# Introduction to Accelerator Physics
# Old Dominion University

# Linear Accelerator Lattice Optics

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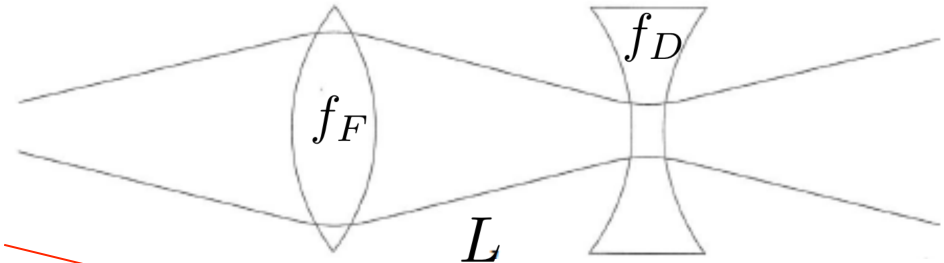
# Matrix Example: Strong Focusing

- Consider a doublet of thin quadrupoles separated by drift L




Thin quadrupole matrices

$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

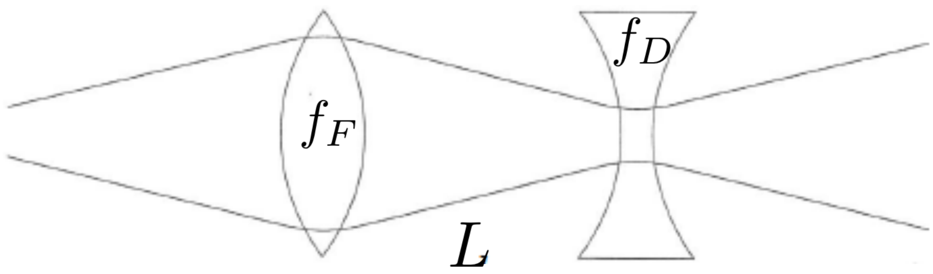
$$\frac{1}{f_{\text{doublet}}} = \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D}$$

$$f_D = f_F = f \quad \Rightarrow \quad \frac{1}{f_{\text{doublet}}} = -\frac{L}{f^2}$$

There is **net focusing** given by this **alternating gradient** system

A fundamental point of optics, and of accelerator **strong focusing**

# Strong Focusing: Another View




$$M_{\mathrm{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix}$$

$$\text{incoming paraxial ray} \quad \begin{pmatrix} x \\ x' \end{pmatrix} = M_{\mathrm{doublet}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \end{pmatrix} x_0$$

For this to be focusing, x' must have opposite sign of x where these are coordinates of transformation of incoming paraxial ray

$$f_F = f_D \quad x' < 0 \quad \mathbf{BUT} \quad x > 0 \text{ iff } f_F > L$$

Equal strength doublet is net focusing under condition that each lens's focal length is greater than distance between them

# More Math: Hill's Equation

- Let's go back to our quadrupole equations of motion for $R \to \infty$

$$x'' + Kx = 0 \qquad y'' - Ky = 0 \qquad K \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)$$

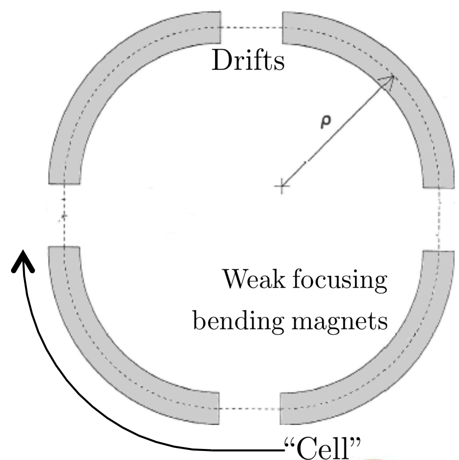
What happens when we let the focusing K vary with s?

Also assume K is **periodic** in s with some periodicity C

$$x'' + K(s)x = 0 \qquad K(s) \equiv \frac{1}{(B\rho)}\left(\frac{\partial B_y}{\partial x}\right)(s) \qquad K(s+C) = K(s)$$

This periodicity can be one revolution around the accelerator or as small as one repeated "cell" of the layout

(Such as a FODO cell in the previous slide)




The simple harmonic oscillator equation with a **periodically** varying spring constant K(s) is known as **Hill's Equation**

# Hill's Equation Solution Ansatz

$$x'' + K(s)x = 0 \qquad K \equiv \frac{1}{(B\rho)} \left( \frac{\partial B_y}{\partial x} \right) (s)$$

- Solution is a quasi-periodic harmonic oscillator

$$x(s) = A\, w(s)\ \cos[\phi(s) + \phi_0]$$

where w(s) is periodic in C but the phase $\phi$ is not!!

