{s}$
- Relativistic charged particle velocities

Relativity Review

- **Accelerators: applied E&M and special relativity**
- Relativistic parameters:

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \sqrt{1 - 1/\gamma^2}$$

  - After this lecture I will try to use β_r and γ_r to avoid confusion with other uses of β and γ in accelerator optics
  - γ=1 (classical mechanics) to ~2.05×10^5 (to date) (where??)

- Total energy U, momentum p, and kinetic energy W

$$U = \gamma mc^2 \qquad p = (\beta\gamma)mc = \beta\left(\frac{U}{c}\right) \qquad W = (\gamma - 1)\, mc^2$$

Relative Relativity

WolframAlpha

LEP energy

Input interpretation:

LEP (Large Electron Positron Collider) | ce

Result:

208 GeV (gigaelectronvolts)

Unit conversions:

0.208 TeV (teraelectronvolts)

2.08×10^{11} eV (electronvolts)

0.03333 μJ (microjoules)

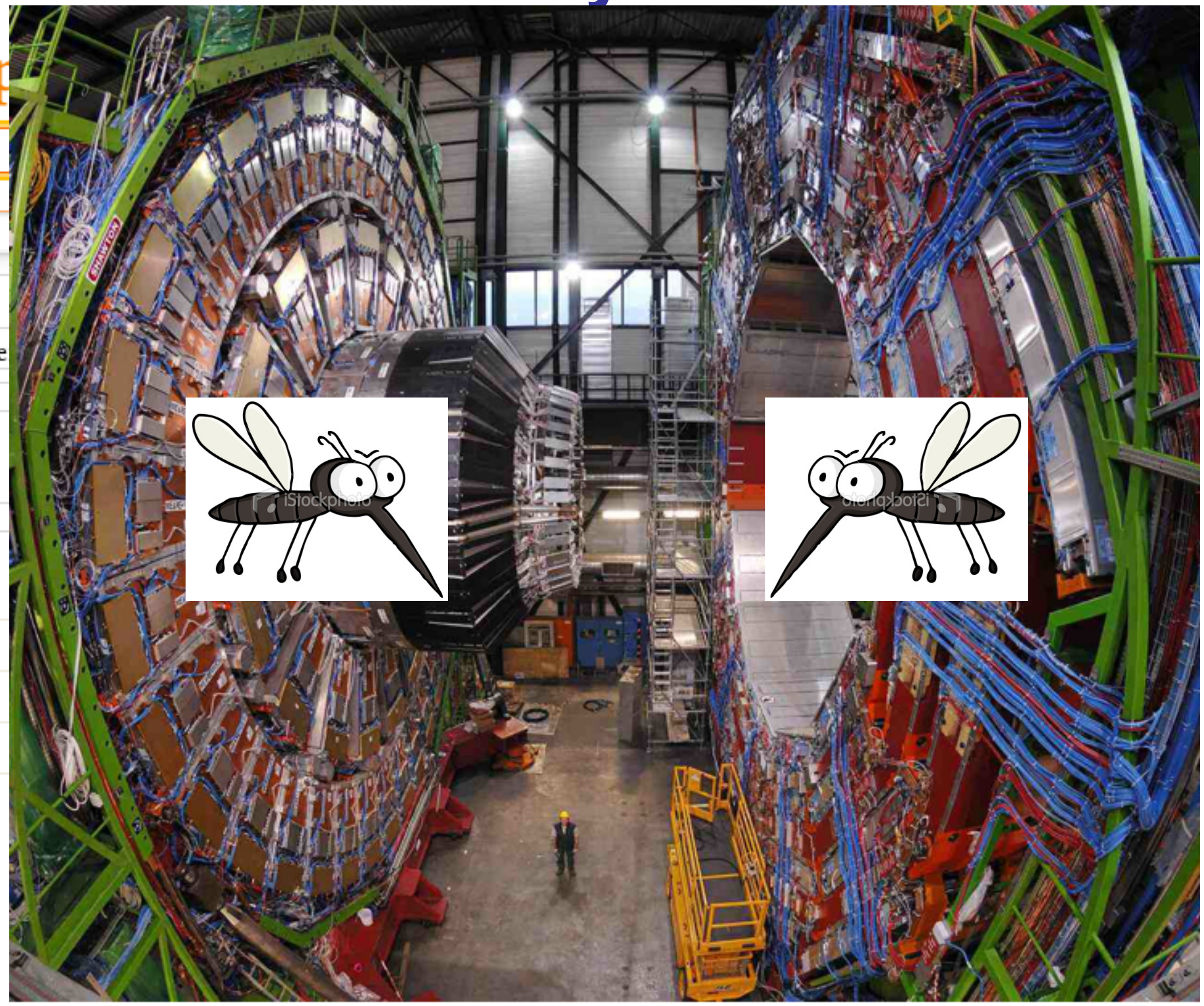
3.333×10^{-8} J (joules)

0.3333 ergs

(unit officially deprecated)

Comparisons as energy:

$\approx$ (0.21 $\approx$ 1/5) ×



approximate kinetic energy of a flying mosquito ($\approx 1.6 \times 10^{-7}$ J)

$\approx$ 2.2 × mass-energy equivalent of a Z boson ($\approx 1.5 \times 10^{-8}$ J)

Convenient Energy Units

$$1\ \mathrm{eV} = (1.602 \times 10^{-19}\ \mathrm{C})(1\ \mathrm{V}) = 1.602 \times 10^{-19}\ \mathrm{J}$$

$$1\ \mathrm{MeV} = 1.602 \times 10^{-13}\ \mathrm{J}$$

$$1\ \mathrm{GeV} = 1.602 \times 10^{-10}\ \mathrm{J}$$

- How much is 1 TeV? (LHC beams to 7 TeV)
  - Energy to raise 1g about 16 μm against gravity
  - Energy to power 100W light bulb 1.6 ns
- But many accelerators have $10^{10\text{-}12}$ particles
  - Single bunch "instantaneous power" of Terawatts over a few ns
- Highest energy cosmic ray
  - ~300 EeV (3×10^{20} eV or 3×10^{8} TeV!)

(125 g hamster at 100 km/hr)

OMG particle

Relativity Review (Again)

- Accelerators are **applied special relativity**
- Relativistic parameters:

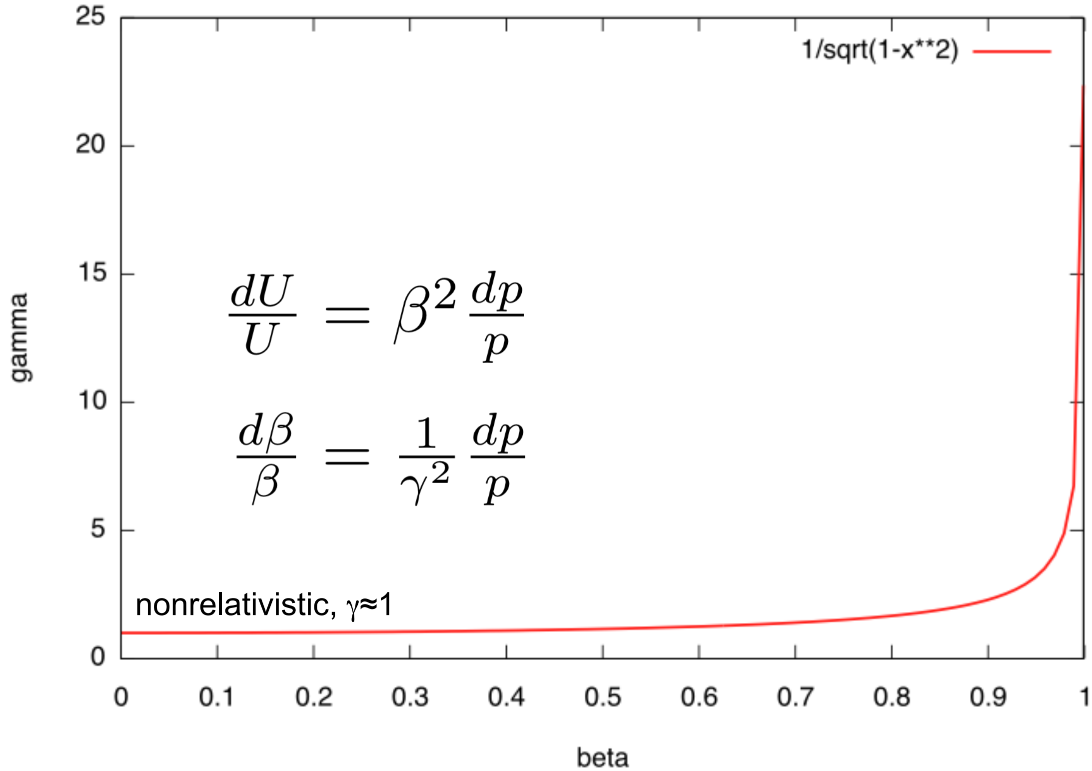
$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \sqrt{1 - 1/\gamma^2}$$

  - After this lecture, will try to use β_r and γ_r to avoid confusion with other lattice parameters
  - γ=1 (classical mechanics) to ~2x10^5 (at LEP)

- Total energy U, momentum p, and kinetic energy W

$$U = \gamma mc^2 \qquad p = (\beta\gamma)mc = \beta\left(\frac{U}{c}\right) \qquad W = (\gamma - 1)\, mc^2$$

Convenient Relativity Relations



ultrarelativistic, $\gamma >> 1$

- Derived in Conte/MacKay, hold for all γ
- In highly relativistic limit $\beta \approx 1$
  - Usually must be careful below $\gamma \approx 5$ or $U \approx 5\ mc^2$

    For high energy electrons this is only U≈2.5 MeV
  - Many accelerator physics phenomena scale with γ^k or $(\beta\gamma)^k$

Relativistic Electromagnetism

- Accelerators are also **applied electromagnetism**
- Classical electromagnetic potentials can be shown to combine to a four-potential (with c=1):

$$A^\alpha \equiv (\Phi, \vec{A})$$

- The field-strength tensor is related to the four-potential

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

  - E/B fields Lorentz transform with factors of γ, $(\beta\gamma)$

J.D. Jackson, Classical Electrodynamics 2nd Ed, Chapter 11 – a classic graduate text!

Relativistic Electromagnetism II

- The relativistic electromagnetic force equation becomes

$$\frac{dp^\alpha}{d\tau} = m\frac{du^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}u_\beta$$

  - Thankfully we can write this in simpler terms

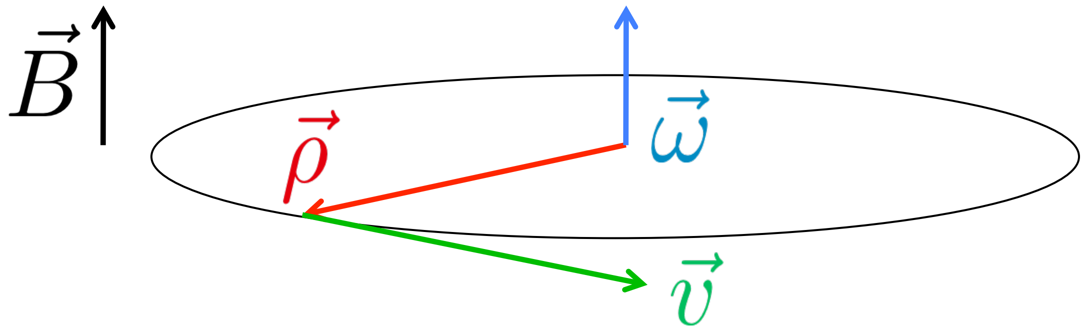
$$\frac{d(\gamma m\vec{v})}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

  - That is, "classical" E&M force equations hold if we treat the momentum as relativistic.
  - If we dot in the velocity, we get energy transfer

$$\frac{d\gamma}{dt} = \frac{q\vec{E}\cdot\vec{v}}{mc^2}$$

  - Unsurprisingly, we can only get energy changes from electric fields, not magnetic fields

