Introduction to Accelerator Physics
2011 Mexican Particle Accelerator School

Lecture 5/7: Dispersion (including FODO), Dispersion Suppressor, Light Source Lattices (DBA, TBA, TME)

Todd Satogata (Jefferson Lab)
satogata@jlab.org
http://www.toddsatogata.net/2011-MePAS

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Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi} - \text{periodic ansatz solution} \quad x(s) = \sqrt{\epsilon\beta(s)}\cos[\phi(s) + \phi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \quad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(s)\sin\Delta\phi_C & \beta(s)\sin\Delta\phi_C \\ -\gamma(s)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(s)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

betatron phase advance $\Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$ $\quad \mathrm{Tr}\, M = 2\cos\Delta\phi_C$

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

$$\text{Courant} - \text{Snyder invariant} \quad \mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$$

Dispersion

- There is one more important lattice parameter to discuss
- **Dispersion** $\eta(s)$ is defined as the change in particle position with fractional momentum offset $\delta \equiv \Delta p/p_0$

Dispersion originates from momentum dependence of dipole bends

- Add explicit momentum dependence to equation of motion again

$$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

Particular solution of inhomogeneous differential equation with periodic ρ(s)

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\delta_0$$

$$x'(s) = C'(s)x_0 + S'(s)x_0' + D'(s)\delta_0$$

$$D(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau$$

Use initial conditions etc to find

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix}$$

The trajectory has two parts: $\boxed{x(s) = \mathrm{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}}$

Dispersion Continued

- Substituting and noting dispersion is periodic, $\eta_x(s+C) = \eta_x(s)$

$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta_0 \end{pmatrix} \qquad \text{achromat} : D = D' = 0$$

  - If we take $\delta_0 = 1$ we can solve this in a clever way

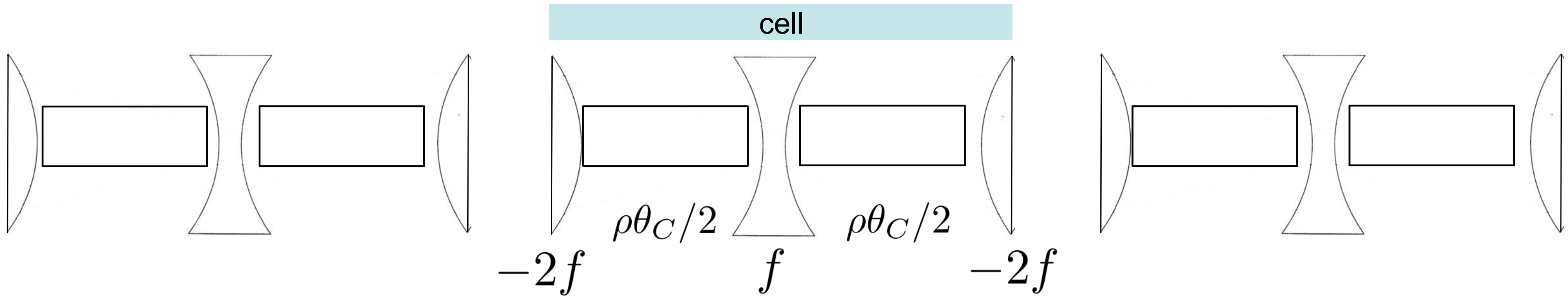
$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(S) \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

$$(I - M) \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} \quad \Rightarrow \quad \boxed{\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}}$$

  - Solving gives

$$\boxed{\begin{aligned} \eta(s) &= \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\Delta\phi)} \\ \eta'(s) &= \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\Delta\phi)} \end{aligned}}$$

FODO Cell Dispersion



- A periodic lattice without dipoles has no **intrinsic** dispersion
- Consider FODO with long dipoles and thin quadrupoles
  - Each dipole has total length $\rho\theta_C/2$ so each cell is of length $L = \rho\theta_C$
  - Assume a large accelerator with many FODO cells so $\theta_C \ll 1$

$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\text{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_C}{8} \\ 0 & 1 & \frac{\theta_C}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{\text{FODO}} = M_{-2f} M_{\text{dipole}} M_f M_{\text{dipole}} M_{-2f}$$

$$M_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_C \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_C \\ 0 & 0 & 1 \end{pmatrix}$$