Substitute this educated guess ("ansatz") to find

$$x' = Aw' \cos[\phi + \phi_0] - Aw\phi' \sin[\phi + \phi_0]$$

$$x'' = A(w'' - w\phi'^2) \cos[\phi + \phi_0] - A(2w'\phi' + w\phi'') \sin[\phi + \phi_0]$$

$$x'' + K(s)x = -A(2w'\phi' + w\phi'') \sin(\phi + \phi_0) + A(w'' - w\phi'^2 + Kw) \cos(\phi + \phi_0) = 0$$

For $w(s)$ and $\phi(s)$ to be independent of $\phi_0$, coefficients of sin and cos terms must vanish identically

# Courant-Snyder Parameters

$$2ww'\phi' + w^2\phi'' = (w^2\phi')' = 0 \quad \Rightarrow \quad \phi' = \frac{k}{w(s)^2}$$

$$w'' - (k^2/w^3) + Kw = 0 \quad \Rightarrow \quad w^3(w'' + Kw) = k^2$$

- Notice that in both equations $w^2 \propto k$ so we can scale this out and define a new set of functions, **Courant-Snyder Parameters** or **Twiss Parameters**

$$\beta(s) \equiv \frac{w^2(s)}{k}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s)$$

$$\gamma(s) \equiv \frac{1 + \alpha(s)^2}{\beta(s)}$$

$$\Rightarrow$$

$$\phi' = \frac{1}{\beta(s)} \qquad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$K\beta = \gamma + \alpha'$$

$\beta(s), \alpha(s), \gamma(s)$ are all periodic in $C$

$\phi(s)$ is **not** periodic in $C$

# Towards The Matrix Solution

- What is the matrix for this Hill's Equation solution?

$$x(s) = A\sqrt{\beta(s)}\cos\phi(s) + B\sqrt{\beta(s)}\sin\phi(s)$$

Take a derivative with respect to s to get $x' \equiv \frac{dx}{ds}$

$$\phi' = \frac{1}{\beta(s)} \qquad x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\phi(s) - [A + \alpha(s)B]\sin\phi(s)\}$$

Now we can solve for A and B in terms of initial conditions $(x(0), x'(0))$

$$x_0 \equiv x(0) = A\sqrt{\beta(0)} \qquad x'_0 \equiv x'(0) = \frac{1}{\sqrt{\beta(0)}}[B - \alpha(0)A]$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x'_0 + \alpha(0)x_0]$$

And take advantage of the periodicity of $\beta, \alpha$ to find $x(C), x'(C)$

# Hill's Equation Matrix Solution

$$x(s) = A\sqrt{\beta(s)}\cos\phi(s) + B\sqrt{\beta(s)}\sin\phi(s)$$

$$x'(s) = \frac{1}{\sqrt{\beta(s)}}\{[B - \alpha(s)A]\cos\phi(s) - [A + \alpha(s)B]\sin\phi(s)\}$$

$$A = \frac{x_0}{\sqrt{\beta(0)}} \qquad B = \frac{1}{\sqrt{\beta(0)}}[\beta(0)x_0' + \alpha(0)x_0]$$

$$x(C) = [\cos\phi(C) + \alpha(0)\sin\phi(C)]x_0 + \beta(0)\sin\phi(C)\, x_0'$$

$$x'(C) = -\gamma(0)\sin\phi(C)\, x_0 + [\cos\phi(C) - \alpha(0)\sin\phi(C)]\, x_0'$$

We can write this down in a matrix form where $\Delta\phi_C$ is the betatron phase advance through one period C

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(0)\sin\Delta\phi_C & \beta(0)\sin\Delta\phi_C \\ -\gamma(0)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(0)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

# Interesting Observations

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(0)\sin\Delta\phi_C & \beta(0)\sin\Delta\phi_C \\ -\gamma(0)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(0)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

- $\Delta\phi_C$ is independent of s: this is the **betatron phase advance** of this periodic system
- Determinant of matrix M is still 1!
- Looks like a rotation and some scaling
- M can be written down in a beautiful and deep way

