Introduction to Accelerator Physics
2011 Mexican Particle Accelerator School

Lecture 4/7: Stability, FODO Cells, More Lattice Functions, Emittance, Chromaticity (and maybe Dispersion)

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Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi} - \text{periodic ansatz solution} \quad x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \quad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(s)\sin\Delta\phi_C & \beta(s)\sin\Delta\phi_C \\ -\gamma(s)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(s)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

betatron phase advance $\Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$ $\quad \mathrm{Tr}\, M = 2\cos\Delta\phi_C$

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\Delta\phi_C}$$

Transport Matrix Stability Criteria

- For long systems (rings) we want $M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ stable as $n \to \infty$
  - If 2x2 M has eigenvectors (V_1, V_2) and eigenvalues (λ_1, λ_2):

$$M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = A\lambda_1^n V_1 + B\lambda_2^n V_2$$

  - M is also unimodular (det M=1) so $\lambda_{1,2} = e^{\pm i\phi}$ with complex ϕ
  - For $\lambda_{1,2}^n$ to remain bounded, ϕ must be real

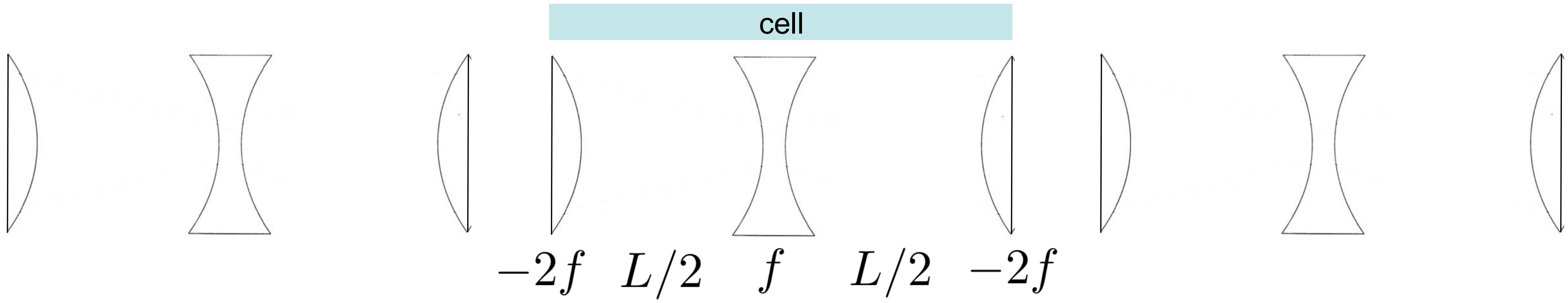
  - We can always transform M into diagonal form with the eigenvalues on the diagonal (since det M=1); this does not change the trace of the matrix

$$e^{i\phi} + e^{-i\phi} = 2\cos\phi = \mathrm{Tr}\ M$$

  - The **stability requirement** for these types of matrices is then

$$\phi \text{ real} \quad \Rightarrow \quad \boxed{-1 \leq \frac{1}{2}\mathrm{Tr}\ M \leq 1}$$

Periodic Example: FODO Cell Phase Advance



- Select periodicity between centers of focusing quads
  - A natural periodicity if we want to calculate maximum β(s)

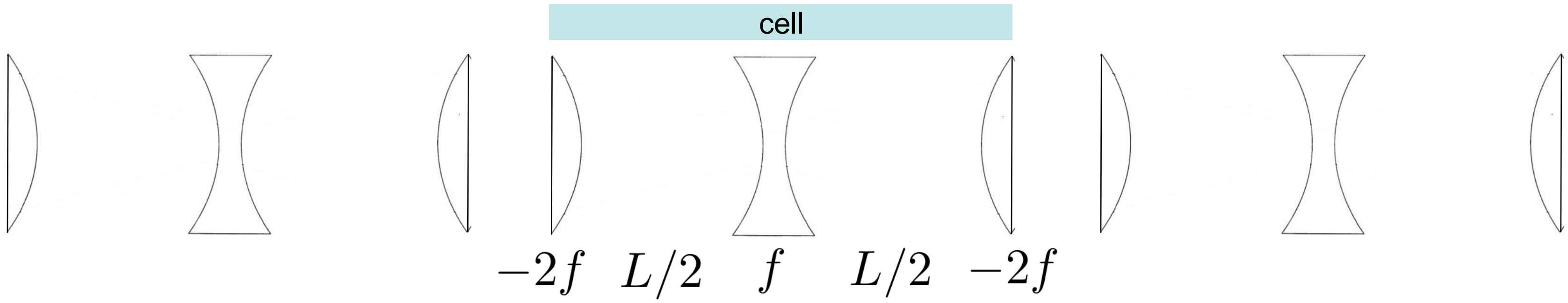
$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \qquad \mathrm{Tr}\, M = 2\cos\Delta\phi_C = 2 - \frac{L^2}{4f^2}$$

$$1 - \frac{L^2}{8f^2} = \cos\Delta\phi_C = 1 - 2\sin^2\frac{\Delta\phi_C}{2} \qquad \Rightarrow \qquad \sin\frac{\Delta\phi_C}{2} = \pm\frac{L}{4f}$$