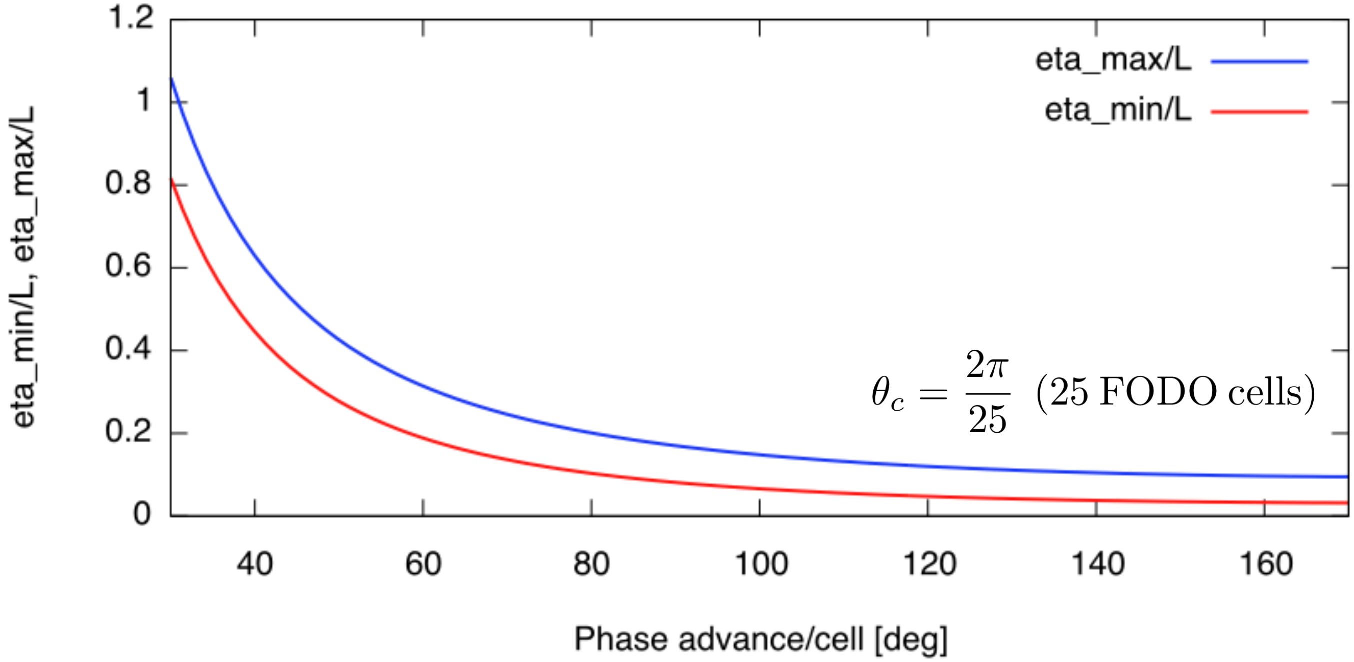
FODO Cell Dispersion

- Like $\hat{\beta}$ before, this choice of periodicity gives us $\hat{\eta}_x$

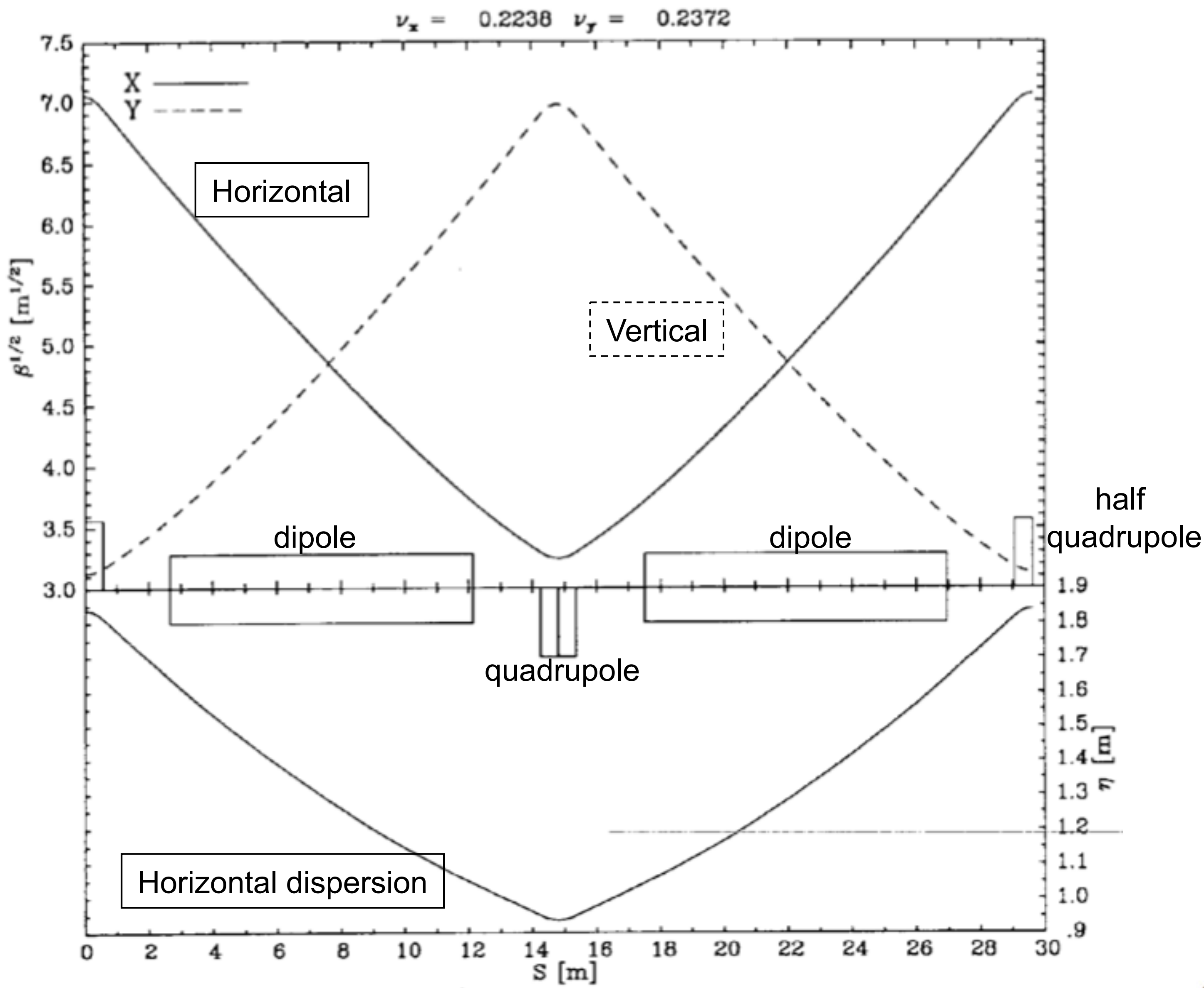
$$\hat{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right] \qquad \eta'_x = 0 \text{ at max}$$

- Changing periodicity to defocusing quad centers gives $\check{\eta}_x$

$$\check{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1-\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right]$$