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\Delta\phi_C}$$

and remember $x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$

# Convenient Calculations

- If we know the transport matrix, we can find the lattice parameters

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(0)\sin\Delta\phi_C & \beta(0)\sin\Delta\phi_C \\ -\gamma(0)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(0)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\text{betatron phase advance per cell } \Delta\phi_C = \frac{1}{2}\text{Tr } M$$

$$\beta(0) = \frac{m_{12}}{\sin\Delta\phi_C}$$

$$\alpha(0) = \frac{m_{11} - \cos\Delta\phi_C}{\sin\Delta\phi_C}$$

$$\gamma(0) \equiv \frac{1+\alpha^2(0)}{\beta(0)}$$

# General Non-Periodic Transport Matrix

- We can parameterize a general non-periodic transport matrix from $s_1$ to $s_2$ using the lattice parameters

$$M(s_2) = \begin{pmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}[\cos\Delta\phi + \alpha(s_1)\sin\Delta\phi] & \sqrt{\beta(s_1)\beta(s_2)}\sin\Delta\phi \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\Delta\phi + [1+\alpha(s_1)\alpha(s_2)]\sin\Delta\phi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}[\cos\Delta\phi - \alpha(s_2)\sin\Delta\phi] \end{pmatrix}$$

- This does not have a pretty form like the periodic matrix

However both can be expressed as $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

The most common use of this matrix is the $m_{12}$ term:

$$\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)}\sin(\Delta\phi)\, x'(s_1)$$

Effect of angle kick on downstream position

# (Deriving the Non-Periodic Transport Matrix)

$$x(s) = Aw(s)\cos\phi(s) + Bw(s)\sin\phi(s)$$

$$x'(s) = A\left(w'(s)\cos\phi(s) - \frac{\sin\phi(s)}{w(s)}\right) + B\left(w'(s)\sin\phi(s) + \frac{\cos\phi(s)}{w(s)}\right)$$

Calculate A, B in terms of initial conditions $(x_0, x_0')$ and $(w_0, \phi_0)$

$$A = \left(w_0'\sin\phi_0 + \frac{\cos\phi_0}{w_0}\right)x_0 - (w_0\sin\phi_0)x_0'$$

$$B = -\left(w_0'\cos\phi_0 - \frac{\sin\phi_0}{w_0}\right)x_0 + (w_0\cos\phi_0)x_0'$$

Substitute (A,B) and put into matrix form: $\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$

$$m_{11}(s) = \frac{w(s)}{w_0}\cos\Delta\phi - w(s)w_0'\sin\Delta\phi$$

$$\Delta\phi \equiv \phi(s) - \phi_0$$

$$w(s) = \sqrt{\beta(s)}$$

$$m_{12}(s) = w(s)w_0\sin\Delta\phi$$

$$m_{21}(s) = -\frac{1 + w(s)w_0w'(s)w_0'}{w(s)w_0}\sin\Delta\phi - \left[\frac{w_0'}{w(s)} - \frac{w'(s)}{w_0}\right]\cos\Delta\phi$$

$$m_{22}(s) = \frac{w_0}{w(s)}\cos\Delta\phi + w_0w'\sin\Delta\phi$$

# Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi-periodic ansatz solution} \quad x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \quad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(0)\sin\Delta\phi_C & \beta(0)\sin\Delta\phi_C \\ -\gamma(0)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(0)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

betatron phase advance $\quad \Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)} \quad$ $\mathrm{Tr}\, M = 2\cos\Delta\phi_C$

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\Delta\phi_C}$$

# Transport Matrix Stability Criteria

- For long systems (rings) we want $M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ stable as $n \to \infty$
  - If 2x2 M has eigenvectors $(V_1, V_2)$ and eigenvalues $(\lambda_1, \lambda_2)$:

$$M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = A\lambda_1^n V_1 + B\lambda_2^n V_2$$

  - M is also unimodular (det M=1) so $\lambda_{1,2} = e^{\pm i\phi}$ with complex $\phi$
  - For $\lambda_{1,2}^n$ to remain bounded, $\phi$ must be real