Constant Magnetic Field, Particle Energy

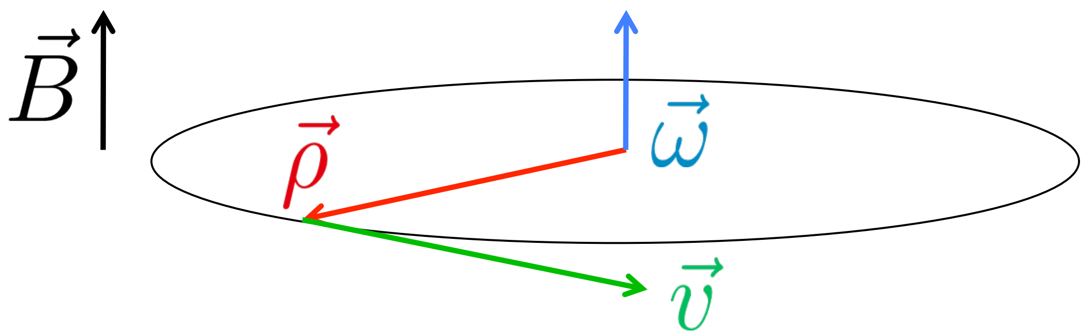


- In a constant magnetic field, constant-energy charged particles move in circular arcs of radius ρ with constant angular velocity ω:

$$\vec{F} = \frac{d}{dt}(\gamma m \vec{v}) = \gamma m \frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$$

$$\vec{v} = \vec{\omega} \times \vec{\rho} \qquad \Rightarrow \qquad q\vec{v} \times \vec{B} = \gamma m \vec{\omega} \times \frac{d\vec{\rho}}{dt} = \gamma m \vec{\omega} \times \vec{v}$$

Constant Magnetic Field, Particle Energy II



- For $\vec{B} \perp \vec{v}$ we then have:

$$qvB = \frac{\gamma m v^2}{\rho} \qquad p = \gamma m(\beta c) = q(B\rho)$$

$$\omega = \frac{v}{\rho} = \frac{qB}{\gamma m}$$

$$\frac{p}{q} = (B\rho)$$

Rigidity: Bending Radius vs Momentum

Beam

$$\frac{p}{q} = (B\rho)$$

Accelerator (magnets, geometry)

- This is such a useful expression in accelerator physics that it has its own name: **rigidity**
- Ratio of momentum to charge
  - How hard (or easy) is a particle to deflect?
  - Often expressed in [T-m] (easy to calculate B)
  - (Be careful when q≠e)
- A **very useful expression** for particle bending:

$$p[\mathrm{GeV/c}] \approx 0.3\, B[\mathrm{T}]\, \rho[\mathrm{m}]\, q[\mathrm{e}]$$

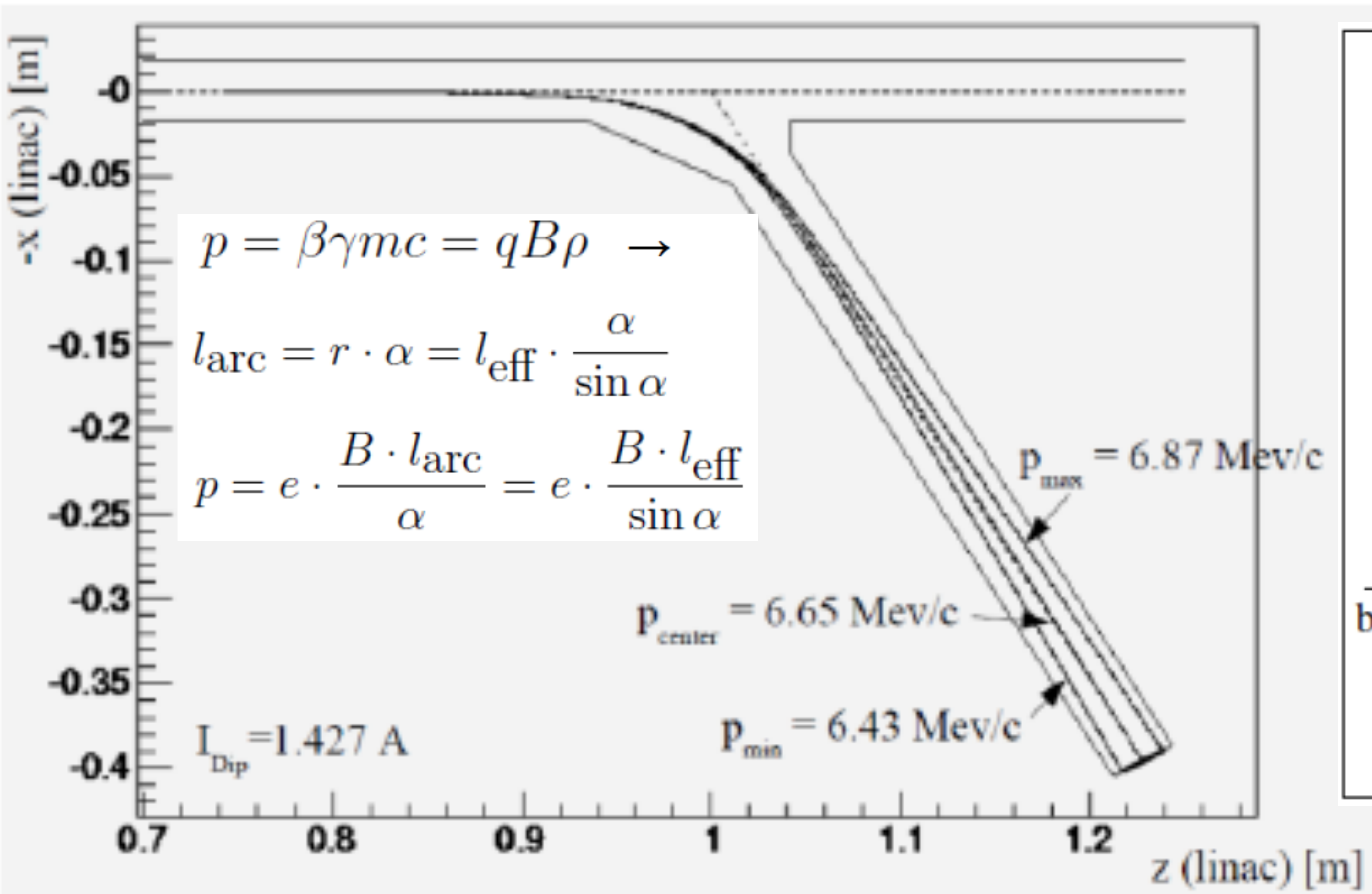
Application: Particle Spectrometer

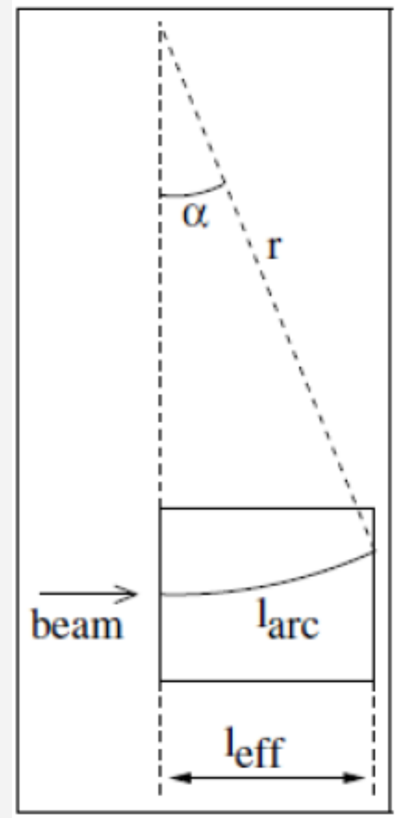
- Identify particle momentum by measuring bend angle α from a calibrated magnetic field B

$$p = \beta\gamma mc = qB\rho \quad \rightarrow$$

$$l_{\mathrm{arc}} = r \cdot \alpha = l_{\mathrm{eff}} \cdot \frac{\alpha}{\sin\alpha}$$

$$p = e \cdot \frac{B \cdot l_{\mathrm{arc}}}{\alpha} = e \cdot \frac{B \cdot l_{\mathrm{eff}}}{\sin\alpha}$$

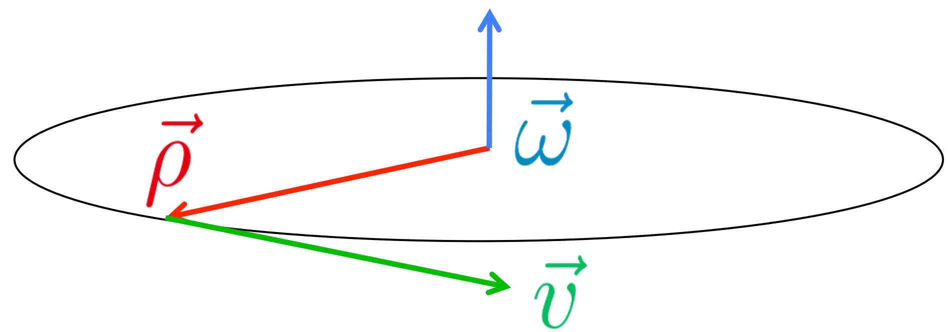




Yujong Kim, 2010 accelerator physics lectures

Cyclotron Frequency

$$\omega = \frac{v}{\rho} = \frac{qB}{\gamma m}$$



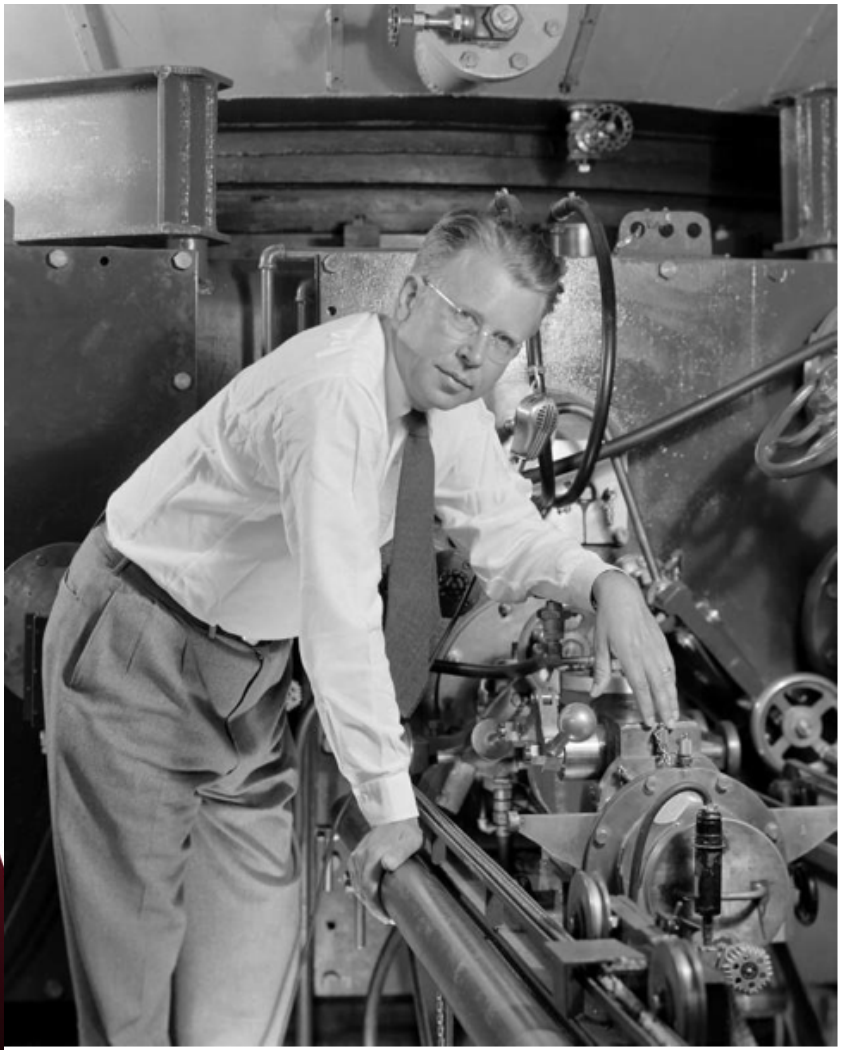
- Another very useful expression is the particle angular frequency in a constant field: **cyclotron frequency**

- In the nonrelativistic approximation

$$\omega_{\mathrm{nonrelativistic}} \approx \frac{qB}{m}$$

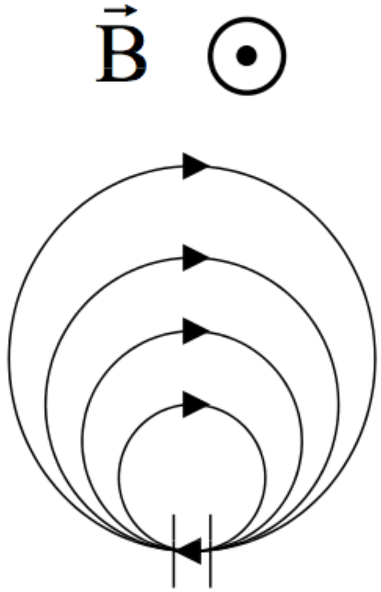
Here revolution frequency is independent of radius or energy!