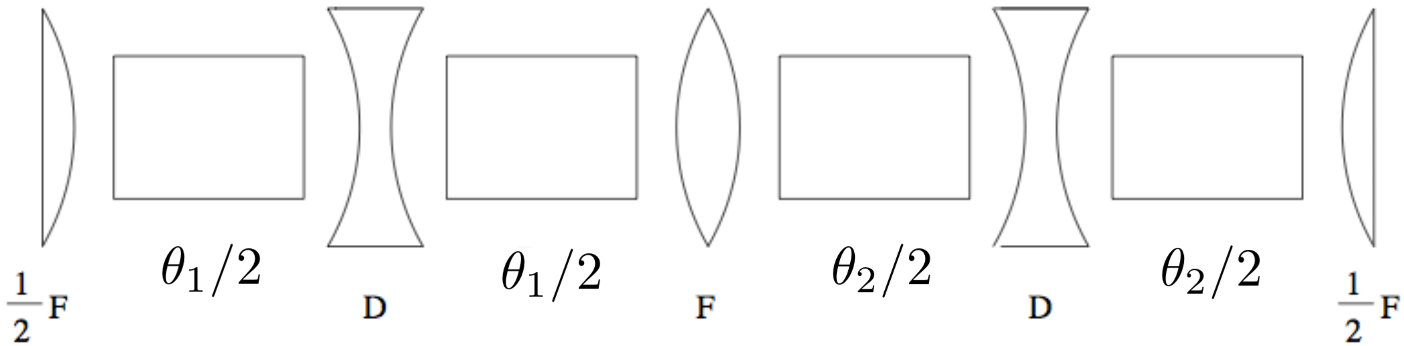


RHIC FODO Cell



Dispersion Suppressor

- The FODO dispersion solution is non-zero everywhere
  - But in straight sections we often want $\eta_x = \eta'_x = 0$
    - e.g. to keep beam small in wigglers/undulators in light source
  - We can "match" between these two conditions with with a **dispersion suppressor**, a **non-periodic** set of magnets that transforms FODO (η_x, η'_x) to zero.



  - Consider two FODO cells with different total bend angles θ_1, θ_2
    - Same quadrupole focusing to not disturb $\beta_x, \Delta\phi_x$ much
    - We want this to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0, 0)$
    - $\alpha_x = 0$ at ends to simplify periodic matrix

(FODO Dispersion Suppressor)

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\Delta\phi_x & \beta_x \sin 2\Delta\phi_x & D(s) \\ -\frac{\sin 2\Delta\phi_x}{\beta_x} & \cos 2\Delta\phi_x & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_x \\ 0 \\ 1 \end{pmatrix}$$

multiply matrices $\Rightarrow$

$$D(s) = \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

$$D'(s) = \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

$$\hat{\eta}_x = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)(\theta_1 + \theta_2)$$

$$\boxed{\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta \qquad \theta_2 = \left(\frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta}$$

$\boxed{\theta = \theta_1 + \theta_2}$ two cells, one FODO bend angle $\rightarrow$ reduced bending

Homework

- For this dispersion suppressor, is it possible to make θ_1 or θ_2 equal to zero?
  - If so, what condition must be applied to the FODO cell?
  - What does the magnet layout of the resulting dispersion suppressor look like?

- Sketch $\eta_x(s)$ through the dispersion suppressor

- Describe how to make an achromatic section out of two dispersion suppressors of this type

Light Source Beam Size and Optics

- Small emittance electron beam desirable for high brightness synchrotron radiation
  - Emittance in a synchrotron light source is balance between
    - quantum excitation due to synchrotron light photon emission
    - adiabatic damping from energy-restoring RF electric field
  - Natural equilibrium emittance is (see Wolski):

$$C_q = 3.832 \times 10^{-13}\ \mathrm{m}$$

$$\boxed{\epsilon_0 = C_q \gamma_r^2 \frac{I_5}{j_x I_2}}$$

$j_x \approx 1$ horizontal damping partition number

$$I_2 \equiv \oint \frac{ds}{\rho^2(s)}$$

geometry/beam energy

$$I_5 \equiv \oint \frac{\mathcal{H}_x}{|\rho(s)|^3} ds$$

optics: which optics are best??

"curly H function"

$$\boxed{\mathcal{H}_x(s) \equiv \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_x' + \beta_x {\eta'}_x^2}$$

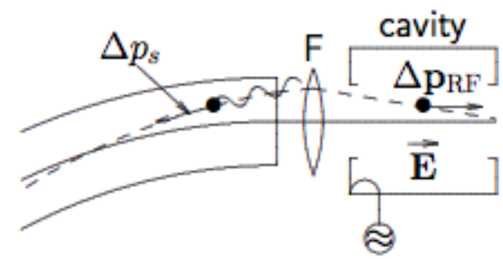
A. Wolski, Joint US-CERN-Japan-Russia school on particle accelerators, April 2011
http://cas.web.cern.ch/cas/JAS/Erice-2011/Lectures/StorageRingDesign2-Handout.pdf