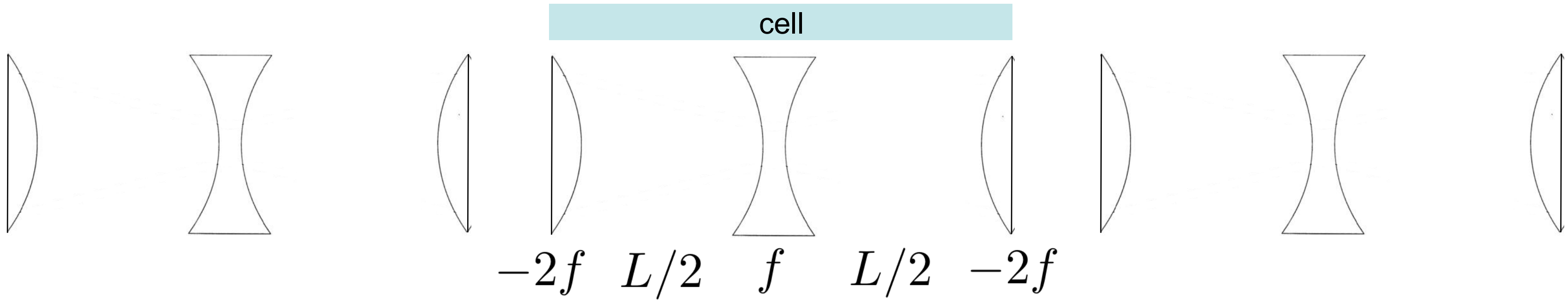
  - We can always transform M into diagonal form with the eigenvalues on the diagonal (since det M=1); this does not change the trace of the matrix

$$e^{i\phi} + e^{-i\phi} = 2\cos\phi = \mathrm{Tr}\, M$$

  - The **stability requirement** for these types of matrices is then

$$\phi \text{ real} \quad \Rightarrow \quad \boxed{-1 \leq \frac{1}{2}\mathrm{Tr}\, M \leq 1}$$

# Periodic Example: FODO Cell Phase Advance




- Select periodicity between centers of focusing quads
  - A natural periodicity if we want to calculate maximum β(s)

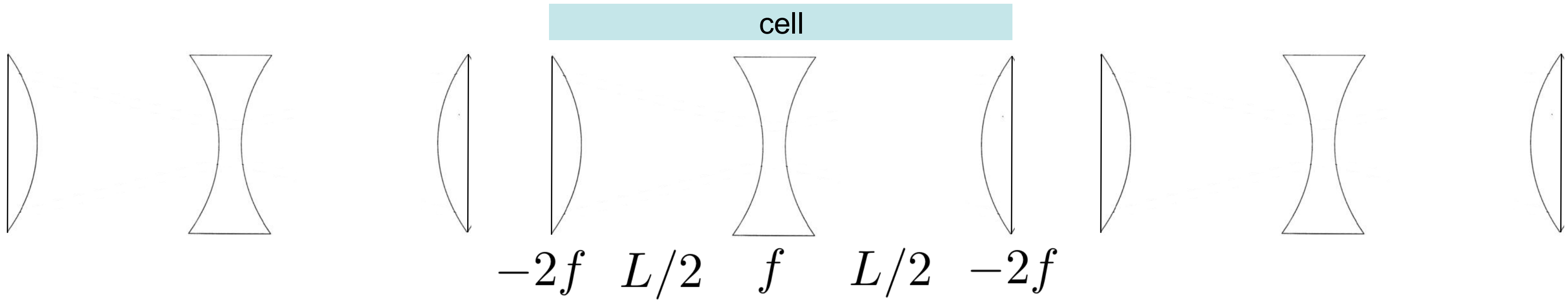
$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \qquad \mathrm{Tr}\, M = 2\cos\Delta\phi_C = 2 - \frac{L^2}{4f^2}$$

$$1 - \frac{L^2}{8f^2} = \cos\Delta\phi_C = 1 - 2\sin^2\frac{\Delta\phi_C}{2} \quad \Rightarrow \quad \sin\frac{\Delta\phi_C}{2} = \pm\frac{L}{4f}$$

  - $\Delta\phi_C$ only has real solutions (stability) if $\frac{L}{4} < f$

# Periodic Example: FODO Cell Beta Max/Min




- What is $\hat{\beta}$?
  - A natural periodicity if we want to calculate maximum β(s)

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix} \Leftarrow \quad M_{12} = \beta \sin \Delta\phi_C$$