Lawrence and the Cyclotron



Ernest Orlando Lawrence

- Can we repeatedly spiral and accelerate particles through the same potential gap?



Electric field accelerating gap $\Delta\Phi$

Cyclotron Frequency Again

- Recall that for a constant B field

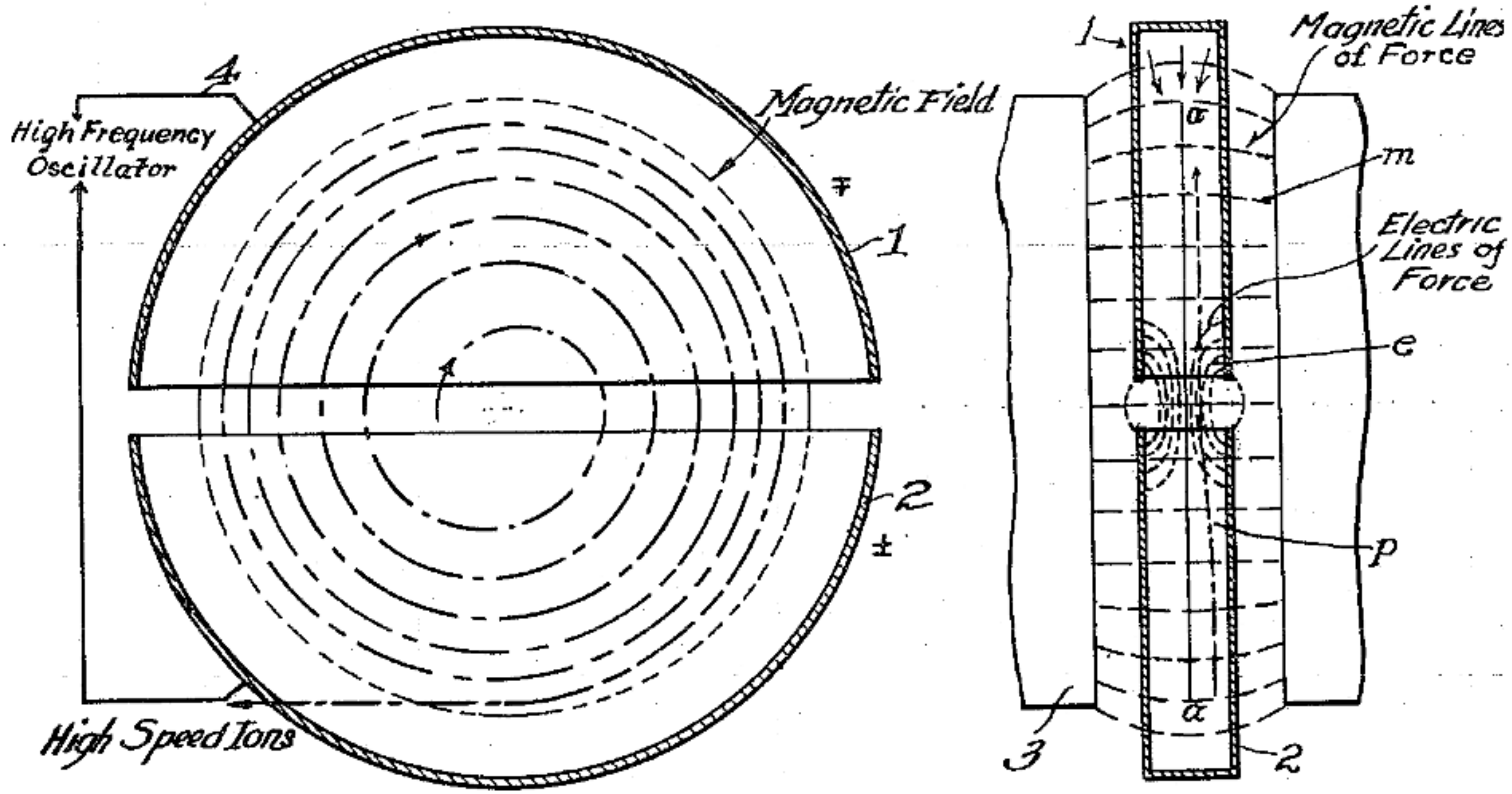
$$p = \gamma m v = q(B\rho) \quad \Rightarrow \quad \rho = \left(\frac{\gamma m}{qB}\right) v$$

  - Radius/circumference of orbit scale with velocity
    - Circulation time (and frequency) are independent of v
  - Apply AC electric field in the gap at frequency f_{rf}
    - Particles accelerate until rising γ pulls them off resonance

$$\omega = \frac{v}{\rho} = \frac{qB}{\gamma m} \qquad f_{\mathrm{rf}} = \frac{\omega}{2\pi} = \frac{qB}{2\pi\gamma m}$$

    - Note a first appearance of “bunches”, not DC beam
    - BUT works best with heavy particles (hadrons, not electrons)

A Patentable Idea

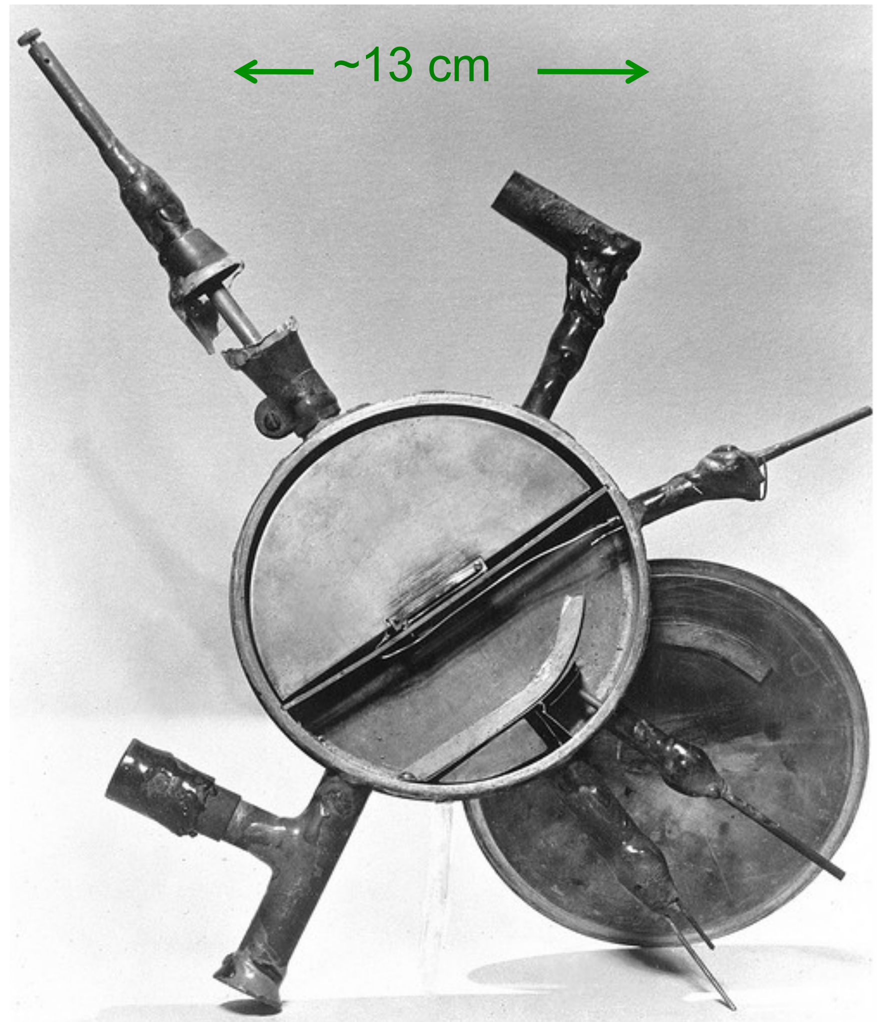


- 1934 patent 1948384
  - Two accelerating gaps per turn!

All The Fundamentals of an Accelerator

- Large static magnetic fields for guiding (~1T)
- HV RF electric fields for accelerating
  - (RF phase focusing)
- p/H source, injection, extraction, vacuum

- 13 cm: 80 keV
- 28 cm: 1 MeV
- 69 cm: ~5 MeV
- … 223 cm: ~55 MeV



(Berkeley)

Parameterizing Particle Motion: Coordinates

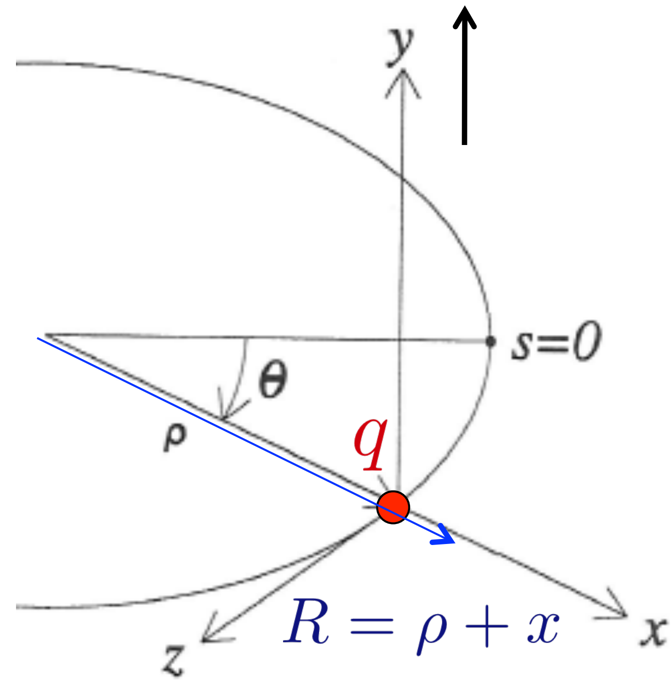
- Now we derive more general equations of motion
- We need a **local coordinate system** $(\hat{x}, \hat{y}, \hat{z} \equiv \hat{s})$ relative to the design particle trajectory
  - s is the direction of design particle motion
  - y is the main magnetic field direction
  - x is the radial direction
  - ρ is not a coordinate, but the design bending radius in magnetic field B_0
- Can express total radius R as

$$R = \rho + x \qquad \theta = \frac{s}{R} = \frac{\beta c t}{R}$$

- Also define local trajectory angle

$$x' \equiv \frac{dx}{ds} = \frac{1}{R}\frac{dx}{d\theta}$$

$\vec{B}_0 = B_0 \hat{y}$



Parameterizing Particle Motion: Approximations

- We will make a few reasonable approximations:

  0) No local currents (beam travels in a near-vacuum)

  1) Paraxial approximation: $x', y' \ll 1 \quad \text{or} \quad p_x, p_y \ll p_s$

  2) Perturbative coordinates: $x, y \ll \rho$

  3) Transverse linear B field: $\vec{B} = B_0 \hat{y} + (x\hat{y} + y\hat{x}) \left(\frac{\partial B_y}{\partial x} \right)$

     - Note this obeys Maxwell's equations in free space

  (A0)

