Introduction to Accelerator Physics
2011 Mexican Particle Accelerator School

Lecture 6/7: Transition Energy, Longitudinal Motion, Cross Sections, Touschek Effect

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Transition Energy

- Relativistic particle motion in a synchrotron creates some weird effects
  - For particles moving around with frequency ω in circumference C

$$\omega = \frac{2\pi\beta_r c}{C} \quad \Rightarrow \quad \frac{d\omega}{\omega} = \frac{d\beta_r}{\beta_r} - \frac{dC}{C} = \left(\frac{1}{\gamma_r^2} - \frac{1}{\gamma_{\rm tr}^2}\right)\frac{dp}{p_0}$$

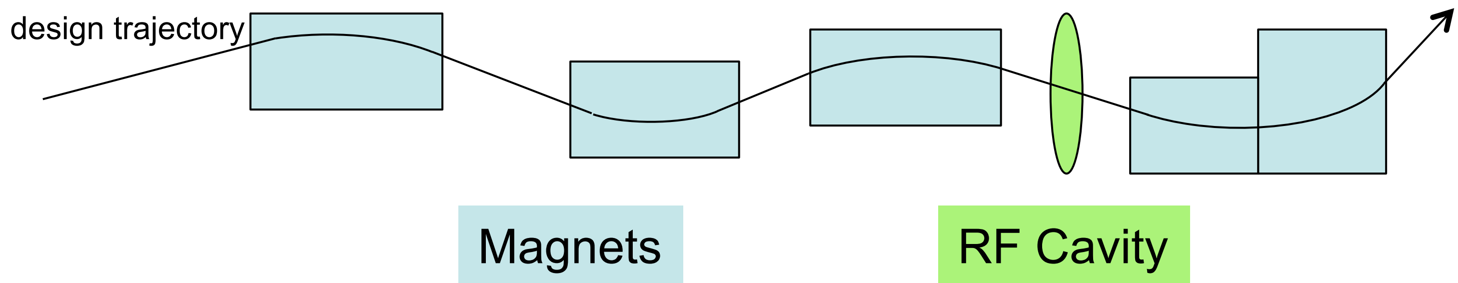
  - At $\gamma_r = \gamma_{\rm tr}$ we have the condition that particle revolution frequency does not depend on its momentum
    - Reminiscent of a cyclotron but now we're strong focusing and at constant radius!

electron ring	At $\gamma_r > \gamma_{\rm tr}$ higher momentum gives **lower** revolution frequency
electron linac	At $\gamma_r < \gamma_{\rm tr}$ higher momentum gives **higher** revolution frequency

$$\text{momentum compaction } \alpha_P \equiv \frac{dC}{C}/\delta = \frac{p_0}{C}\frac{dC}{dp}$$

$$\text{transition gamma } \gamma_{\rm tr} \equiv \frac{1}{\sqrt{\alpha_P}}$$

Changing Pace: Longitudinal Motion and Energy



- Up to now we have considered **transverse motion** in our accelerator, mostly in systems with **periodic transverse focusing**
- But what about **longitudinal motion**? If we don't provide some longitudinal focusing, particles different than design momentum will move away from the design particle over time
  - Momentum spread corresponds to a velocity spread

$$\delta \equiv \frac{dp}{p_0} = \gamma^2 \frac{d\beta}{\beta_0}$$

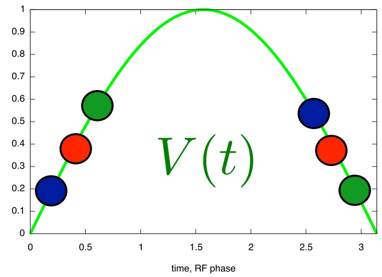
  - For typical numbers $\delta \approx 10^{-3}$, $\gamma \approx 10^4 \;\Rightarrow d\beta \approx 10^{-11} c = 3\ \mathrm{mm/s}$
  - Our bunch spreads and loses energy to synchrotron radiation