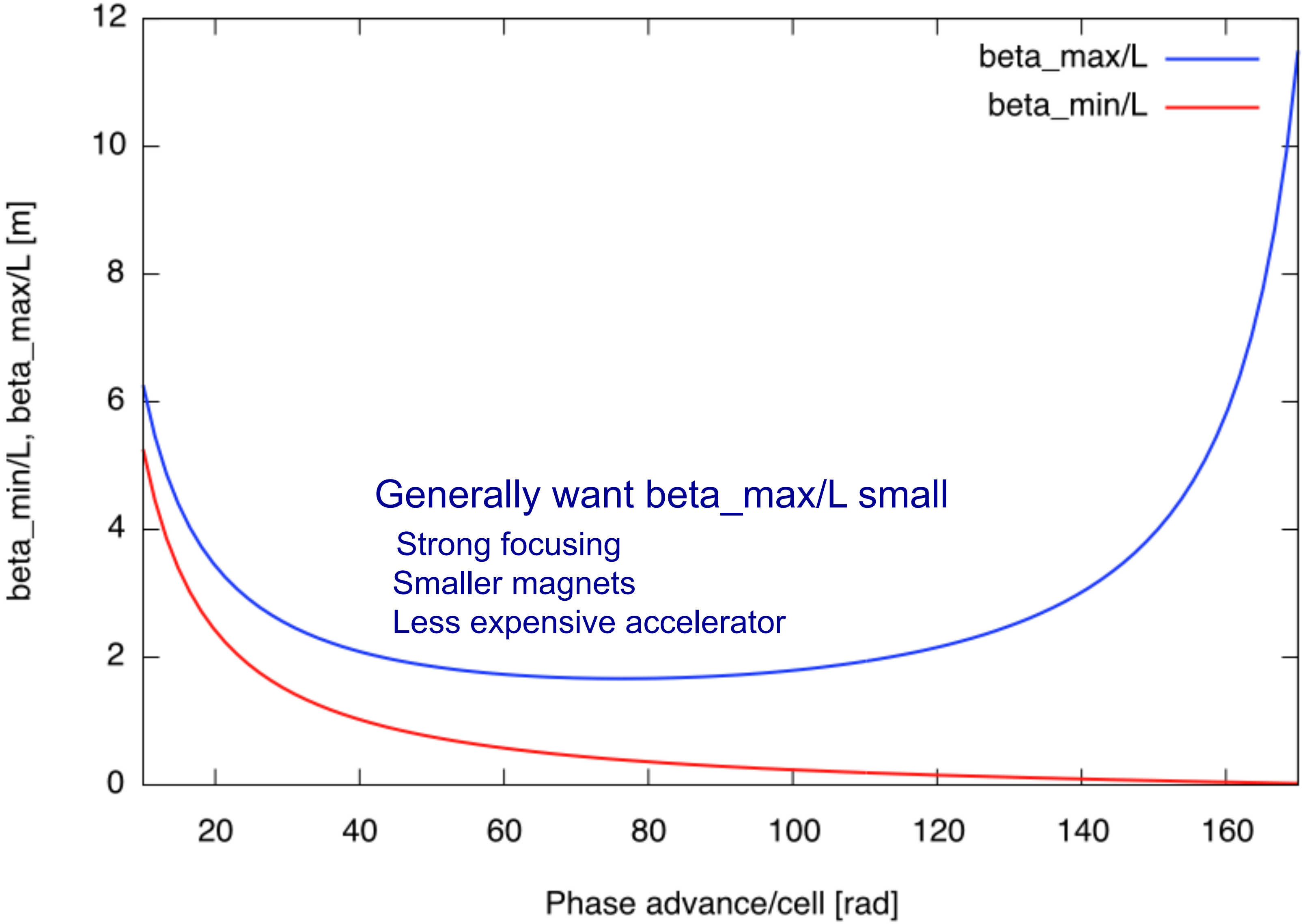
$$\hat{\beta} \sin \Delta\phi_C = \frac{L^2}{4f} + L = L\left(1 + \sin \frac{\Delta\phi_C}{2}\right)$$

$$\boxed{\hat{\beta} = \frac{L}{\sin \Delta\phi_C}\left(1 + \sin \frac{\Delta\phi_C}{2}\right)}$$

  - Follow a similar strategy reversing F/D quadrupoles to find the minimum β(s) within a FODO cell (center of D quad)

$$\boxed{\check{\beta} = \frac{L}{\sin \Delta\phi_C}\left(1 - \sin \frac{\Delta\phi_C}{2}\right)}$$