  - $\Delta\phi_C$ only has real solutions (stability) if $\frac{L}{4} < f$

Periodic Example: FODO Cell Beta Max/Min



- What is $\hat{\beta}$?
  - A natural periodicity if we want to calculate maximum β(s)

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \Leftarrow \quad M_{12} = \beta \sin \Delta\phi_C$$

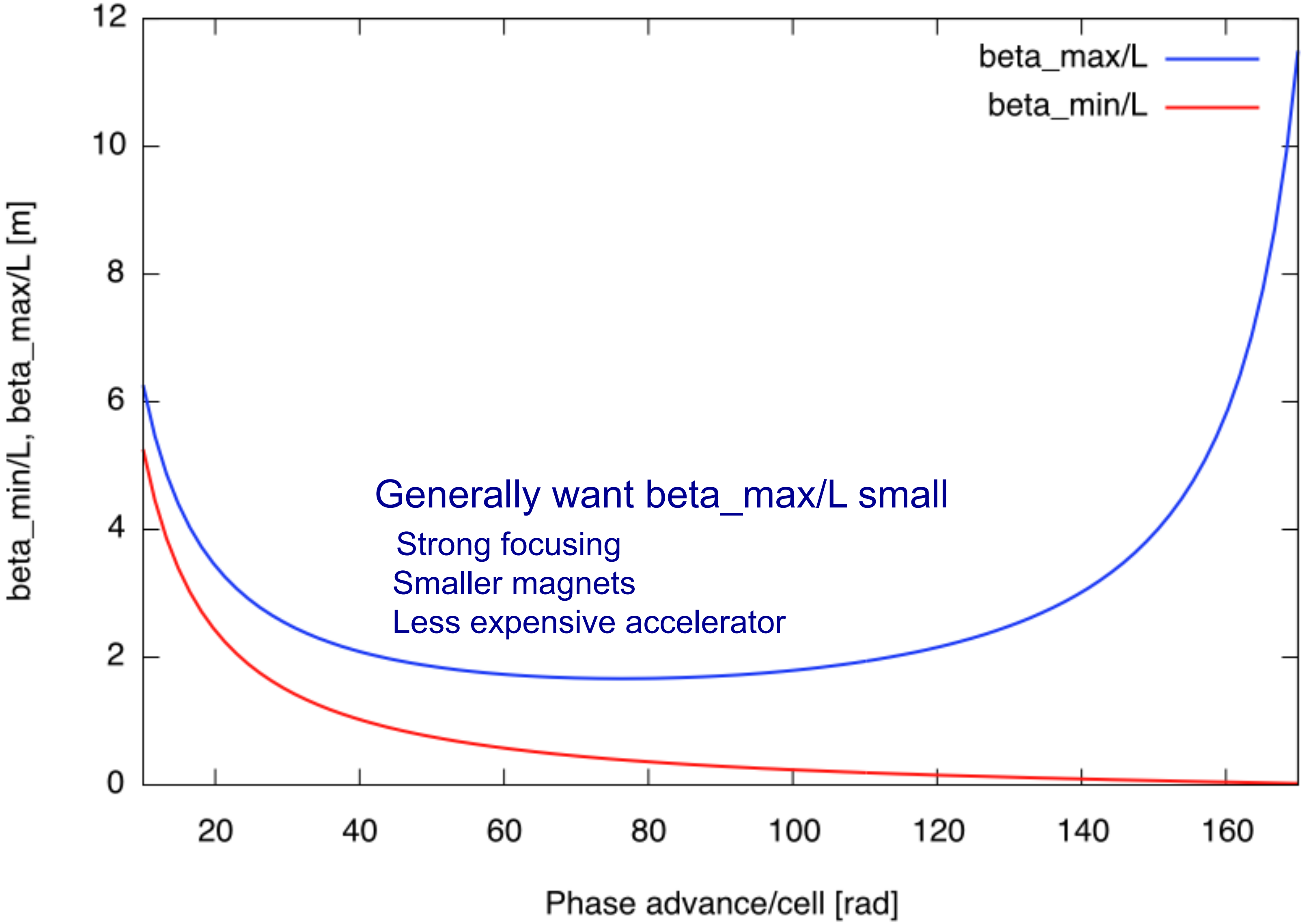
$$\hat{\beta} \sin \Delta\phi_C = \frac{L^2}{4f} + L = L\left(1 + \sin \frac{\Delta\phi_C}{2}\right)$$

$$\boxed{\hat{\beta} = \frac{L}{\sin \Delta\phi_C}\left(1 + \sin \frac{\Delta\phi_C}{2}\right)}$$