Quantum excitation — *Longitudinal - beam energy spread*

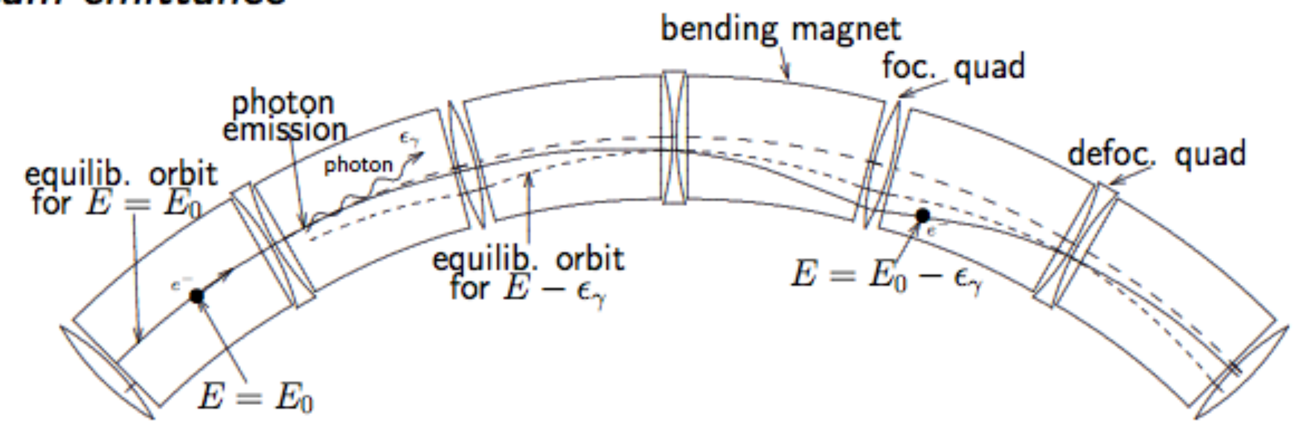
The radiation is emitted in quanta $\epsilon = \hbar\omega$ and excites energy oscillation which are damped giving a Gaussian distribution with variance σ_E^2.

$$\tau_s = E_0 T_0 / U_s \text{ damping}$$
$$g(t) = \mathrm{e}^{-t/\tau_s} \cos(\omega_s t)$$
$$\dot{n} = 5\sqrt{3} e^2 \gamma / (6\epsilon_0 h \rho)$$
$$\langle \epsilon_\gamma^2 \rangle = \frac{11}{27} \epsilon_{\gamma c}^2 \text{ excitation}$$

$$\left(\frac{\sigma_E}{E_0}\right)^2 = \int g^2(t) dt\, \dot{n}\, \langle \epsilon^2 \rangle \approx \frac{55}{32\sqrt{3}} \frac{h\gamma^2}{2\pi\rho m_0 c} \propto h$$



Transverse - beam emittance



Photon emitted at finite dispersion excites betatron oscillations, nominal orbit different at new E

$$\epsilon_x \approx \frac{55}{32\sqrt{3}} \frac{h\gamma^2}{2\pi\rho m_0 c} \langle \mathcal{H} \rangle \propto h$$
$$\langle \mathcal{H} \rangle \approx \langle D_x^2 / \beta_x \rangle$$

IMPAS11-72

A. Hoffman, IMPAS 2011 slide 72

FODO Cell Equilibrium Emittance

- Wolski uses almost everything we have developed so far to find I_5/I_2 in periodic lattice cells, such as a FODO cell
  - For small cell bend angle and equal strength quadrupoles

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\left(1 + \frac{\rho^2}{4f^2}\right)^{-\frac{3}{2}}$$

$$\rho \gg 2f \quad \Rightarrow \quad \frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\frac{8f^3}{\rho^3}$$

$$4f \gg L \quad \Rightarrow \quad \frac{I_5}{I_2} \approx \frac{8f^3}{\rho^3}$$

$$\boxed{\epsilon_0 \approx C_q \gamma_r^2 \left(\frac{2f}{L}\right)^3 \theta^3} \qquad \text{FODO cell equilibrium emittance}$$

FODO Cell Equilibrium Emittance

$$\boxed{\epsilon_0 \approx C_q \gamma_r^2 \left(\frac{2f}{L}\right)^3 \theta^3} \qquad \text{FODO cell equilibrium emittance}$$

- Some observations: equilibrium emittance is…
  - proportional to square of beam energy
  - proportional to cube of FODO cell bending angle
    - increasing number of FODO cells in ring reduces emittance
  - proportional to cube of the quadrupole focal length
    - lower f, stronger quadrupoles means lower emittance
  - inversely proportional to cube of cell (or dipole) length
    - longer FODO cells produce lower equilibrium emittance
    - sensible: reduces required dipole field, quantum excitation
    - in conflict with desire to increase number of FODO cells
  - minimum is when $\Delta\phi \approx 137^\circ$

  - We now follow Wolski's presentation…