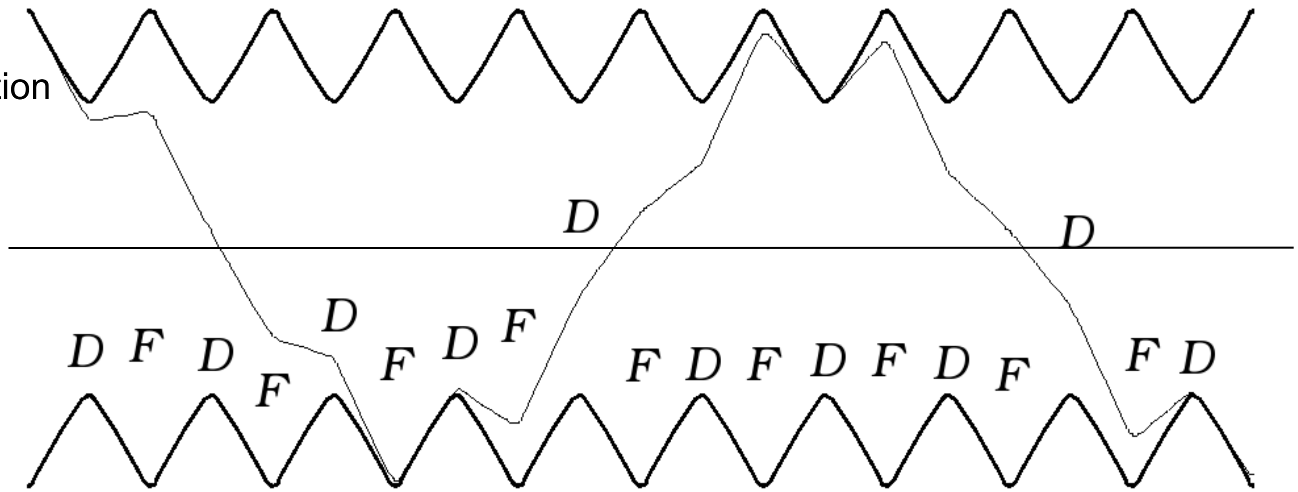
# FODO Betatron Functions vs Phase Advance




Generally want beta_max/L small

- Strong focusing
- Smaller magnets
- Less expensive accelerator

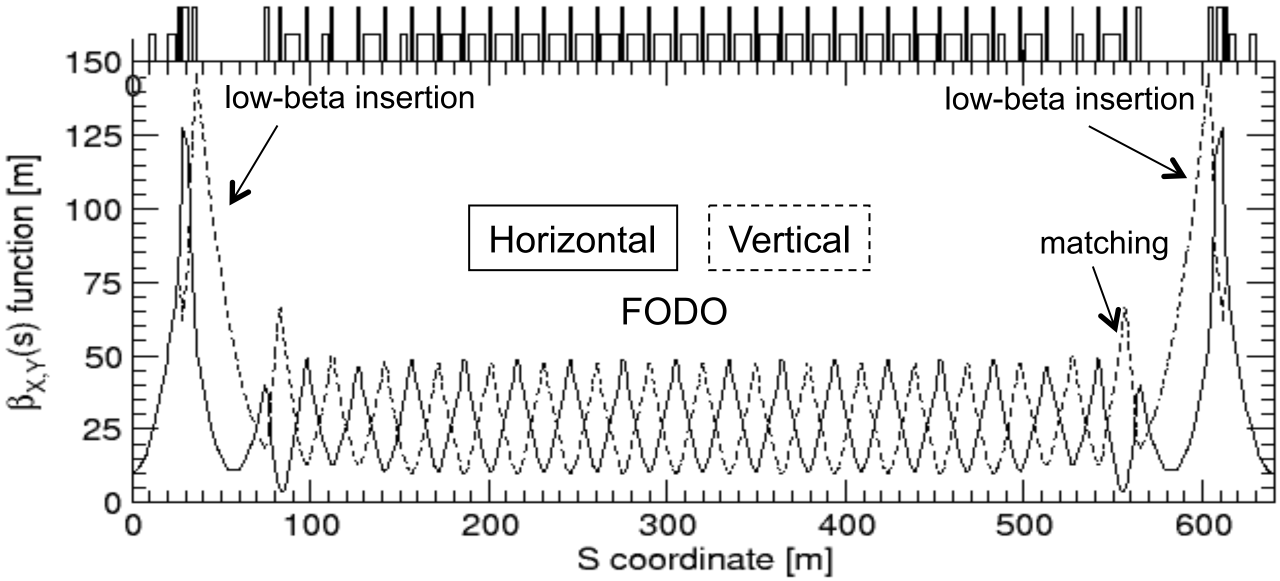
# FODO Beta Function, Betatron Motion




- This is a picture of a FODO lattice, showing contours of $\pm\sqrt{\beta(s)}$ since the particle motion goes like $x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$
  - This also shows a particle oscillating through the lattice
  - Note that $\sqrt{\beta(s)}$ provides an "envelope" for particle oscillations
    - $\sqrt{\beta(s)}$ is sometimes called the envelope function for the lattice
  - Min beta is at defocusing quads, max beta is at focusing quads
  - 6.5 periodic FODO cells per betatron oscillation