  4) Negligible E field: $\gamma \approx \text{constant}$

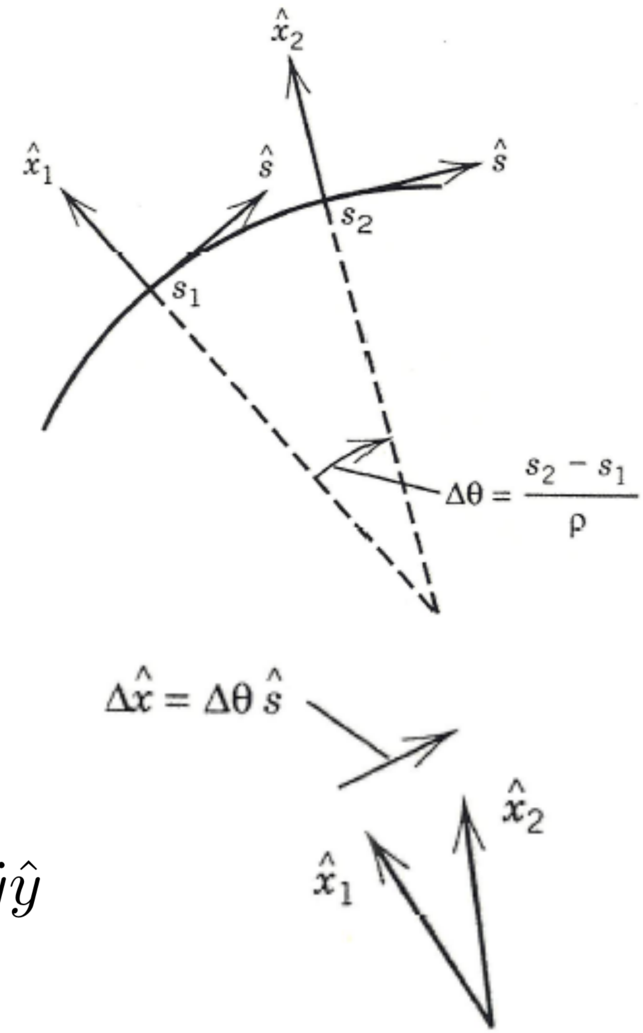
     - Equivalent to assuming adiabatically changing B fields relative to $dx/dt, dy/dt$

Parameterizing Particle Motion: Acceleration

- Lorentz force equation of motion is

$$q\vec{v} \times \vec{B} = \frac{d(\gamma m \vec{v})}{dt} = \gamma m \dot{\vec{v}} \qquad \text{(A4)}$$

- Calculate velocity and acceleration in our coordinate system



$$\vec{v} = \dot{R}\hat{x} + R\dot{\hat{x}} + \dot{y}\hat{y} = \dot{R}\hat{x} + R\dot{\theta}\hat{s} + \dot{y}\hat{y}$$

$$\dot{\vec{v}} = \ddot{R}\hat{x} + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{s} + R\dot{\theta}\dot{\hat{s}} + \ddot{y}\hat{y}$$

$$\dot{\hat{s}} = -\dot{\theta}\hat{x} = -\frac{v}{R}\hat{x}$$

so $$\dot{\vec{v}} = (\ddot{R} - R\dot{\theta}^2)\hat{x} + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{s} + \ddot{y}\hat{y}$$

$$\dot{\vec{v}} = \boxed{\left(\ddot{x} - \frac{v^2}{R}\right)\hat{x}} + \frac{2\dot{x}v}{R}\hat{s} + \boxed{\ddot{y}\hat{y}}$$

Parameterizing Particle Motion: Eqn of Motion

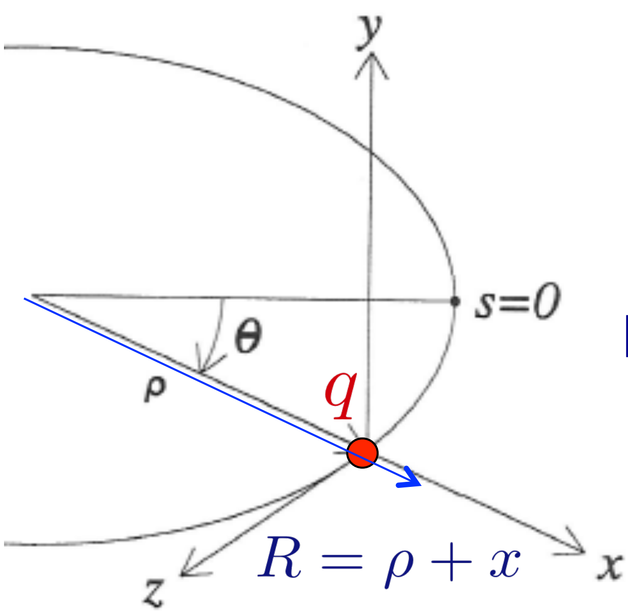
- Component equations of motion

Vertical:

$$F_y = q\beta c B_x = \gamma m \ddot{y} \qquad \ddot{y} - \frac{q\beta c B_x}{\gamma m} = 0$$

$$t = \frac{R}{\beta c}\theta \Rightarrow \frac{d}{dt} = \frac{\beta c}{R}\frac{d}{d\theta} \qquad \frac{d^2 y}{d\theta^2} - \frac{qB_x}{\beta c \gamma m} R^2 = 0$$

$$\boxed{\frac{d^2 y}{d\theta^2} - \frac{qB_x}{p} R^2 = 0}$$



Horizontal:

$$F_x = -q\beta c B_y = \gamma m \left(\ddot{x} - \frac{v^2}{R} \right)$$

$$\frac{d^2 x}{d\theta^2} = -\frac{qB_y R}{p} + R$$

$$\boxed{\frac{d^2 x}{d\theta^2} + \left(\frac{qB_y}{p} R - 1 \right) R = 0}$$

Equations of Motion

- Apply our paraxial and linearization approximations

$$p = qB_0\rho \qquad R = \rho\left(1+\frac{x}{\rho}\right) \quad \text{(A2)} \qquad B_y = B_0 + \left(\frac{\partial B_y}{\partial x}\right)x \quad \text{(A3)} \qquad B_x = \left(\frac{\partial B_y}{\partial x}\right)y \quad \text{(A3)}$$

Horizontal: $$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p}R - 1\right)R = 0$$

$$\frac{d^2x}{d\theta^2} + \left[\left(1+\frac{1}{B_0}\frac{\partial B_y}{\partial x}x\right)\left(1+\frac{x}{\rho}\right) - 1\right]\rho\left(1+\frac{x}{\rho}\right) = 0 \quad \text{(A2)}$$

$$\Rightarrow \boxed{\frac{d^2x}{d\theta^2} + (1-n)x = 0} \qquad \text{where} \qquad \boxed{n \equiv -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)}$$

Vertical: $$\frac{d^2y}{d\theta^2} - \frac{qB_x}{p}R^2 = 0 \qquad \Rightarrow \boxed{\frac{d^2y}{d\theta^2} + ny = 0}$$

Homework (Wednesday)

- Expand the horizontal equation of motion to second order in x
  - Does it reduce to the stated equation at first order?
  - Use expansions for R, B that are still first order!

- Expand the horizontal and vertical equations of motion to second order in x, y, δ (p. 43 of these slides)
  - Use expansions for R, B that are still first order!

Simple Equations of Motion!

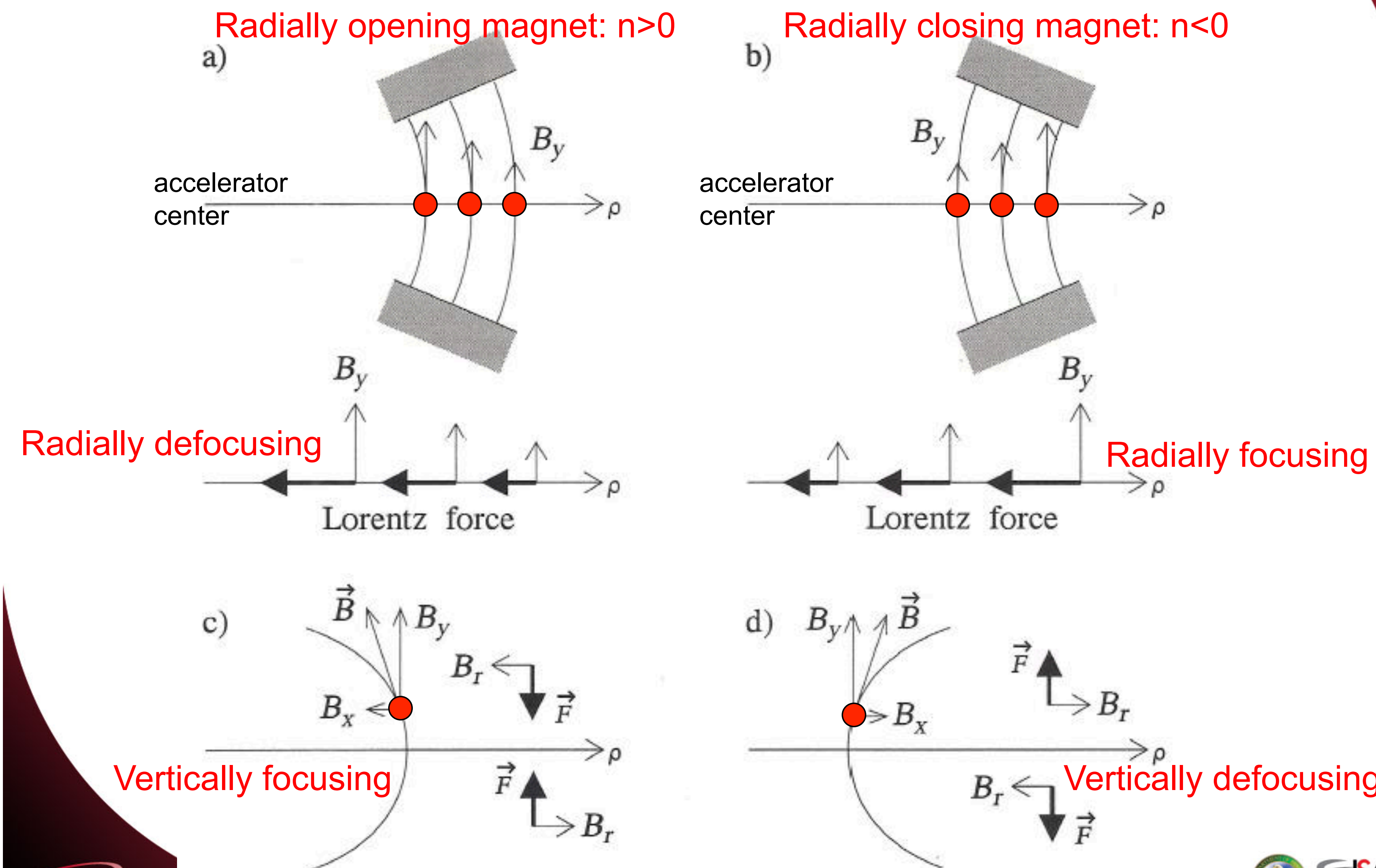
$$\frac{d^2x}{d\theta^2} + (1-n)x = 0 \qquad \frac{d^2y}{d\theta^2} + ny = 0$$

- These are like simple harmonic oscillator equations
  (Not surprising since we linearized 2^{nd} order differential equations)

- These are known as the **weak focusing** equations
  If n does not depend on θ, stability is only possible in both planes if **0<n<1**
  This is known as the weak focusing criterion

electron velocity INTO page

Weak Focusing Forces



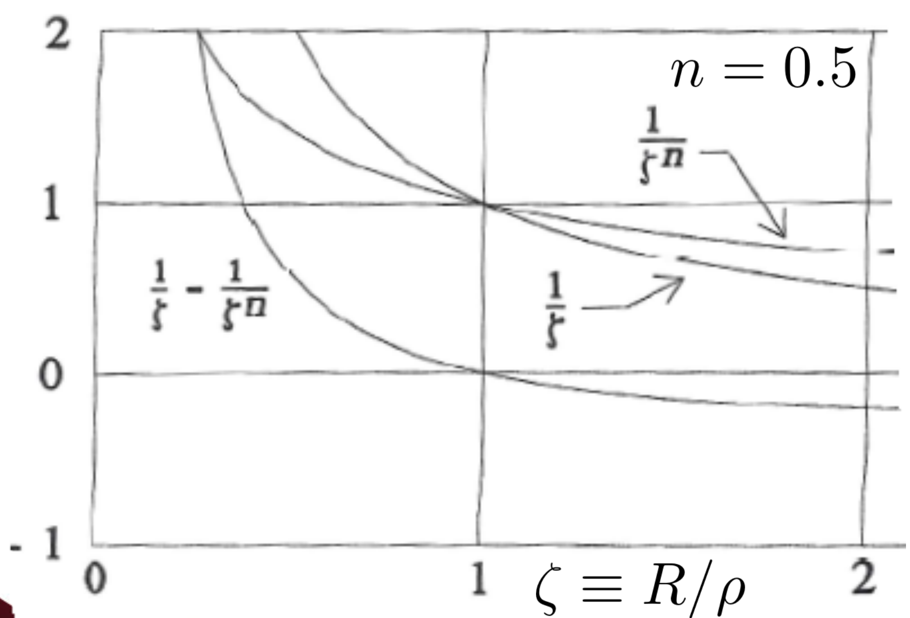
But Wasn't 0<n<1 Stable?