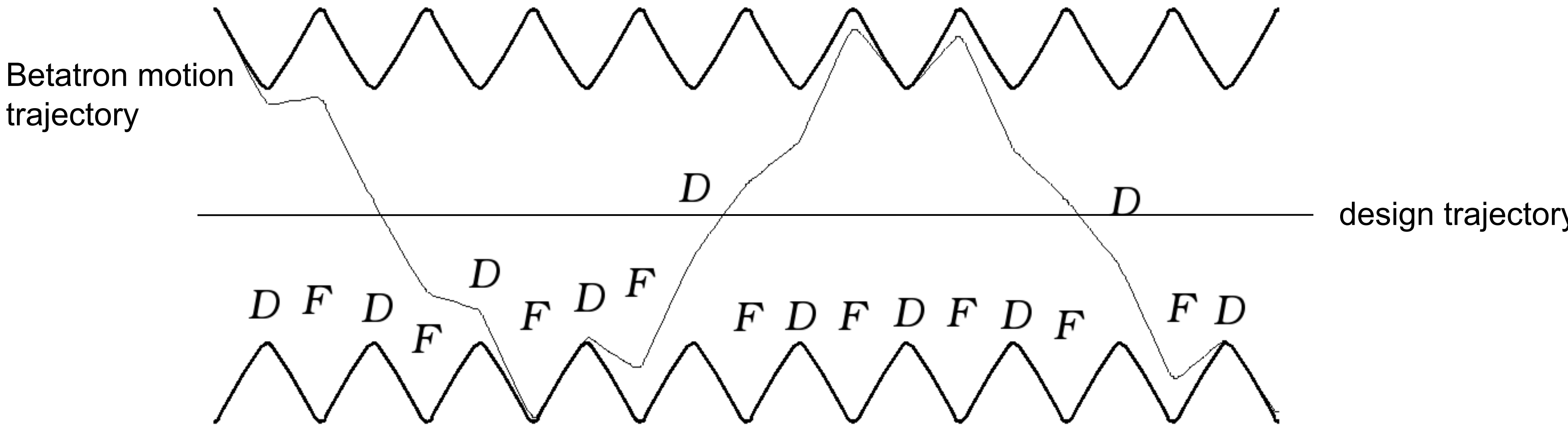
  - Follow a similar strategy reversing F/D quadrupoles to find the minimum β(s) within a FODO cell (center of D quad)

$$\boxed{\check{\beta} = \frac{L}{\sin \Delta\phi_C}\left(1 - \sin \frac{\Delta\phi_C}{2}\right)}$$

FODO Betatron Functions vs Phase Advance



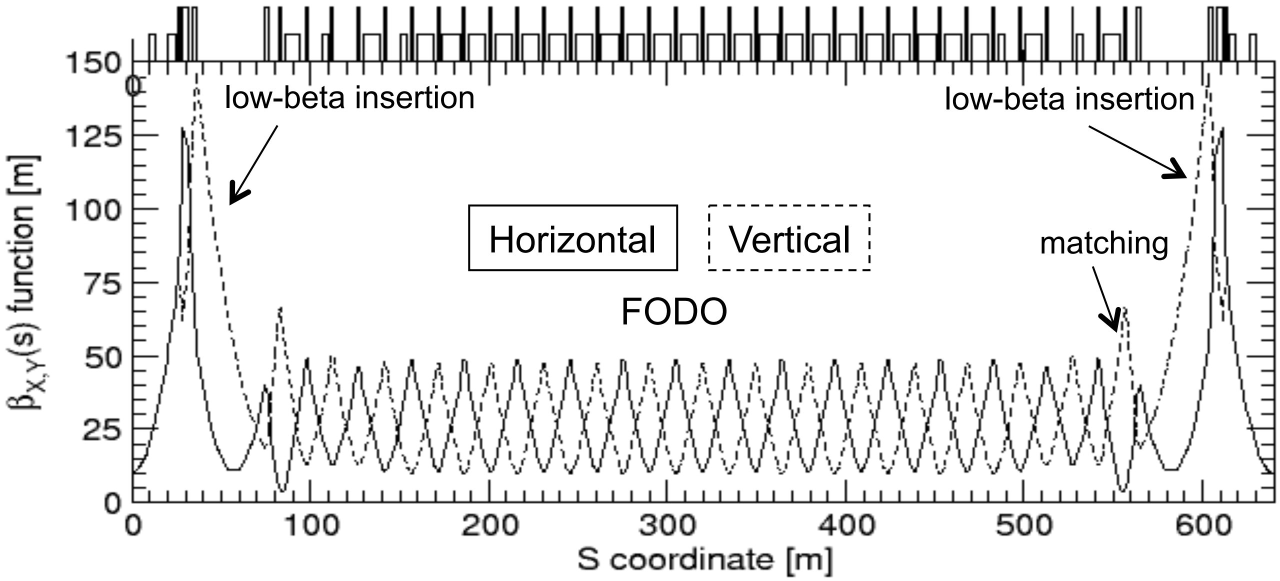
FODO Beta Function, Betatron Motion



- This is a picture of a FODO lattice, showing contours of $\pm\sqrt{\beta(s)}$ since the particle motion goes like $x(s) = A\sqrt{\beta(s)}\cos[\phi(s)+\phi_0]$
  - This also shows a particle oscillating through the lattice
  - Note that $\sqrt{\beta(s)}$ provides an “envelope” for particle oscillations
    - $\sqrt{\beta(s)}$ is sometimes called the envelope function for the lattice
  - Min beta is at defocusing quads, max beta is at focusing quads
  - 6.5 periodic FODO cells per betatron oscillation