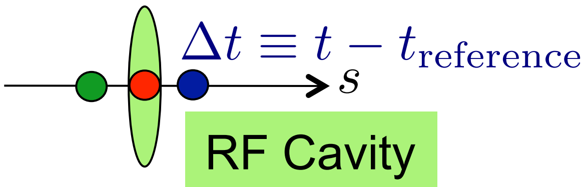
RF Fields

- We need to accomplish two things
  - Add longitudinal energy to the beam to keep p_0 constant
  - Add longitudinal focusing
- RF is also used in accelerating systems to not just balance losses from synchrotron radiation, but
  - Accelerate the beam as a whole: $E_s \neq 0$
  - Keep the beam bunched (focusing, **phase stability)**: $\frac{dE_s}{ds} \neq 0$
- Use sinusoidally varying RF voltage in **short** RF cavities
  - Run at **harmonic number** h of revolution frequency, $\omega_{\rm rf} = h\omega_{\rm rev}$

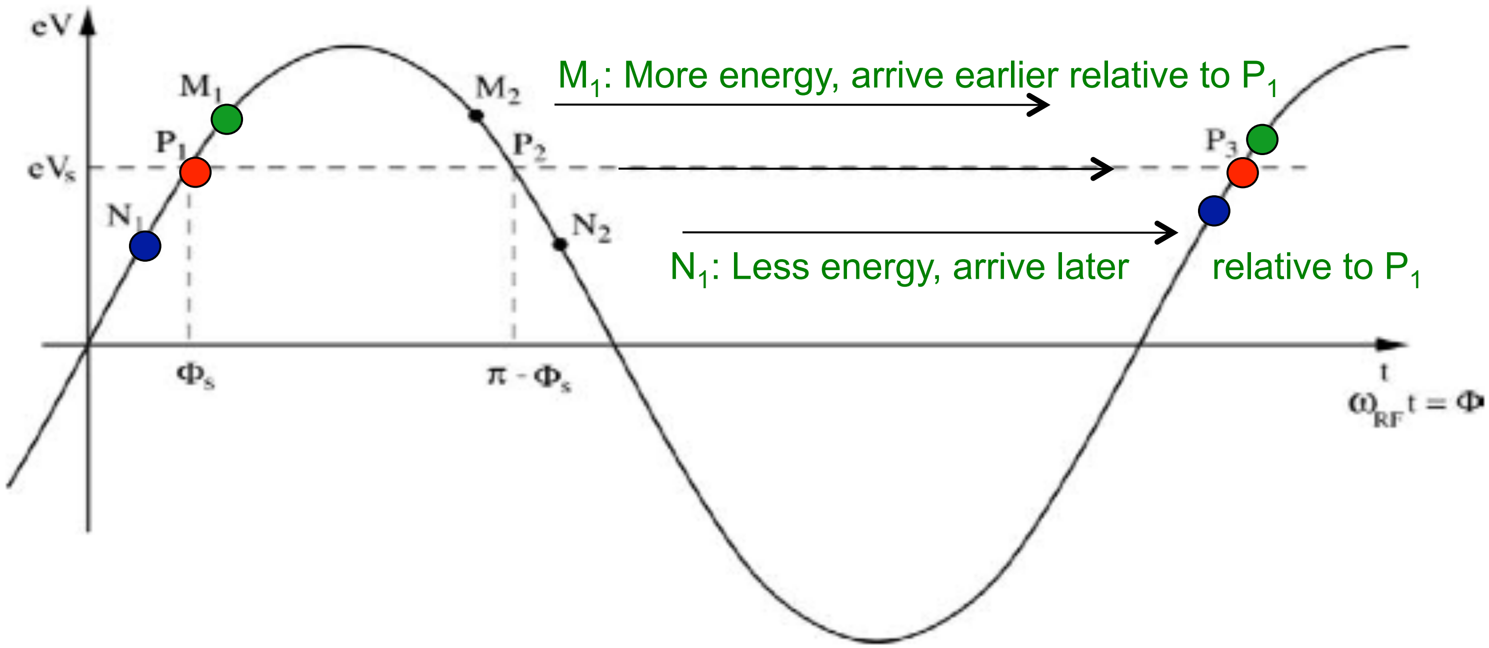
$$\vec{E}(s,t) = \hat{s}E(s,t) = \hat{s}V\sin(\omega_{\rm rf}t + \phi_s)\sum_{n=-\infty}^{\infty}\delta(s - nL)$$