- This seems to indicate n>0 is horizontally unstable!
- Horizontal motion is a combination of two forces

  Centrifugal mv^2/R and centripetal Lorentz qvB_y

  Both forces cancel **by definition** for the design trajectory

$$F_{tot} = \frac{mv^2}{R} - qvB_y \approx \frac{mv^2}{\rho}\left(\frac{\rho}{R}\right) - qvB_0\left(\frac{\rho}{R}\right)^n = qvB_0\left(\frac{1}{\zeta} - \frac{1}{\zeta^n}\right)$$

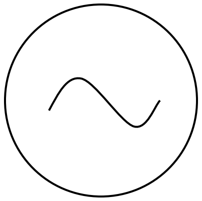


where $\zeta \equiv \frac{R}{\rho}$

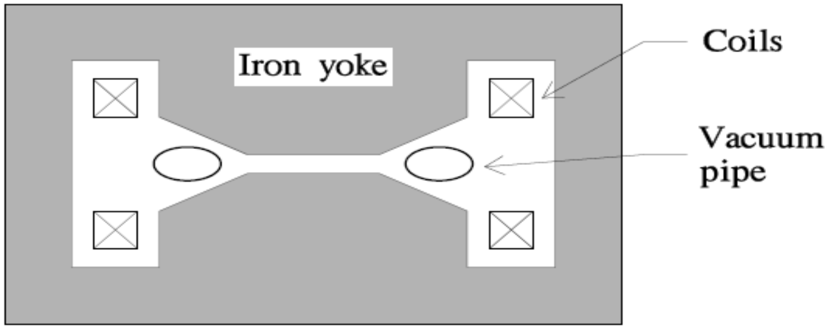
$F_{tot} > 0$ for $\zeta < 0$

$F_{tot} < 0$ for $\zeta > 0$

The Betatron



$I(t)=I_0 \cos(2\pi\omega_I t)$



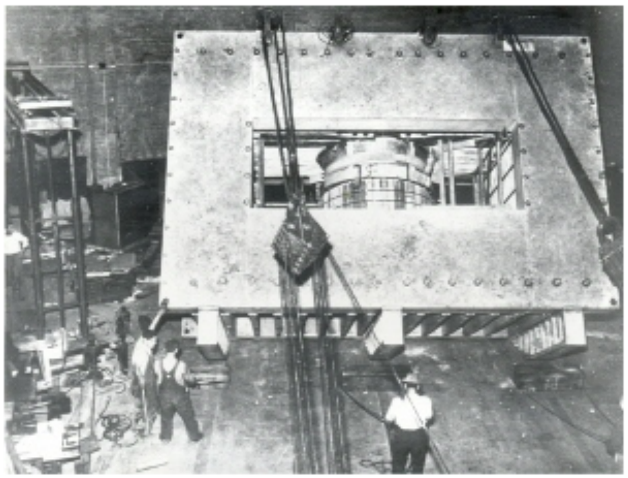
- Weak focusing formalism was originally developed for the Betatron
- Apply Faraday's law with time-varying current in coils
- Beam sees time-varying accelerating electric field too!
- Early proofs of stability: focusing and "betatron" motion

Donald Kerst

UIUC 2.5 MeV

Betatron, 1940

Don't try this at home!!



UIUC 312 MeV

betatron, 1949

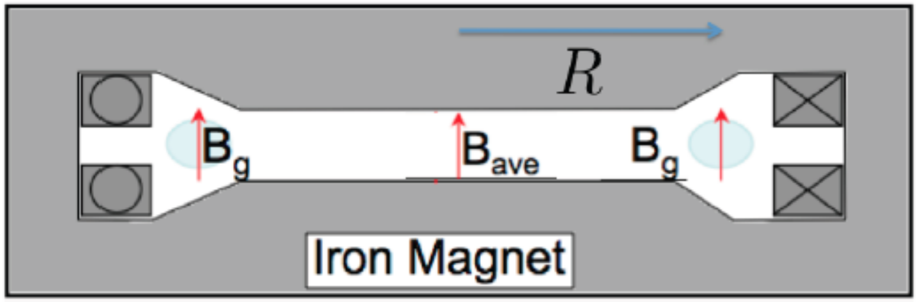
Really don't try this at home!!

The Betatron

apply sinusoidally varying current to toroidal conductors

$$I = I_0 \sin(\omega t)$$

$$B_{\text{ave}} = B_0 \sin(\omega t)$$



cylindrically symmetric about center vertical axis

$$\text{Magnetic flux} \quad \phi \approx \pi R^2 B_{\text{ave}} = \pi R^2 B_0 \sin(\omega t)$$

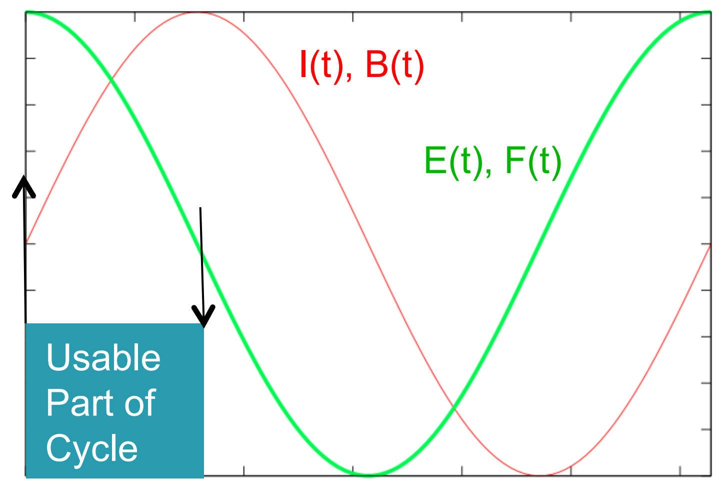
$$\text{Faraday's law} \quad \mathcal{E} = -\frac{d\phi}{dt} \qquad \oint \vec{E} \cdot d\vec{r} = 2\pi R E = -\frac{d\phi}{dt} = -\pi R^2 B_0 \omega \cos(\omega t)$$

$$\text{Force on electron} \quad F = qE = -\frac{qR}{2}\frac{dB_{\text{ave}}}{dt} = \frac{eRB_0\omega}{2}\cos(\omega t)$$

Circular motion

$$p/q = B_g R \quad \Rightarrow \quad F = \frac{dp}{dt} = eR\frac{dB_g}{dt}$$

$$\text{Betatron Field Condition} \quad B_g = \frac{B_0}{2}$$



Back to Solutions of Equations of Motion

$$\frac{d^2x}{d\theta^2} + (1-n)x = 0 \qquad \frac{d^2y}{d\theta^2} + ny = 0$$

- Assume azimuthal symmetry (n does not depend on θ)
- Solutions are simple harmonic oscillator solutions

$$x(\theta) = A\cos(\theta\sqrt{1-n}) + B\sin(\theta\sqrt{1-n})$$

$$\frac{dx}{d\theta} = \sqrt{1-n}[-A\sin(\theta\sqrt{1-n}) + B\cos(\theta\sqrt{1-n})]$$

- Constants A,B are related to initial conditions (x_0, x_0')

$$x_0 = x(\theta = 0) = A \qquad x_0' = \frac{1}{\rho}\left(\frac{dx}{d\theta}\right)(\theta = 0) = \frac{\sqrt{1-n}}{\rho}B$$

$$A = x_0 \qquad B = \frac{\rho}{\sqrt{1-n}}x_0'$$

Solutions of Equations of Motion

- Write down solutions in terms of initial conditions

$$x(\theta) = \cos(\theta\sqrt{1-n})\, x_0 + \frac{\rho}{\sqrt{1-n}} \sin(\theta\sqrt{1-n})\, x_0'$$

$$x'(\theta) = \frac{1}{\rho}\frac{dx}{d\theta} = -\frac{\sqrt{1-n}}{\rho} \sin(\theta\sqrt{1-n})\, x_0 + \cos(\theta\sqrt{1-n})\, x_0'$$

- This can be (very) conveniently written as matrices (including both horizontal and vertical)

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}} \sin(\theta\sqrt{1-n}) \\ -\frac{\sqrt{1-n}}{\rho} \sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}} \sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho} \sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Transport Matrices

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \boxed{\begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}}\sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho}\sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix}} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \boxed{M_V(\theta)} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

- M_V here is an example of a **transport matrix**
  - Linear: derived from linear equations
    - Will be concatenated to make further transformations

$$M_V(\theta_1 + \theta_2) = M_V(\theta_1) M_V(\theta_2)$$

  - Depends only on "length" θ, radius ρ, and "field" n
    - Acts to transform or transport coordinates to a new state
    - Our accelerator "lattices" will be built out of these matrices
  - Unimodular: det(M_V)=1
    - More strongly, it's symplectic: $S = M_V^T \, S \, M_V \quad \text{where} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
    - Hamiltonian dynamics, phase space conservation (Liouville)
    - These matrices here are scaled rotations!

Sinusoidal Solutions, Betatron Phases

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}}\sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho}\sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

- Sinusoidal simple harmonic oscillators solutions
  - Particles move in transverse **betatron oscillations** around the **design trajectory** $(x, x') = (y, y') = 0$
- We define **betatron phases**

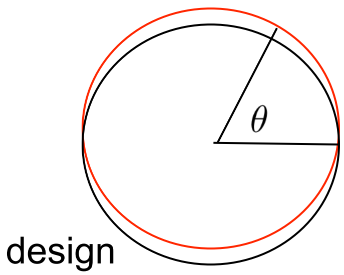
$$\phi_x(s) \equiv \theta\sqrt{1-n} = \frac{s}{\rho}\sqrt{1-n} \qquad \phi_y(s) \equiv \theta\sqrt{n} = \frac{s}{\rho}\sqrt{n}$$

Write matrix equation in terms of s rather than θ

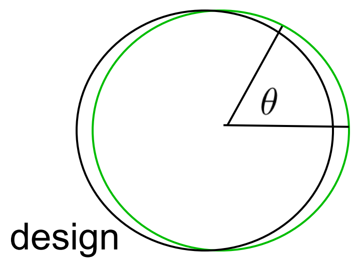
$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos\phi_x(s) & \frac{\rho}{\sqrt{1-n}}\sin\phi_x(s) \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi_x(s) & \cos\phi_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Visualization of Betatron Oscillations

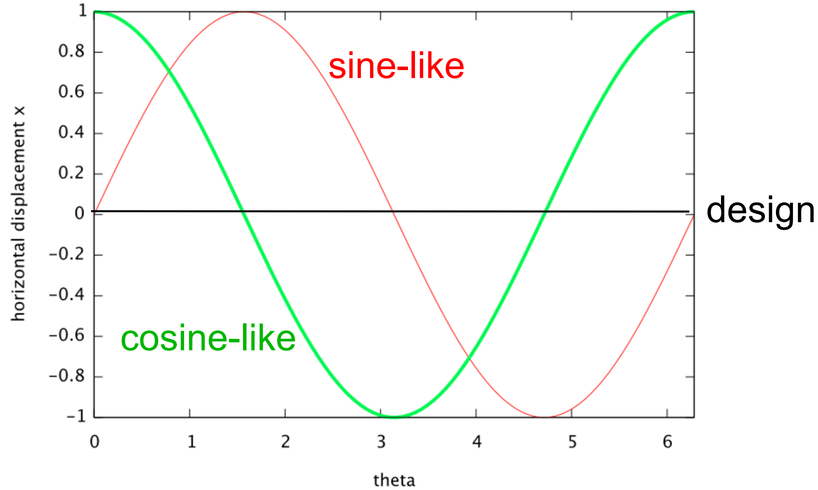
- Simplest case: constant uniform vertical field (n=0)



sine-like
$x_0 = 0, x_0' \neq 0$

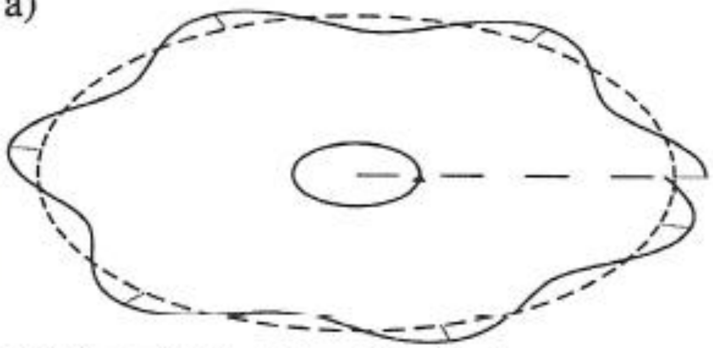


cosine-like
$x_0 \neq 0, x_0' = 0$



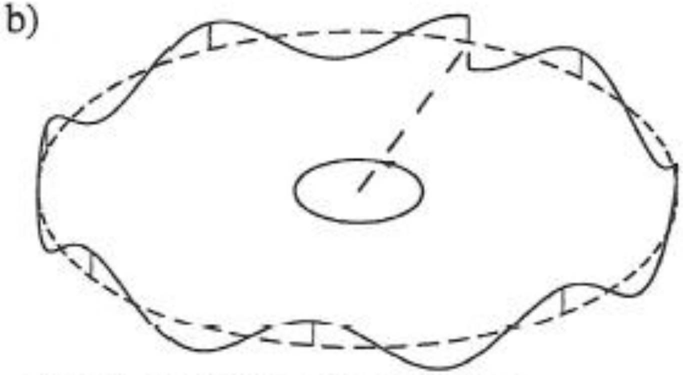
- More complicated **strong focusing**

a)



Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$,
i.e., 6.3 oscillations per turn.

b)



Vertical Betatron Oscillation
with tune: $Q_V = 7.5$,
i.e., 7.5 oscillations per turn.