$$\Rightarrow \quad \Delta\phi_C = 360^\circ/6.5 \approx 55^\circ$$

Example: RHIC FODO Lattice



- 1/6 of one of two RHIC synchrotron rings, injection lattice
  - FODO cell length is about L=30m
  - Phase advance per FODO cell is about $\Delta\phi_C = 77^\circ = 1.344\,\mathrm{rad}$

$$\hat{\beta} = \frac{L}{\sin\Delta\phi_C}\left(1 + \sin\frac{\Delta\phi_C}{2}\right) \approx 53\ \mathrm{m}$$

☺

$$\check{\beta} = \frac{L}{\sin\Delta\phi_C}\left(1 - \sin\frac{\Delta\phi_C}{2}\right) \approx 8.7\ \mathrm{m}$$

General Non-Periodic Transport Matrix

- We can parameterize a general non-periodic transport matrix from s_1 to s_2 using the lattice parameters

$$M(s_2) = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}[\cos\Delta\phi + \alpha(s_1)\sin\Delta\phi] & \sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\phi \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\phi + [1+\alpha(s_1)\alpha(s_2)]\sin\Delta\phi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\phi - \alpha(s_2)\sin\Delta\phi] \end{pmatrix}$$

- This does not have a pretty form like the periodic matrix

However both can be expressed as

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

The most common use of this matrix is the m_{12} term:

$$\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\sin(\Delta\phi)\, x'(s_1)$$

Effect of angle kick on downstream position

(Deriving the Non-Periodic Transport Matrix)

$$x(s) = Aw(s)\cos\phi(s) + Bw(s)\sin\phi(s)$$

$$x'(s) = A\left(w'(s)\cos\phi(s) - \frac{\sin\phi(s)}{w(s)}\right) + B\left(w'(s)\sin\phi(s) + \frac{\cos\phi(s)}{w(s)}\right)$$

Calculate A, B in terms of initial conditions (x_0, x_0') and (w_0, ϕ_0)

$$A = \left(w_0'\sin\phi_0 + \frac{\cos\phi_0}{w_0}\right)x_0 - (w_0\sin\phi_0)x_0'$$

$$B = -\left(w_0'\cos\phi_0 - \frac{\sin\phi_0}{w_0}\right)x_0 + (w_0\cos\phi_0)x_0'$$

Substitute (A,B) and put into matrix form: $\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$

$$m_{11}(s) = \frac{w(s)}{w_0}\cos\Delta\phi - w(s)w_0'\sin\Delta\phi$$

$$\Delta\phi \equiv \phi(s) - \phi_0$$

$$w(s) = \sqrt{\beta(s)}$$

$$m_{12}(s) = w(s)w_0\sin\Delta\phi$$

$$m_{21}(s) = -\frac{1 + w(s)w_0w'(s)w_0'}{w(s)w_0}\sin\Delta\phi - \left[\frac{w_0'}{w(s)} - \frac{w'(s)}{w_0}\right]\cos\Delta\phi$$

$$m_{22}(s) = \frac{w_0}{w(s)}\cos\Delta\phi + w_0w'\sin\Delta\phi$$

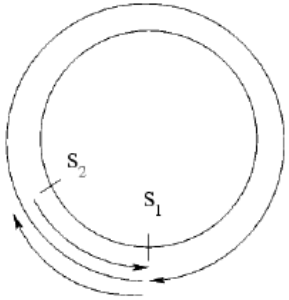
Propagating Lattice Parameters

- If I have $(\beta, \alpha, \gamma)(s_1)$ and I have the transport matrix $M(s_1, s_2)$ that transports particles from $s_1 \to s_2$, how do I find the new lattice parameters $(\beta, \alpha, \gamma)(s_2)$?

$$M(s_1, s_1 + C) = I \cos\mu + J \sin\mu = \begin{pmatrix} \cos\mu + \alpha(s_1)\sin\mu & \beta(s_1)\sin\mu \\ -\gamma(s_1)\sin\mu & \cos\mu - \alpha(s_1)\sin\mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The J(s) matrices at s_1, s_2 are related by