$$\Delta U = qV\sin(\omega_{\rm rf}\Delta t + \phi_s)$$



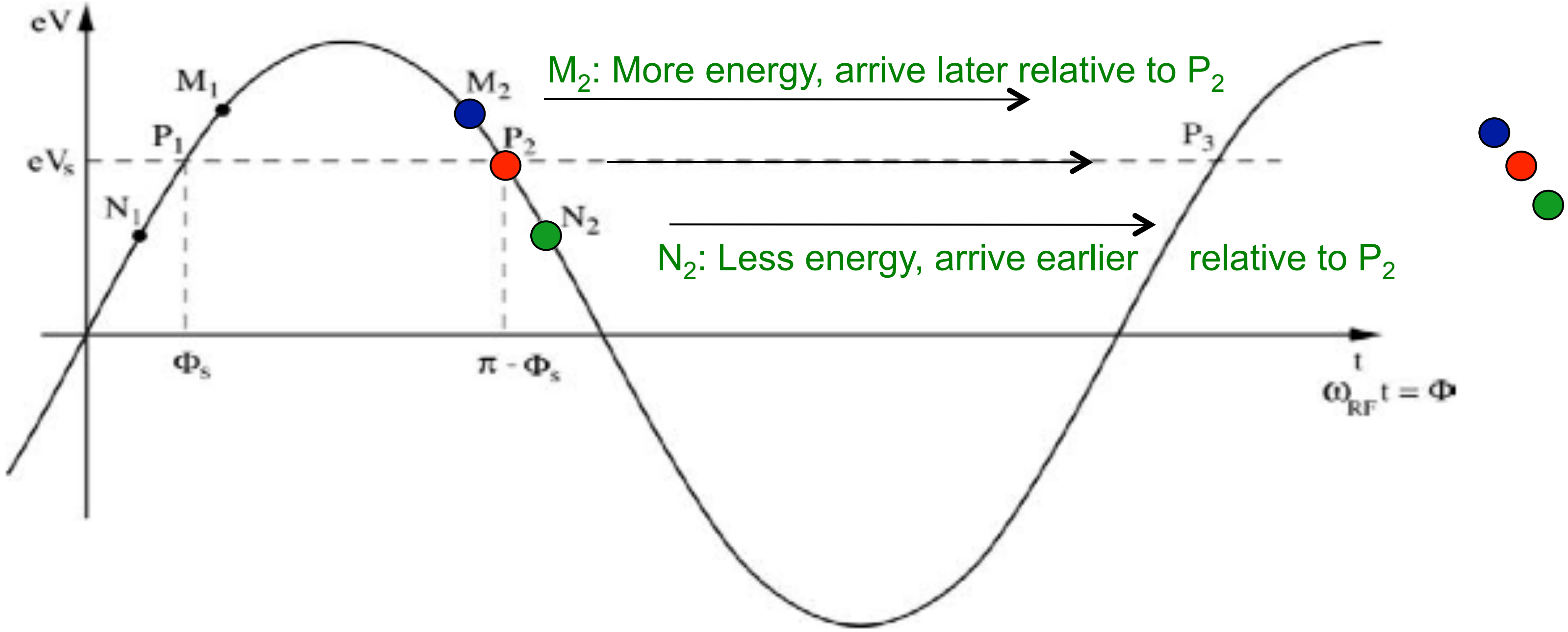


Phase Stability in a Linac



- Consider a series of accelerating gaps (or a ring with one gap)
  - **By design** synchronous phase Φ_s gains just enough energy to balance radiation losses and hit same phase Φ_s in the next gap
  - P_1 are our design particles: they "ride the wave" exactly in phase
- If increased energy means increased frequency ("below transition", e.g. linac)
  - M_1,N_1 will move towards P_1 (local stability) => **phase stability**
  - M_2,N_2 will move away from P_2 (local instability)

Phase Stability in an Electron Synchrotron

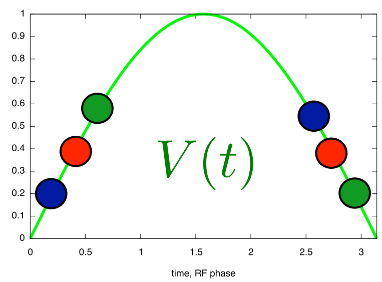


- If increased energy means decreased frequency ("above transition")
  - P_2 are our design particles: they "ride the wave" exactly in phase
  - M_1,N_1 will move away from P_1 (local instability)
  - M_2,N_2 will move towards P_2 (local stability) => **phase stability**
  - All synchrotron light sources run in this regime ($\gamma_r \gg 1$)
  - Note ϕ_s is given by maximum RF voltage and required energy gain per turn