Visualization of Betatron Oscillations, Tunes

- What happens for 0<n<1?
  - Example picture below has 5 “turns” with sin(0.89 θ)
  - The betatron oscillation precesses, not strictly periodic
  - **Betatron tune $Q_{X,Y}$**: number of cycles made for every revolution or turn around accelerator

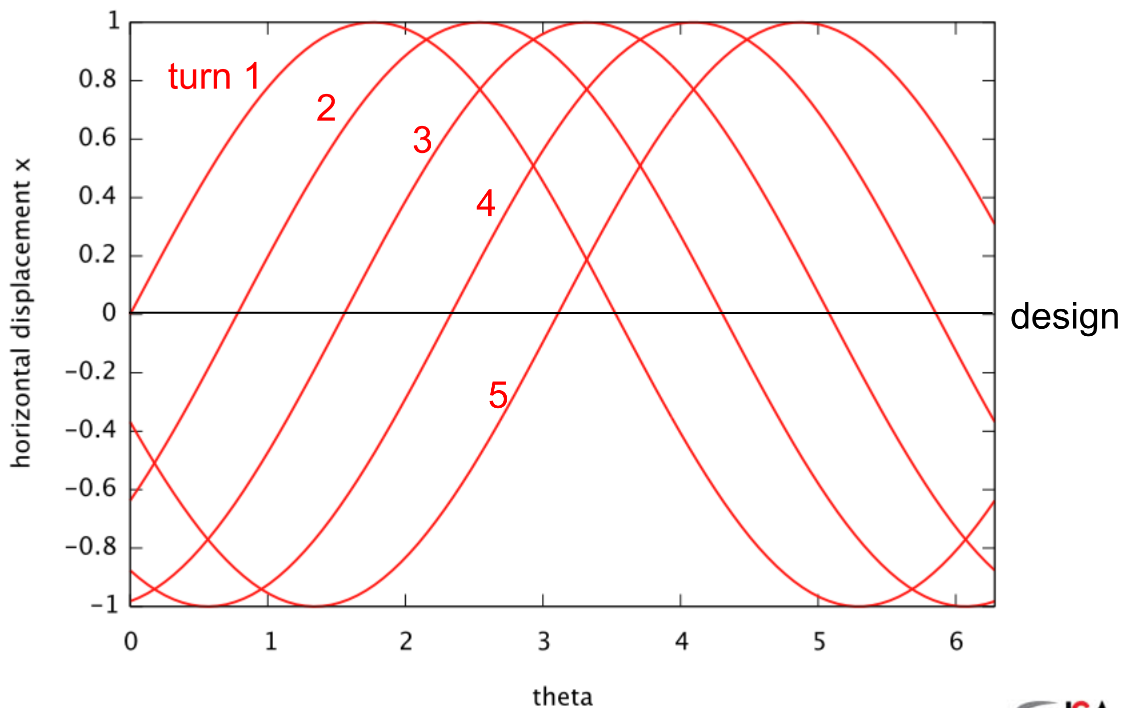
$$Q_x = \frac{1}{2\pi}\sqrt{1-n}(2\pi) = \sqrt{1-n}$$

$$Q_y = \frac{1}{2\pi}\sqrt{n}(2\pi) = \sqrt{n}$$

Frequency of betatron oscillations relative to turns around accelerator

For weak focusing:

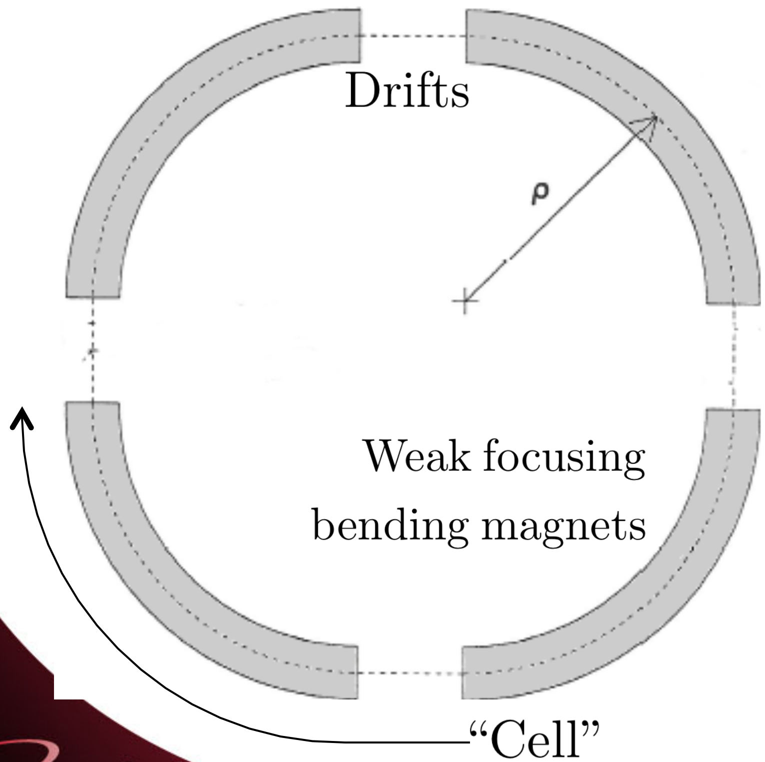
$$Q_x^2 + Q_y^2 = 1$$



Transport Matrices: Piecewise Solutions

$$\begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \boxed{\begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{\rho}{\sqrt{n}}\sin(\theta\sqrt{n}) \\ -\frac{\sqrt{n}}{\rho}\sin(\theta\sqrt{n}) & \cos(\theta\sqrt{n}) \end{pmatrix}} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} = \boxed{M_V(\theta)} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

- Linear transport matrices make piecewise solutions of equations of motion accessible



"Cell" transport matrix:

$$M_{\text{cell}} = M(\theta)\, M_{\text{drift}}$$

"One turn" transport matrix:

$$M_{\text{one turn}} = (M(\theta)\, M_{\text{drift}})^4$$

Build accelerator optics out of "Lego" transport matrices

Transport Matrices: Accelerator Legos

(A3)

- With linear fields, there are two basic types of Legos
  - Dipoles $$\vec{B} = B_0\hat{y} + (x\hat{y} + y\hat{x})\left(\frac{\partial B_y}{\partial x}\right) \qquad B_0 \neq 0$$
    - Often long magnets to bend design trajectory
    - Entrance/exit locations can become important
    - May or may not include focusing (“combined function”)
    - Special case: drift when all B components are zero
  - Quadrupoles $$\vec{B} = (x\hat{y} + y\hat{x})\left(\frac{\partial B_y}{\partial x}\right) \qquad B_0 = 0$$
    - Design trajectory is straight! (no fields at x=y=0)
    - Act to focus particles moving off of design trajectory
    - Special case: “thin lens” approximation
    - We’ll talk about quadrupoles tomorrow (Thursday)

Transport Matrices: Dipole

- We have already derived a very general **transport matrix for a dipole magnet with focusing**

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\phi_x(s) & \frac{\rho}{\sqrt{1-n}}\sin\phi_x(s) & 0 & 0 \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi_x(s) & \cos\phi_x(s) & 0 & 0 \\ 0 & 0 & \cos\phi_y(s) & \frac{\rho}{\sqrt{n}}\sin\phi_y(s) \\ 0 & 0 & -\frac{\sqrt{n}}{\rho}\sin\phi_y(s) & \cos\phi_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

$$\phi_x(s) \equiv \theta\sqrt{1-n} = \frac{s}{\rho}\sqrt{1-n} \qquad \phi_y(s) \equiv \theta\sqrt{n} = \frac{s}{\rho}\sqrt{n}$$

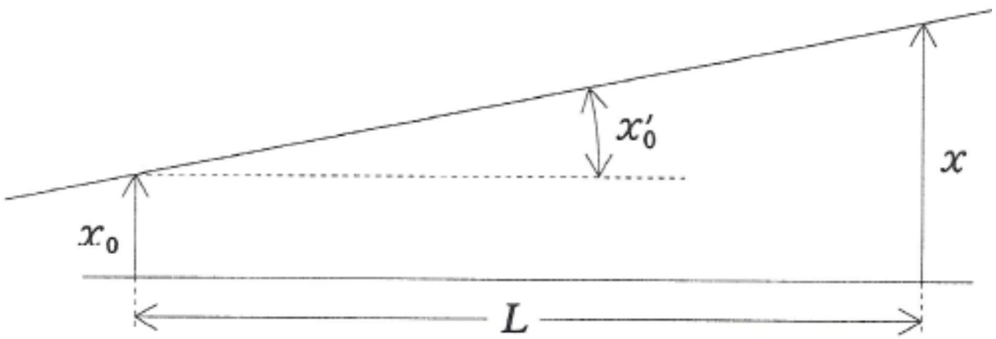
- Taking n->0 (and being careful) gives the **transport matrix for a dipole of bend angle θ without focusing**

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \rho\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Transport Matrices: Drifts

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \rho\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

- For n=0, there is no horizontal field or vertical force
  - The vertical transport matrix here is for a **field-free drift**
  - This applies in both x,y planes when there is no field



$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

What About Momentum?

- So far we have assumed that the design trajectory particle and our particle have the **same momentum**
- How do equations change if we break this assumption?
  - Expect only horizontal motion changes to first order (A2)

$$p = p_0(1+\delta) \text{ where } \delta \equiv \frac{\Delta p}{p_0} \ll 1 \qquad p_0 = \text{design particle momentum}$$

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p}R - 1\right)R = 0 \quad \Rightarrow \quad \frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0(1+\delta)}R - 1\right)R = 0$$

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0}(1-\delta)R - 1\right)R = 0$$ (A1)

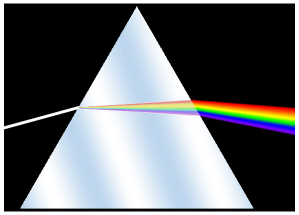
$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0}R - 1\right)R = \delta\frac{R^2 qB_y}{p_0} = \rho\delta$$ (A2)

$$\boxed{\frac{d^2x}{d\theta^2} + (1-n)x = \rho\delta}$$

Add inhomogeneous term to original δ=0 equation of motion

Solutions of Dispersive Equations of Motion

$$\frac{d^2x}{d\theta^2} + (1-n)x = \rho\delta \qquad \frac{d^2y}{d\theta^2} + ny = 0$$



- This momentum effect is called **dispersion**
  - Similar to prism light dispersion in classical optics
- Solutions are simple harmonic oscillator solutions
  - But now we add a specific inhomogeneous solution

$$x(\theta) = A\cos(\theta\sqrt{1-n}) + B\sin(\theta\sqrt{1-n}) + \frac{\rho}{1-n}\delta$$

inhomogeneous term!

$$\frac{dx}{d\theta} = \sqrt{1-n}[-A\sin(\theta\sqrt{1-n}) + B\cos(\theta\sqrt{1-n})]$$

- Constants A,B again related to initial conditions (x_0, x_0')

$$A = x_0 - \frac{\rho}{1-n}\delta \qquad B = \frac{\rho}{\sqrt{1-n}}x_0'$$

δ is constant (A4)

Solutions of Dispersive Equations of Motion

- Write down solutions in terms of initial conditions

$$x'(\theta) = -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n})x_0 + \cos(\theta\sqrt{1-n})x'_0 + \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n})\delta_0$$

$$x'(\theta) = \frac{1}{\rho}\frac{dx}{d\theta} = -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n})x_0 + \cos(\theta\sqrt{1-n})x'_0 + \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n})\delta_0$$

$$\delta = \delta_0$$

- This can be now be "conveniently" written in terms of a 3x3 matrix:

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \\ \delta(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) & \frac{\rho}{1-n}[1-\cos(\theta\sqrt{1-n})] \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) & \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

As usual, this can be simplified for n=0 (pure dipole)

Note that δ has become a "coordinate"!