$$J(s_2) = M(s_1, s_2) J(s_1) M^{-1}(s_1, s_2)$$

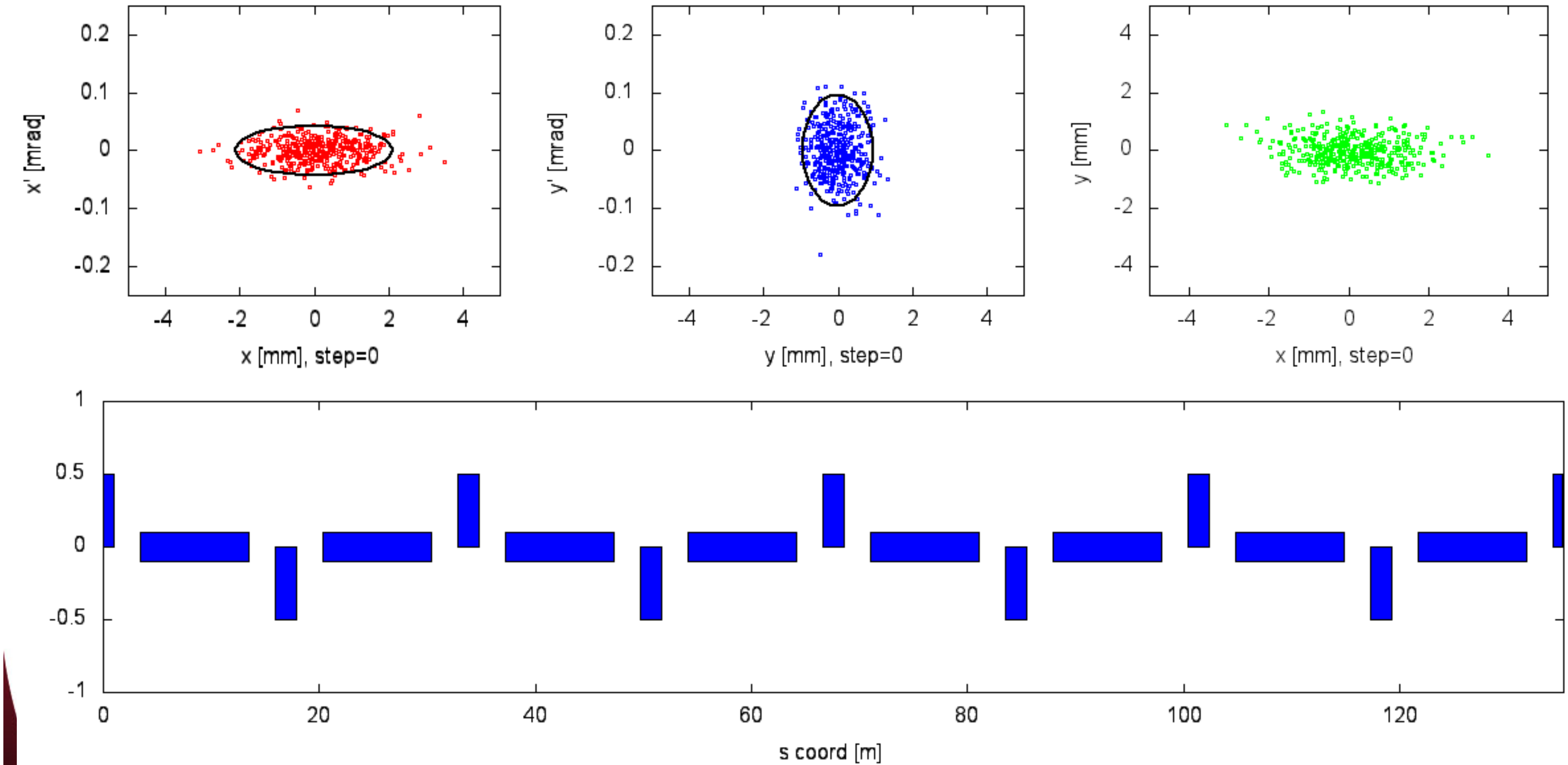


Then expand, using det M=1

$$J(s_2) = \begin{pmatrix} \alpha(s_2) & \beta(s_2) \\ -\gamma(s_2) & -\alpha(s_2) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \alpha(s_1) & \beta(s_1) \\ -\gamma(s_1) & -\alpha(s_1) \end{pmatrix} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

$$\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}$$

=========== What's the Ellipse? ===========



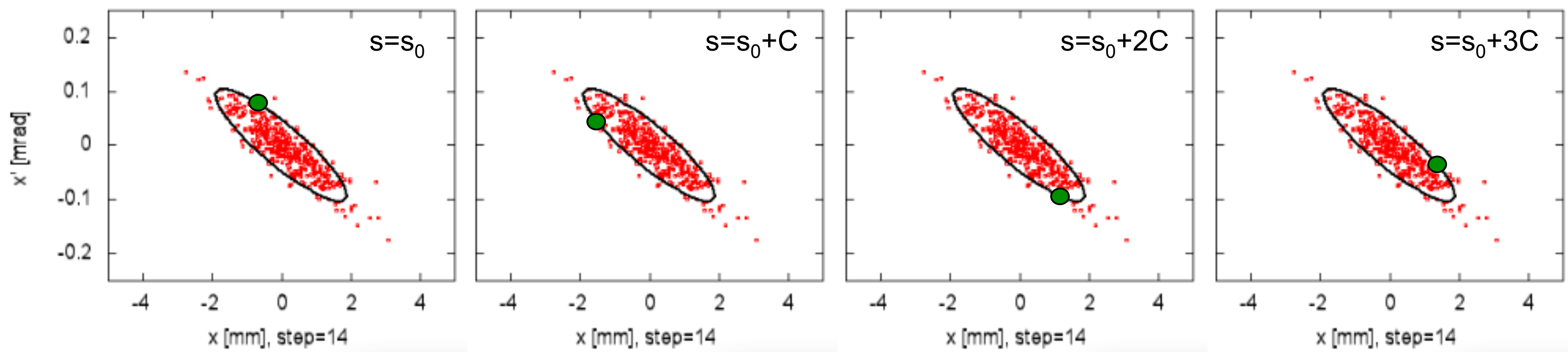
- Area of an ellipse that envelops a given percentage of the beam particles in phase space is related to the **emittance**