$$\Rightarrow \quad \Delta\phi_C = 360°/6.5 \approx 55°$$

# Example: RHIC FODO Lattice




- 1/6 of one of two RHIC synchrotron rings, injection lattice
  - FODO cell length is about L=30m
  - Phase advance per FODO cell is about $\Delta\phi_C = 77^\circ = 1.344\,\mathrm{rad}$

$$\hat{\beta} = \frac{L}{\sin\Delta\phi_C}\left(1+\sin\frac{\Delta\phi_C}{2}\right) \approx 53\ \mathrm{m}$$

☺

$$\check{\beta} = \frac{L}{\sin\Delta\phi_C}\left(1-\sin\frac{\Delta\phi_C}{2}\right) \approx 8.7\ \mathrm{m}$$

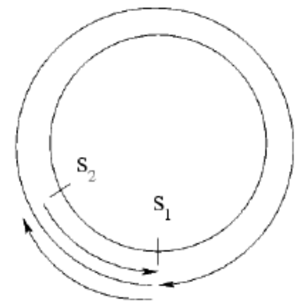
# Propagating Lattice Parameters

- If I have $(\beta,\alpha,\gamma)(s_1)$ and I have the transport matrix $M(s_1,s_2)$ that transports particles from $s_1 \to s_2$, how do I find the new lattice parameters $(\beta,\alpha,\gamma)(s_2)$?

$$M(s_1,s_1+C) = I\cos\mu + J\sin\mu = \begin{pmatrix} \cos\mu+\alpha(s_1)\sin\mu & \beta(s_1)\sin\mu \\ -\gamma(s_1)\sin\mu & \cos\mu-\alpha(s_1)\sin\mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The J(s) matrices at $s_1$, $s_2$ are related by