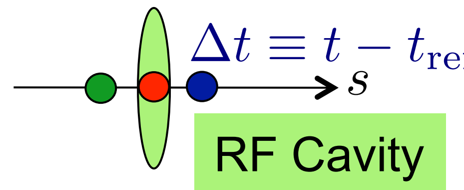
Synchrotron Oscillations

$$\vec{E}(s,t) = \hat{s}E(s,t) = \hat{s}V\sin(\omega_{\rm rf}t+\phi_s)\sum_{n=-\infty}^{\infty}\delta(s-nL)$$

$$\Delta U = qV\sin(\omega_{\rm rf}\Delta t+\phi_s)$$



$$\Delta t \equiv t - t_{\rm reference}$$



- The electric force is sinusoidal so we expect particle motion to look something like a pendulum
  - Define coordinate **synchrotron phase** of a particle $\varphi \equiv \phi - \phi_s$
  - We can go through tedious relativistic mathematics to find a biased pendulum equation

$$\ddot{\varphi} + \frac{h\omega_{\rm ref}^2\eta_{\rm tr}qV}{2\pi\beta_r^2U_{\rm ref}}\left[\sin(\phi_s+\varphi)-\sin(\phi_s)\right] = 0$$

where $\omega_{\rm rf} = h\omega_{\rm rev}$ $\qquad \eta_{\rm tr} \equiv \left(\frac{1}{\gamma_r^2}-\frac{1}{\gamma_{\rm tr}^2}\right)$

Linearized Synchrotron Oscillations

$$\ddot{\varphi} + \frac{h\omega_{\rm ref}^2 \eta_{\rm tr} qV}{2\pi \beta_r^2 U_{\rm ref}} [\sin(\phi_s + \varphi) - \sin(\phi_s)] = 0$$

- If these synchrotron phase oscillations are small, this motion looks more like (surprise!) a simple harmonic oscillator

$$\sin(\phi_s + \varphi) \approx \varphi \cos(\phi_s) + \sin(\phi_s)$$

$$\ddot{\varphi} + \Omega_s^2 \varphi = 0$$

$$\Omega_s \equiv \omega_{\rm ref} \sqrt{\frac{h\eta_{\rm tr} \cos(\phi_s)}{2\pi \beta_r^2 \gamma_r} \frac{qV}{mc^2}}$$

synchrotron frequency

$$Q_s \equiv \frac{\Omega_s}{\omega_{ref}} = \sqrt{\frac{h\eta_{\rm tr} \cos(\phi_s)}{2\pi \beta_r \gamma_r^2} \frac{qV}{mc^2}}$$

synchrotron tune

Note that $\eta_{\rm tr} \cos(\phi_s) > 0$ is required for phase stability.

Example: ALS synchrotron frequency on order of few 10^{-3}

$(\varphi, \dot{\varphi} \equiv d\varphi/dt)$ are natural phase space coordinates

Large Synchrotron Oscillations

- Sometimes particles achieve large momentum offset δ and therefore get a large phase offset φ relative to design
  - For example, electron-electron scattering (Touschek)
  - Then our longitudinal motion equation becomes

$$\ddot{\varphi} + \frac{\Omega_s^2}{\cos\phi_s}[\sin(\varphi+\phi_s) - \sin(\phi_s)] = 0 \qquad \varphi \equiv \phi - \phi_s$$

$$\frac{d(\dot{\phi}^2)}{dt} = 2\ddot{\phi}\frac{d\phi}{dt} \quad \Rightarrow \quad d\left(\dot{\phi}^2\right) = \frac{2\Omega_s^2}{\cos\phi_s}(-\sin\phi\, d\phi) + 2\Omega_s^2 \tan\phi_s\, d\phi$$