We can express this in terms of our lattice functions!

Invariants and Ellipses

$$x(s) = A\sqrt{\beta(s)}\cos[\phi(s)+\phi_0]$$

- We assumed A was constant, an **invariant of the motion** (A4)
  - A can be expressed in terms of initial coordinates to find
$$\mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$$
  - This is known as the **Courant-Snyder invariant**: for all s,
$$\mathcal{W} = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$
  - Similar to total energy of a simple harmonic oscillator
  - $\mathcal{W}$ looks like an elliptical area in (x, x') phase space
  - Our matrices look like scaled rotations (ellipses) in phase space



Emittance

- The area of the ellipse inscribed by any given particle in phase space as it travels through our accelerator is called the **emittance** ϵ: it is constant (A4) and given by

$$\epsilon = \pi\mathcal{W} = \pi[\gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2]$$

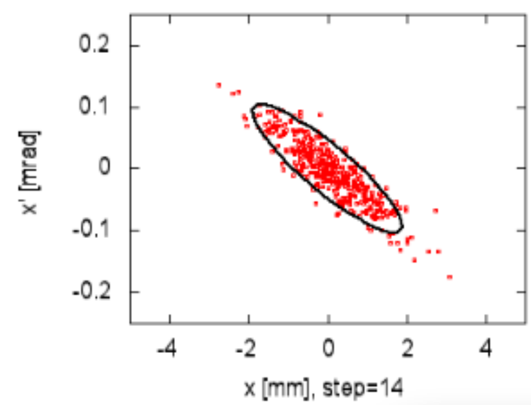
Emittance is often quoted as the area of the ellipse that would contain a certain fraction of all (Gaussian) beam particles

e.g. RMS emittance contains 39% of 2D beam particles

Related to RMS beam size σ_{RMS}

$$\sigma_{\mathrm{RMS}} = \sqrt{\epsilon\beta(s)}$$

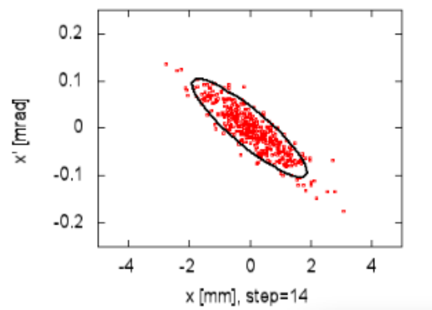
Yes, this RMS beam size depends on s!

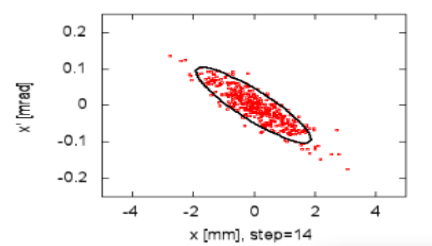


RMS emittance convention is fairly standard for electron rings, with units of mm-mrad

Adiabatic Damping and Normalized Emittance

- But assumption (A4) is violated when we accelerate!
  - When we accelerate, invariant emittance is not invariant!
  - We are defining areas in (x, x') phase space
  - The definition of x doesn't change as we accelerate
  - But $x' \equiv dx/ds = p_x/p_0$ **does** since p_0 changes!
  - p_0 scales with relativistic beta, gamma: $p_0 \propto \beta_r \gamma_r$
  - This has the effect of compressing x' phase space by $\beta_r \gamma_r$