Example: 180 Degree Dipole Magnet

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \\ \delta(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta\sqrt{1-n}) & \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) & \frac{\rho}{1-n}[1-\cos(\theta\sqrt{1-n})] \\ -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n}) & \cos(\theta\sqrt{1-n}) & \frac{1}{\sqrt{1-n}}\sin(\theta\sqrt{1-n}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

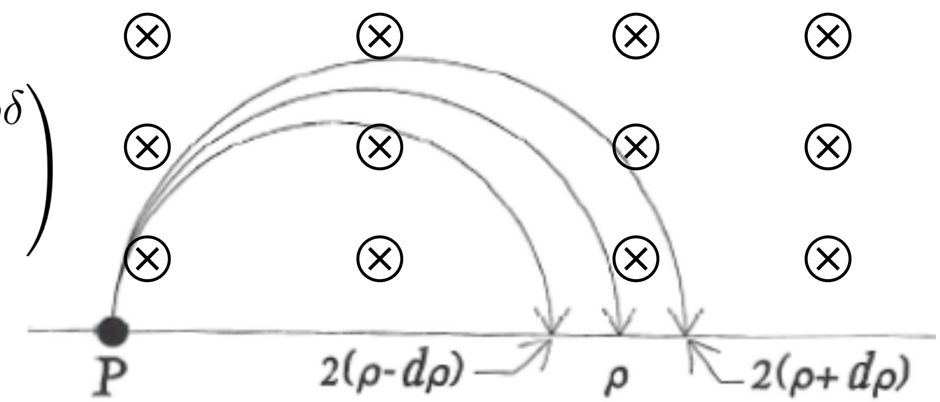
$$n = 0 \quad \Rightarrow \quad M_H(\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$n = 0,\ \theta = \pi \quad \Rightarrow \quad M_H(\theta) = \begin{pmatrix} -1 & 0 & 2\rho \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{For initial coordinates} \begin{pmatrix} 0 \\ 0 \\ \pm\delta \end{pmatrix}$$

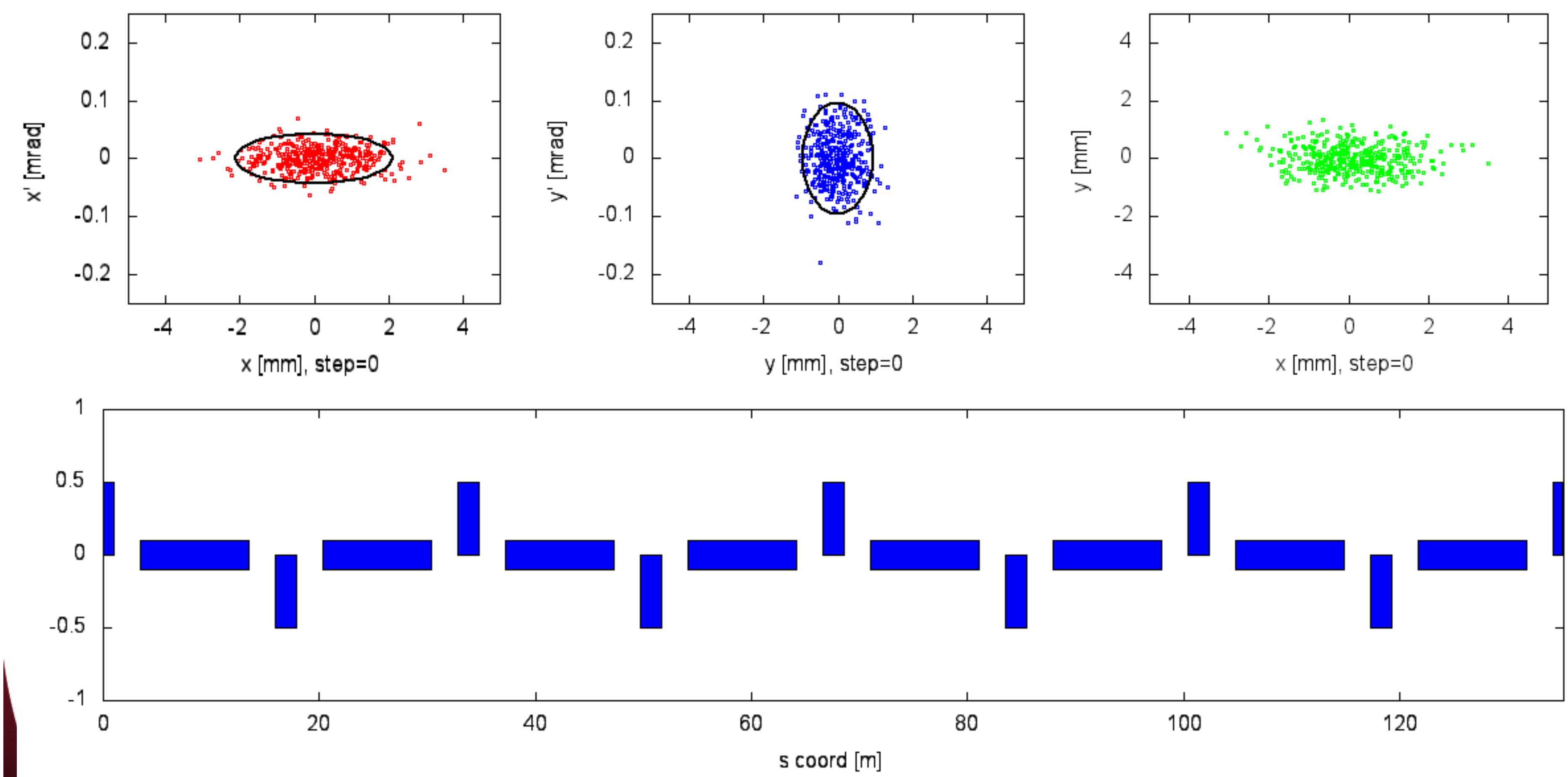
$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2\rho \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \pm\delta \end{pmatrix} = \begin{pmatrix} \pm 2\rho\delta \\ 0 \\ \pm\delta \end{pmatrix}$$

electrons moving through uniform vertical B field



This makes sense from p/q=Bρ!

=========== The Future ===========



========= Extra Slides =========

Lorentz Lie Group Generators I

- Lorentz transformations can be described by a Lie group where a general Lorentz transformation is

$$A = e^L \qquad \det A = e^{\mathrm{Tr}\, L} = +1$$

where L is 4x4, real, and traceless. With metric g, the matrix gL is also antisymmetric, so L has the general six-parameter form

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

Deep and profound connection to EM tensor $F^{\alpha\beta}$

J.D. Jackson, Classical Electrodynamics 2nd Ed, Section 11.7

Lorentz Lie Group Generators II

- A reasonable basis is provided by six generators
  - Three generate rotations in three dimensions

$$S_{1,2,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  - Three generate boosts in three dimensions

$$K_{1,2,3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Lorentz Lie Group Generators III

- $(S_{1,2,3})^2$ and $(K_{1,2,3})^2$ are diagonal.
- $(\epsilon \cdot S)^3 = -\epsilon \cdot S$ and $(\epsilon \cdot K)^3 = \epsilon \cdot K$ for any unit 3-vector ϵ
- Nice commutation relations:

$$[S_i, S_j] = \epsilon_{ijk} S_k \quad [S_i, K_j] = \epsilon_{ijk} K_k \quad [K_i, K_j] = -\epsilon_{ijk} S_k$$

- We can then write the Lorentz transformation in terms of two three-vectors (6 parameters) ω, ζ as

$$L = -\omega \cdot S - \zeta \cdot K \qquad A = e^{-\omega \cdot S - \zeta \cdot K}$$

  - Electric fields correspond to boosts
  - Magnetic fields correspond to rotations
  - Deep beauty in Poincare, Lorentz, Einstein connections

(Frames and Lorentz Transformations)

- The lab frame will dominate most of our discussions
  - But not always (synchrotron radiation, space charge…)

- Invariance of space-time interval (Minkowski)

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2$$

- Lorentz transformation of four-vectors
  - For example, time/space coordinates in z velocity boost

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

(Four-Velocity and Four-Momentum)

- The proper time interval $d\tau = dt/\gamma$ is Lorentz invariant
- So we can make a velocity 4-vector

$$cu^\alpha \equiv \left(\frac{dct}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = c\gamma(1, \beta_x, \beta_y, \beta_z)$$

$$\text{Metric } g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

- We can also make a 4-momentum

$$p^\alpha \equiv mcu^\alpha = mc\gamma(1, \beta_x, \beta_y, \beta_z)$$

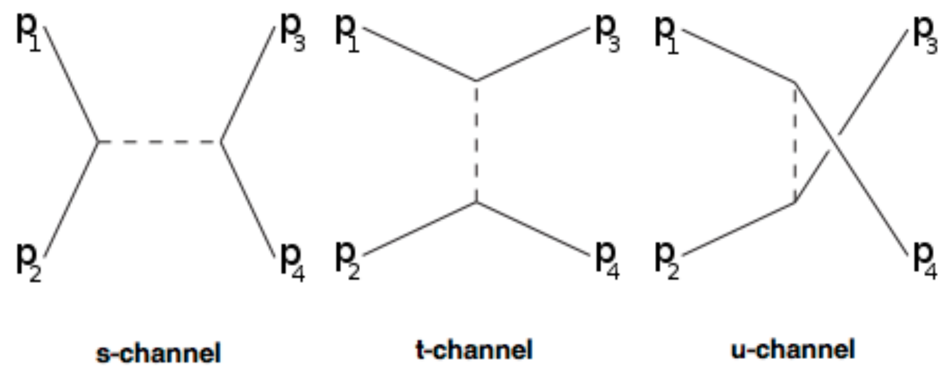
- Double-check that Minkowski norms are invariant

$$u^\alpha u_\alpha = u^\alpha g_{\alpha\beta} u^\beta = \gamma^2(1 - \beta^2) = 1$$

$$p^\alpha p_\alpha = m^2c^2 u^\alpha u_\alpha = m^2c^2$$

(Mandelstam Variables)

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$



$$s + t + u = (m_1^2 + m_2^2 + m_3^2 + m_4^2)c^2$$

- Lorentz-invariant two-body kinematic variables
  - p_{1-4} are four-momenta
- √s is the total available center of mass energy
  - Often quoted for colliders
- Used in calculations of other two-body scattering processes
  - Moller scattering (e-e), Compton scattering (e-γ)

(Relativistic Newton)

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$$

- But now we can define a four-vector force in terms of four-momenta and proper time:

$$F^\alpha \equiv \frac{dp^\alpha}{d\tau}$$

- We are primarily concerned with electrodynamics so now we must make the classical electromagnetic force obey Lorentz transformations

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

(Lorentz Lie Group Generators)

- Lorentz transformations can be described by a Lie group where a general Lorentz transformation is

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where L is 4x4, real, and traceless. With metric g, the matrix gL is also antisymmetric, so L has the general six-parameter form

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Deep and profound connection to EM tensor $F^{\alpha\beta}$

J.D. Jackson, Classical Electrodynamics 2nd Ed, Section 11.7

Livingston, Lawrence, 27"/69 cm Cyclotron

M.S. Livingston and E.O. Lawrence, 1934

The Joy of Physics

- Describing the events of January 9, 1932, Livingston is quoted saying:

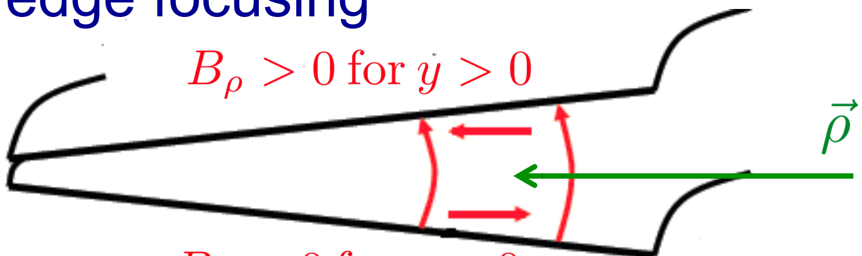
"I recall the day when *I had adjusted the oscillator to a new high frequency*, and, with *Lawrence looking over my shoulder*, tuned the magnet through resonance. As the galvanometer spot swung across the scale, indicating that protons of 1-MeV energy were reaching the collector, *Lawrence literally danced around the room with glee*. The news quickly spread through the Berkeley laboratory, and we were busy all that day demonstrating million-volt protons to eager viewers."