  - Integrate with a constant $\phi_0 \equiv \phi(t=0)$

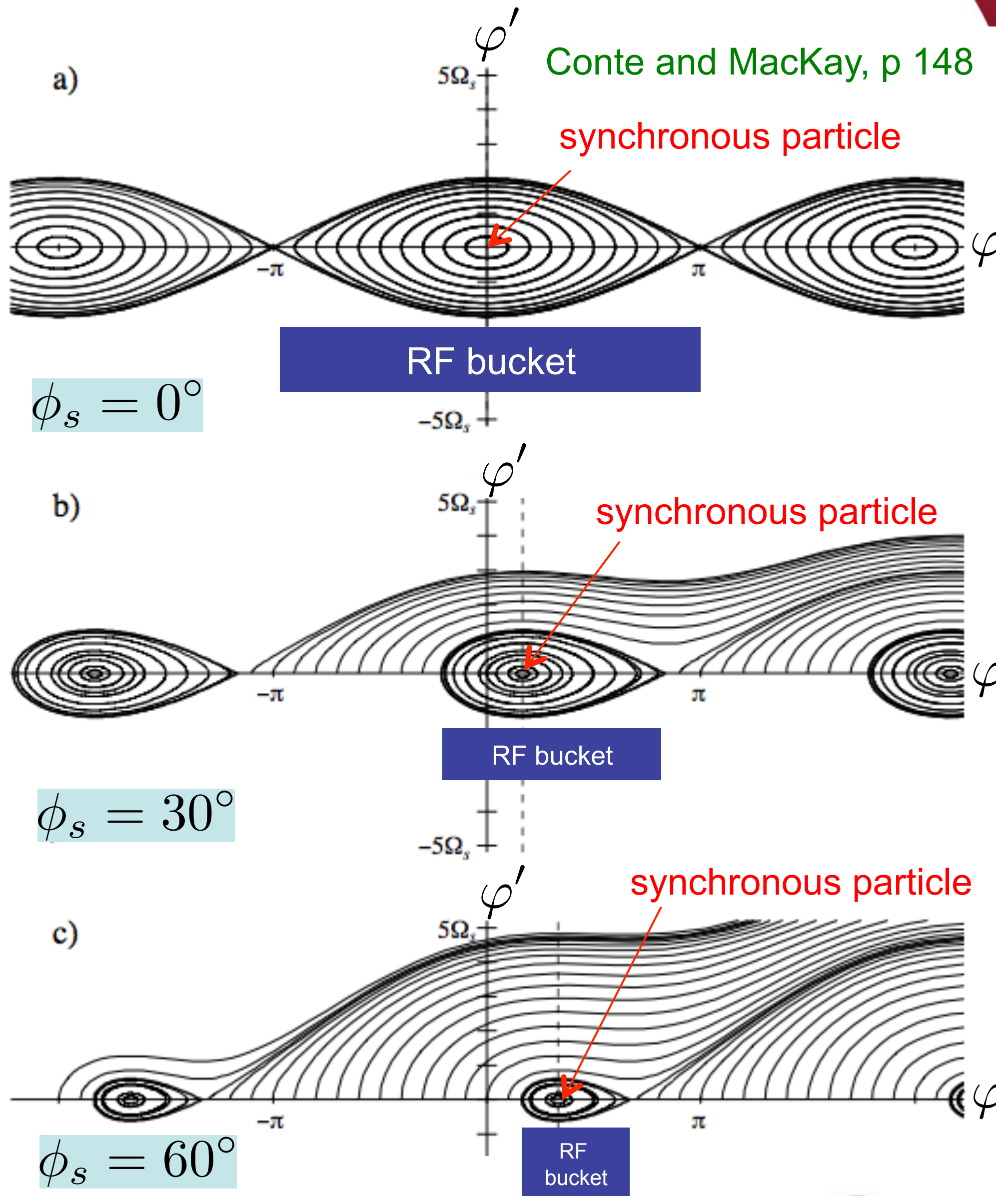
$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\cos\phi_s} + 2(\phi-\phi_0)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}$$

  - This is not closed-form integrable but you can write a computer program to iterate initial conditions to find $(\varphi(t), \varphi'(t))$

Synchrotron Oscillation Phase Space

- Start particles at $\varphi \neq 0$ and $\varphi' = 0$
- φ' is how phase moves
  - Related to momentum offset δ
- Area of locally stable motion is called **RF bucket**
  - Move like stable biased pendula
- Synchronous particle and nearby particles are stable
  - But some particles "spin" through phases like unstable biased pendula
  - $\Rightarrow \varphi', \delta$ grow, particle is lost at momentum aperture