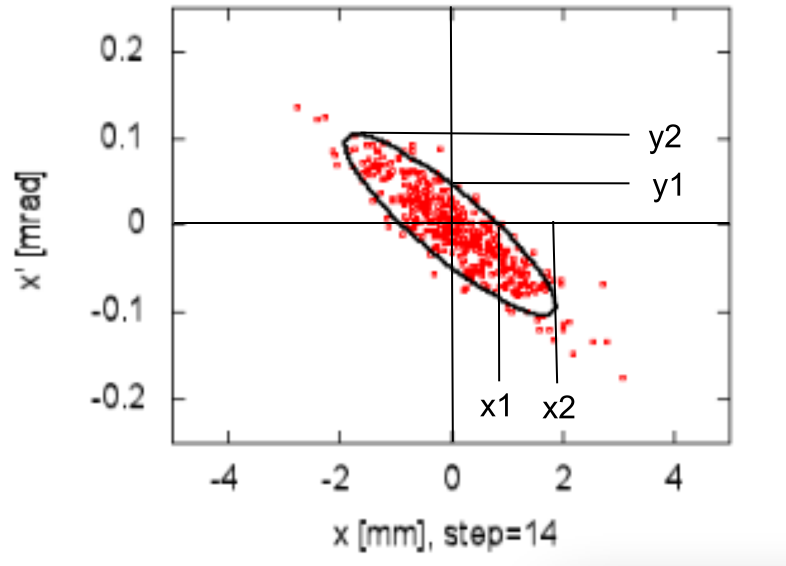


  - **Normalized emittance** is the invariant in this case $\epsilon_N = \beta_r \gamma_r \epsilon$

    unnormalized emittance goes down as we accelerate

    This is called **adiabatic damping**, important in, e.g., linacs

Phase Space Ellipse Geography



- Now we can figure out some things from a phase space ellipse at a given s coordinate:

$$x_1 = \sqrt{\mathcal{W}/\gamma(s)} \qquad x_2 = \sqrt{\mathcal{W}\beta(s)}$$

$$y_1 = \sqrt{\mathcal{W}/\beta(s)} \qquad y_2 = \sqrt{\mathcal{W}\gamma(s)}$$

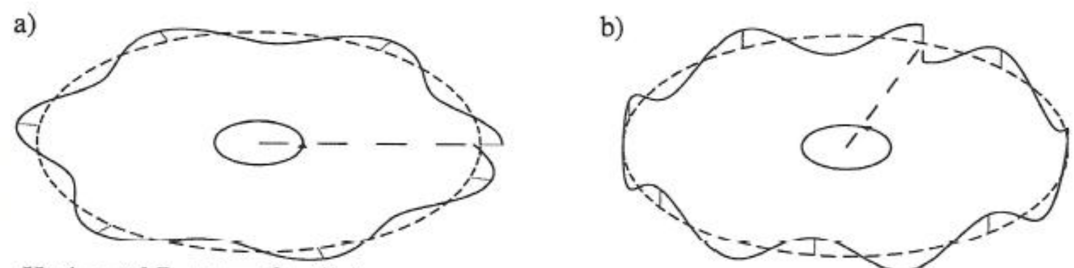
Rings and Tunes

- A synchrotron is by definition a periodic focusing system
  - It is very likely made up of many smaller periodic regions too
  - We can write down a periodic **one-turn matrix** as before

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

  - Recall that we defined **tune** as the total betatron phase advance in one revolution around a ring divided by 2π

$$Q_{x,y} = \frac{\Delta\phi_{x,y}}{\Delta\theta} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$



a) Horizontal Betatron Oscillation with tune: $Q_h = 6.3$, i.e., 6.3 oscillations per turn.

b) Vertical Betatron Oscillation with tune: $Q_v = 7.5$, i.e., 7.5 oscillations per turn.

Tunes

- There are horizontal and vertical tunes
  - turn by turn oscillation frequency

- Tunes are a direct indication of the amount of focusing in an accelerator
  - Higher tune implies tighter focusing, lower $\langle \beta_{x,y}(s) \rangle$

- Tunes are a critical parameter for accelerator performance
  - Linear stability depends greatly on phase advance
  - Resonant instabilities can occur when $nQ_x + mQ_y = k$
  - Often adjusted by changing groups of quadrupoles

$$M_{\text{one turn}} = I\cos(2\pi Q) + J\sin(2\pi Q)$$

Chromaticity

- Just like bending depended on momentum (dispersion), focusing (and thus tunes) depend on momentum
  - The variation of tunes with δ is called **chromaticity**
  - Insert a momentum perturbation is like adding a small extra focusing to our one-turn matrix that depends on the unperturbed focusing K_0

$$M_{\rm one\ turn}(\delta) = \begin{pmatrix} 1 & 0 \\ K_0\delta ds & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi Q) + \alpha\sin(2\pi Q) & \beta\sin(2\pi Q) \\ -\gamma\sin(2\pi Q) & \cos(2\pi Q) - \alpha\sin(2\pi Q) \end{pmatrix}$$

$$M_{\rm one\ turn}(\delta) = \begin{pmatrix} \cos(2\pi Q) + \alpha\sin(2\pi Q) & \beta\sin(2\pi Q) \\ -\gamma\sin(2\pi Q) + K_0\delta[\cos(2\pi Q) + \alpha\sin(2\pi Q)]ds & \cos(2\pi Q) - \alpha\sin(2\pi Q) + K_0\delta\beta\sin(2\pi Q)ds \end{pmatrix}$$

  - This looks painful, but remember the trace is related to the new tune

$$\cos(2\pi Q_{\rm new}) = \frac{1}{2}{\rm Tr}\, M = \cos(2\pi Q) + \frac{K_0\delta}{2}\beta\sin(2\pi Q)ds$$