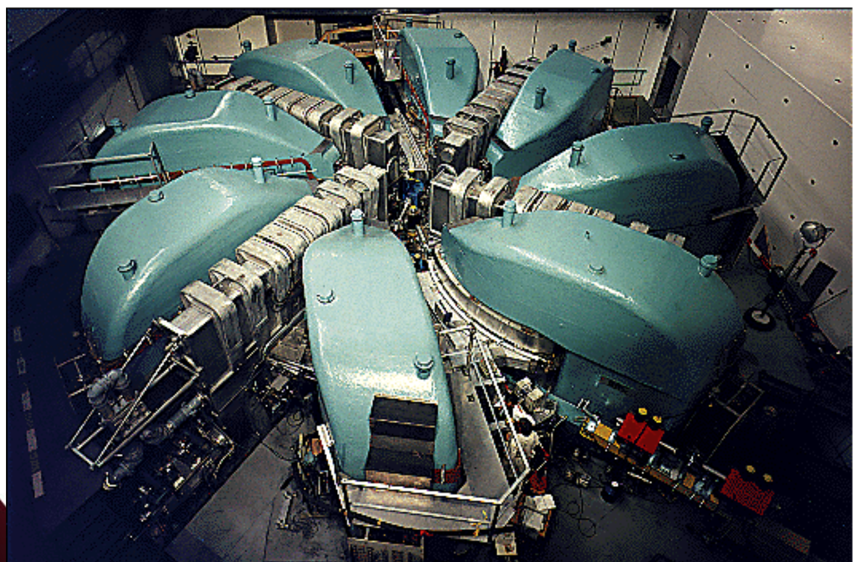
APS Physics History, "Ernest Lawrence and M. Stanley Livingston

Modern Isochronous Cyclotrons

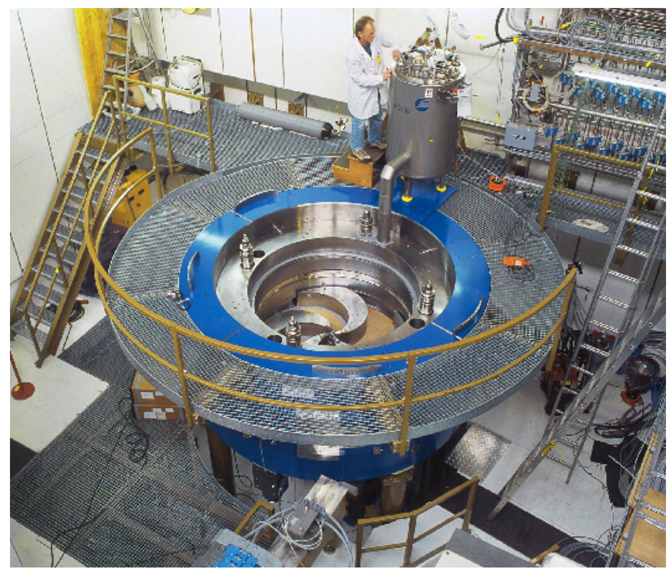
- Higher bending field at higher energies
  - But also introduces vertical defocusing
  - Use bending magnet "edge focusing"
    (later magnet lecture)

$$f_{\rm rf} = \frac{qB(\rho)}{2\pi\gamma(\rho)m}$$





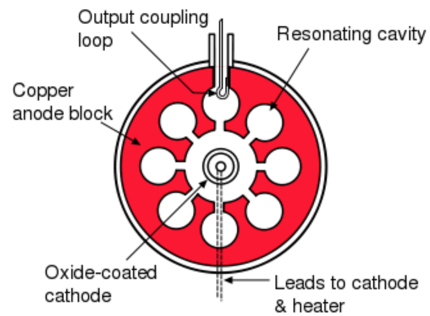
590 MeV PSI Isochronous Cyclotron (1974)



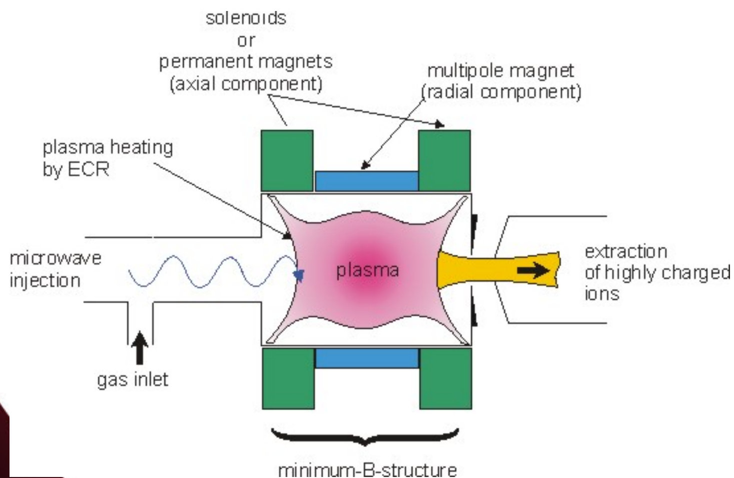
250 MeV PSI Isochronous Cyclotron (2004)

Electrons, Magnetrons, ECRs

Radar/microwave magnetron



Resonant cavity magnetron high-power high-frequency oscillator



ECR plasma/ion source

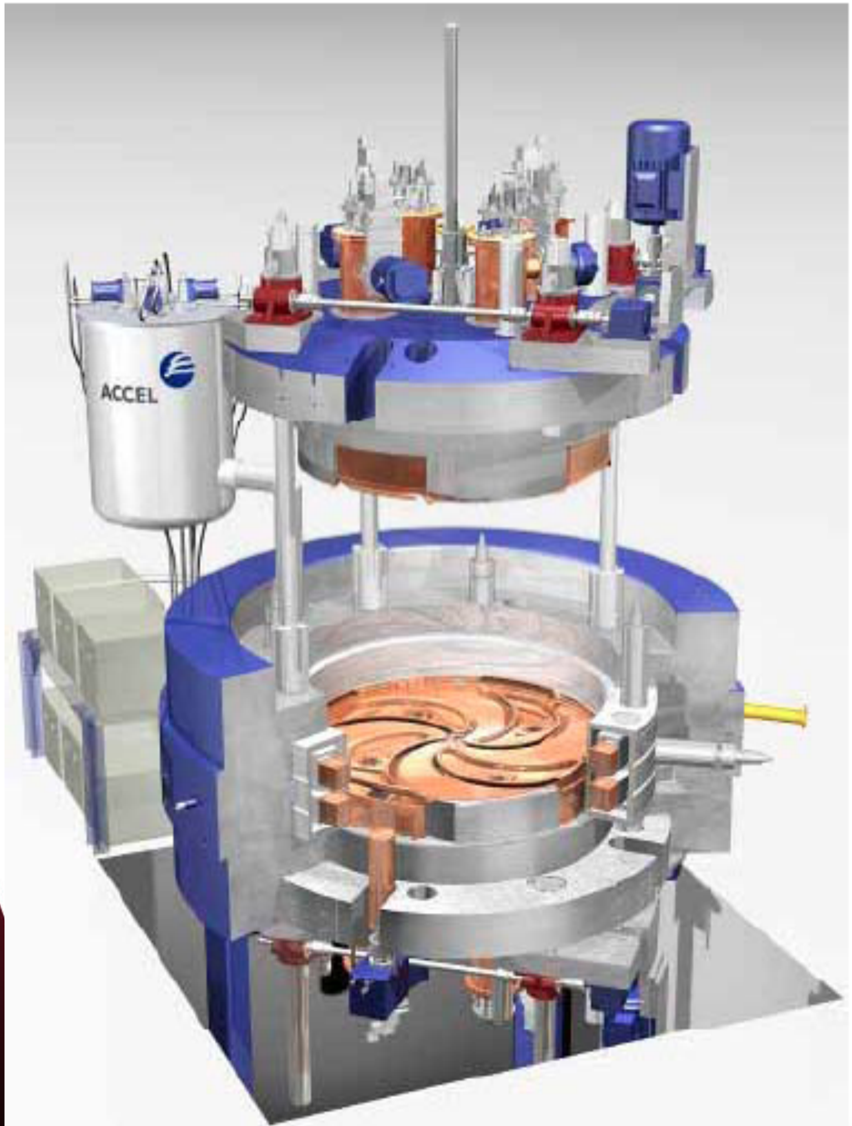
- Cyclotrons aren't very good for accelerating electrons
  - γ changes too quickly!
- But narrow-band response has advantages and uses
  - Microtrons

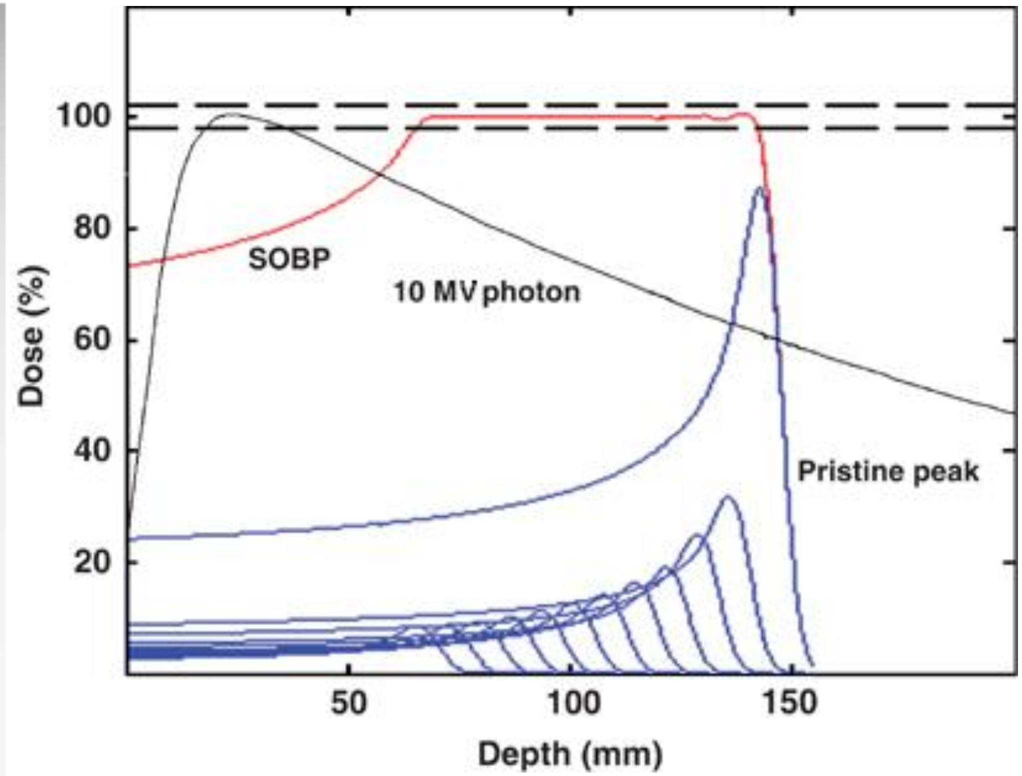
    generate high-power microwaves from circulating electron current
  - ECRs
    - generate high-intensity ion beams and plasmas by resonantly stripping electrons with microwaves

Cyclotrons Today

- Cyclotrons continue to evolve
  - Many contemporary developments
    - Superconducting cyclotrons
    - Synchrocyclotrons (FM modulated RF)
    - Isochronous/Alternating Vertical Focusing (AVF)
    - FFAGs (Fixed Field Alternating Gradient)
  - Versatile with many applications even below ~500 MeV
    - High power (>1MW) neutron production
    - Reliable (medical isotope production, ion radiotherapy)
    - Power+reliability: ~5MW p beam for ADSR?

Accel Radiotherapy Cyclotron





Distinct dose localization advantage for hadrons over X-rays

Also present work on proton and carbon radiotherapy fast-cycling synchrotrons