Conte and MacKay, p 148

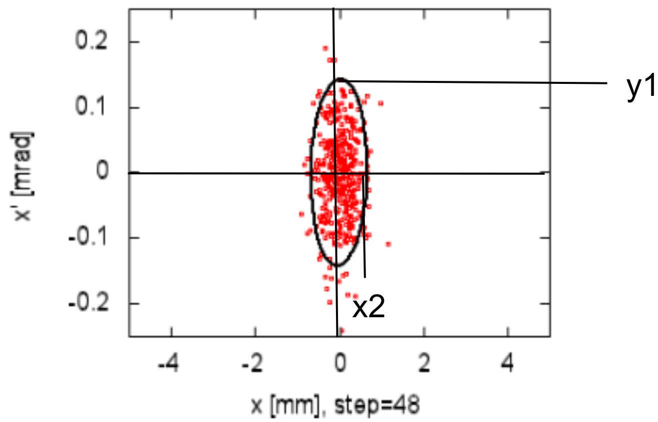


Touschek Scattering

- Electrons within the electron bunches in a synchrotron light storage ring do sometimes interact with each other
  - They're all charged particles, after all

- Fortunately most of these interactions are negligible for high energy, ultrarelativistic electron beams
  - $\gamma \gg 1$ so, e.g., time dilation reduces effect of space charge $\propto \gamma^{-2}$
  - But these are long-distance Coulomb repulsions
  - High angle scattering can lead to sudden large momentum changes for individual electrons
  - Low emittance and high brilliance enhances this effect
    - Tighter distributions of particles => more likelihood of interactions
  - Large momentum changes can move electrons out of the stable RF bucket => particle loss

Rough Order of Magnitude



$$x_2 = \sqrt{\mathcal{W}\beta(s)}$$

$$y_1 = \sqrt{\mathcal{W}/\beta(s)}$$

$$\sigma_{\mathrm{RMS}} = \sqrt{\pi\mathcal{W}\beta(s)}$$

$$\sigma'_{\mathrm{RMS}} = \sqrt{\pi\mathcal{W}/\beta(s)}$$

- You already figured out in homework that $p_x = p_0\sigma'_{\mathrm{x,RMS}}$
- If **all** transverse momentum is transferred into δ then

$$\Delta p = \gamma_r p_x = \gamma_r \frac{p_0\sigma'_{\mathrm{x,RMS}}}{\beta_x}$$

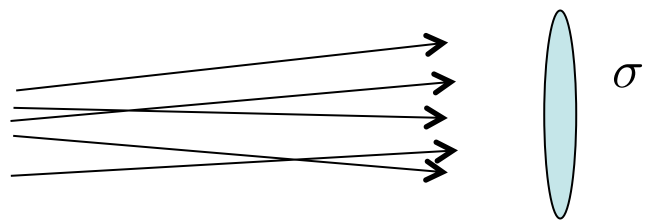
- For realistic numbers of 2 GeV beam (γ~4000), β_x=10m, and $100\mu m$ beam displacement, we find

$$\Delta p \approx 80\ \mathrm{MeV/c} \approx 0.04\, p_0$$

- This scattering mechanism can create electron loss

Cross Section

- **Cross section** is used in high energy physics to express the probability of scattering processes: units of area



$$1 \text{ barn} \equiv 10^{-28}\text{ m}^2 = 10^{-24}\text{ cm}^2$$

- Often expressed as a **differential cross section**, probably of interaction in a given set of conditions (like interaction angle or momentum transfer): $d\sigma/d\Omega$
- In particle colliders, **luminosity** is defined as the rate of observed interactions of a particular type divided by the cross section

$$\mathcal{L} \equiv \frac{\text{event rate}}{\sigma} \quad \text{units } [\text{s}^{-1}\text{ cm}^{-2}]$$

Integrating this over time gives an expected number of events in a given time period to calculate experiment statistics