Chromaticity Continued

$$\cos(2\pi Q_{\text{new}}) = \frac{1}{2}\text{Tr } M = \cos(2\pi Q) + \frac{K_0\delta}{2}\beta\sin(2\pi Q)ds$$

$$\cos(2\pi Q_{\text{new}}) = \cos(2\pi(Q+dQ)) \approx \cos(2\pi Q) - 2\pi\sin(2\pi Q)dQ$$

- These last two terms must be equal, which gives

$$dQ = -\frac{K(s)\delta}{4\pi}\beta(s)ds$$

Integrate around the ring to find the total tune change

$$\Delta Q = -\frac{\delta}{4\pi}\oint K(s)\beta(s)\,ds$$

Natural Chromaticity is defined as

$$\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q}\oint K(s)\beta(s)\,ds$$

The tune Q invariably has some spread from momentum spread

Homework

- Design a circular synchrotron made of 20 identical FODO cells, with bending dipoles in place of the drifts for 500 MeV electrons
  - What is the bend angle of each dipole?
    - For 1.5 T maximum dipole field, how long is each dipole?
    - How long is each FODO cell assuming the quads are thin quads?
  - Assume a reasonable FODO phase advance per cell
    - Treat the dipoles as drifts for the following analysis
  - Calculate
    - The minimum and maximum beta of each FODO cell
    - The tunes Q_X and Q_Y
    - The natural chromaticities ξ_X and ξ_Y (hint: the integral on p.20 becomes a sum for thin quadrupoles)

Dispersion Function $\eta(s)$

The generalized equation of motion of charge particles in magnets supplying bending and focusing effects is given by:

$$x'' + k_x(s)x = \frac{\delta}{\rho(s)} \qquad k_x(s) = \frac{1}{\rho^2} + \frac{q}{p_0}\frac{\partial B_y}{\partial x}, \quad \text{and}$$

$$y'' + k_y(s)y = 0, \qquad k_y(s) = -\frac{q}{p_0}\frac{\partial B_y}{\partial x},$$

$$\delta = \Delta p / p_0$$

Since the generalized solution of the homogeneous equation with $\delta = 0$ is given by

$$\vec{z}(s) = \mathbf{M}(s)\vec{z}_0 \qquad \mathbf{M}(s) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\rightarrow \quad x(s) = C(s)\,x_0 + S(s)\,x'_0$$

Then the generalized solution of the inhomogeneous equation with $\delta \neq 0$ can be written as

$$\rightarrow \quad \begin{aligned} x(s) &= C(s)\,x_0 + S(s)\,x'_0 + D(s)\,\delta_0 \\ x'(s) &= C'(s)\,x_0 + S'(s)\,x'_0 + D'(s)\,\delta_0 \end{aligned}$$

$$D(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}\,d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}\,d\tau$$

Here $D(s)$ is a particular solution with $\delta_0 = 1$ (see Problem 5-8 in Conte's book).

Dispersion Function $\eta(s)$

From the initial conditions (x_0, x'_0) at $s = 0$:

$$x(s) = C(s)\,x_0 + S(s)\,x'_0 + D(s)\,\delta_0$$
$$x'(s) = C'(s)\,x_0 + S'(s)\,x'_0 + D'(s)\,\delta_0$$

$$C(0)x_0 + S(0)x_0' + D(0)\delta_0 = x_0$$
$$C'(0)x_0 + S'(0)x_0' + D'(0)\delta_0 = x_0'$$

$$C(0) = S'(0) = 1$$

$$C'(0) = S(0) = 0$$

$$D(0) = D'(0) = 0$$

Since no change in energy spread is assumed, trajectory equations can be written in matrix form for $\delta \neq 0$:

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

Here, the trajectory $x(s)$ has two parts: a part due to betatron oscillation, $x_\beta(s)$ and the other part due to dispersion $\eta(s) = dx/d\delta$ **or** $x_p(s) = \eta(s)\,\delta$ $x(s) = x_\beta(s) + x_p(s)$

Dispersion Function $\eta(s)$

If we insert the trajectory due to the periodic dispersion $x_p(s) = \eta(s)\,\delta$ in the matrix form:

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

For a periodic cell, $x_0 = \eta(s_0)\delta_0 = \eta(s_0+L)\delta = \eta\delta$ (no energy spread change here).

$$\Rightarrow \quad \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

If we solve equation above, we can find a periodic dispersion function $\eta(s)$ and $\eta'(s)$.

$$\eta(s) = \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\mu)}$$

$$\eta'(s) = \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\mu)}$$

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\,\sin(\pi Q_{\mathrm{H}})} \int_s^{s+L} \frac{\sqrt{\beta(\tau)}}{\rho(\tau)} \cos\left[\phi(\tau) - \phi(s) - \pi Q_{\mathrm{H}}\right] d\tau$$

(see Problem 5-8 in Conte's book & S. Y. Lee's book)

here we used tr(M)= $C + S' = 2\cos\mu$ for $\mathrm{M}(s) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$.