Touschek Scattering Calculations

- Touschek Scattering calculations use the Moller electron elastic interaction cross section in the rest frame of the electrons
  - Then relativistically boost back into the lab frame
  - This is all too involved for this lecture!
    - Really 2nd year graduate level scattering theory calculation
  - See Carlo Bocchetta's talk at CERN Accelerator School
    - http://cas.web.cern.ch/cas/BRUNNEN/Presentations/PDF/Bocchetta/Touschek.pdf
  - As usual we'll just quote the result
  - Touschek loss exponential decay lifetime

$$\tau = \frac{\gamma_r^3 V_{\mathrm{bunch}} \sigma'_{\mathrm{x,RMS}} \delta^2_{\mathrm{acceptance}}}{c r_0^2 N_{\mathrm{bunch}} (\ln(2)\sqrt{\pi})} \frac{1}{C(\epsilon)}$$

$$V_{\mathrm{bunch}} = 8\pi\sigma_x\sigma_y\sigma_z$$

$$C(\epsilon) \approx -[\ln(1.732\epsilon) + 1.5]$$

$$r_0 \approx 2.818 \times 10^{-13}\ \mathrm{cm}$$

$$\delta_{\mathrm{acceptance}}: \quad \frac{\Delta p}{p_0} \text{ at which particles are lost}$$

Touschek Scaling

$$\tau = \frac{\gamma_r^3 V_{\text{bunch}} \sigma'_{\text{x,RMS}} \delta^2_{\text{acceptance}}}{c r_0^2 N_{\text{bunch}} (\ln(2)\sqrt{\pi})} \frac{1}{C(\epsilon)}$$

$$V_{\text{bunch}} = 8\pi\sigma_x\sigma_y\sigma_z$$

$$C(\epsilon) \approx -[\ln(1.732\epsilon) + 1.5]$$

$$r_0 \approx 2.818 \times 10^{-13} \text{ cm}$$

$$\delta_{\text{acceptance}}: \quad \frac{\Delta p}{p_0} \text{ at which particles are lost}$$

- High lifetime is good, low lifetime is bad
  - Higher particle phase space density $N_{\text{bunch}}/V_{\text{bunch}}$ makes loss faster
    - But we want this for higher brilliance!
  - Smaller momentum acceptance makes loss faster
    - But tighter focusing requires sextupoles to correct chromaticity
    - Sextupoles and other nonlinearities reduce $\delta_{\text{acceptance}}$
  - Higher beam energy γ_r makes loss slower
    - Well at least we win somewhere!

Touschek Lifetime Calculations

- Generally one must do some simulation of Touschek losses

Touschek Lifetime Calculations for NSLS-II

Homework: Read and discuss this paper! Which parts do you recognize? Not recognize?

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Abstract

The Touschek effect limits the lifetime for NSLS-II. The basic mechanism is Coulomb scattering resulting in a longitudinal momentum outside the momentum aperture. The momentum aperture results from a combination of the initial betatron oscillations after the scatter and the non-linear properties determining the resultant stability. We find that higher order multipole errors may reduce the momentum aperture, particularly for scattered particles with energy loss. The resultant drop in Touschek lifetime is minimized, however, due to less scattering in the dispersive regions. We describe these mechanisms, and present calculations for NSLS-II using a realistic lattice model including damping wigglers and engineering tolerances.[1]

INTRODUCTION

LINEAR AND NON-LINEAR DYNAMICS MODELING

NSLS-II has a 15-fold periodic DBA lattice. The lattice functions for NSLS-II are shown in Figure 1. The linear lattice results in the equilibrium beam sizes around the ring that enter into Eqn. (1). Non-linear dynamics enter through the parameter $\delta_{acc}(s)$. This is the maximum momentum change that a scattered particle can endure before it is lost. There are two elements to this stability question. The first is the amplitude of the initial orbit which comes from the off-momentum closed orbit (dispersion) and beta functions. These are shown in Figures 2 and 3. The amplitude of the induced betatron oscillation following a scatter with relative energy change $\delta = \frac{\Delta E}{E_0}$ is given by

$$x_2 = (\eta^{(1)}(s_2) + \sqrt{\mathcal{H}(s_1)\beta_x(s_2)})\delta + \eta^{(2)}(s_2)\delta^2 \quad (3)$$

where $\mathcal{H} = \gamma_x\eta_x^2 + 2\alpha_x\eta_x\eta'_x + \beta_x\eta_x'^2$ is the dispersion in-

PAC'09 Conference: http://www.bnl.gov/isd/documents/70446.pdf

Survey Through Bocchetta's paper

- We now take a tour through some parts of Carlo Bocchetta's talk at
  - http://cas.web.cern.ch/cas/BRUNNEN/Presentations/PDF/Bocchetta/Touschek.pdf