$$J(s_2) = M(s_1,s_2)J(s_1)M^{-1}(s_1,s_2)$$

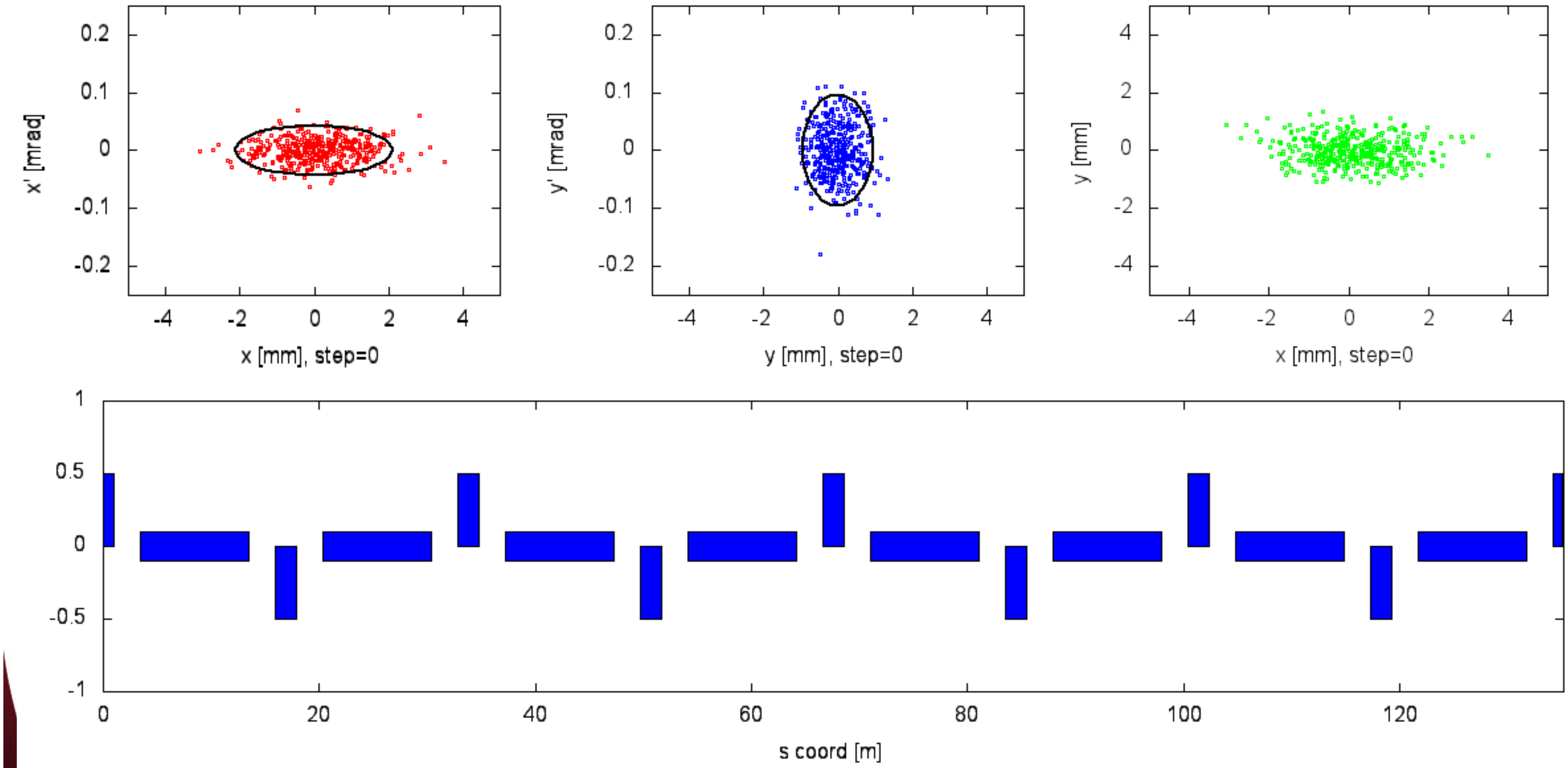



Then expand, using det M=1

$$J(s_2) = \begin{pmatrix} \alpha(s_2) & \beta(s_2) \\ -\gamma(s_2) & -\alpha(s_2) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \alpha(s_1) & \beta(s_1) \\ -\gamma(s_1) & -\alpha(s_1) \end{pmatrix} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

$$\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22}+m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}$$

# =========== What's the Ellipse? ===========




- Area of an ellipse that envelops a given percentage of the beam particles in phase space is related to the **emittance**

We can express this in terms of our lattice functions!

# Invariants and Ellipses

$$x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$$

(A4)

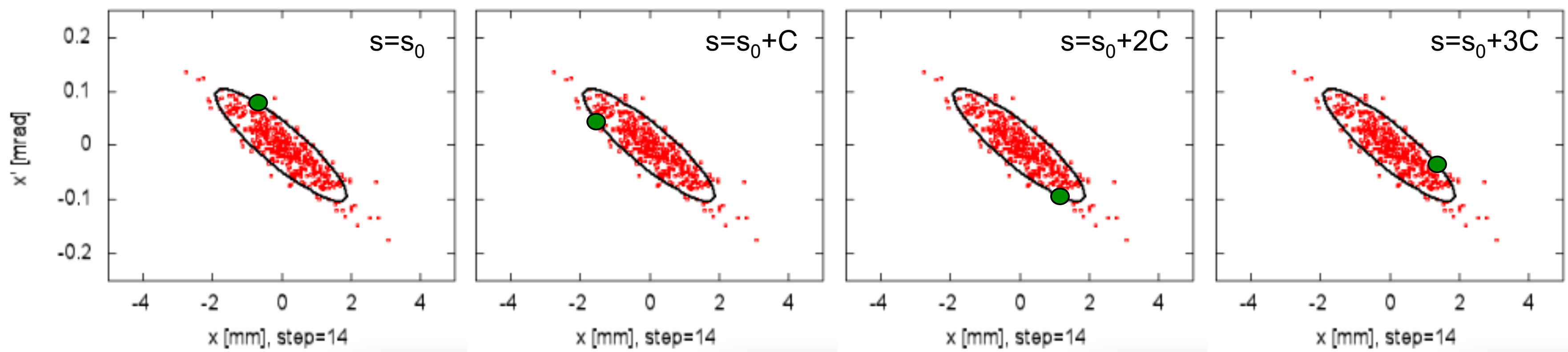
- We assumed A was constant, an **invariant of the motion**
  - A can be expressed in terms of initial coordinates to find

$$\mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$$

  - This is known as the **Courant-Snyder invariant**: for all s,

$$\mathcal{W} = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

  - Similar to total energy of a simple harmonic oscillator
  - $\mathcal{W}$ looks like an elliptical area in $(x, x')$ phase space
  - Our matrices look like scaled rotations (ellipses) in phase space



# Emittance

- The area of the ellipse inscribed by any given particle in phase space as it travels through our accelerator is called the **emittance** $\epsilon$: it is constant (A4) and given by

$$\epsilon = \pi \mathcal{W} = \pi[\gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2]$$

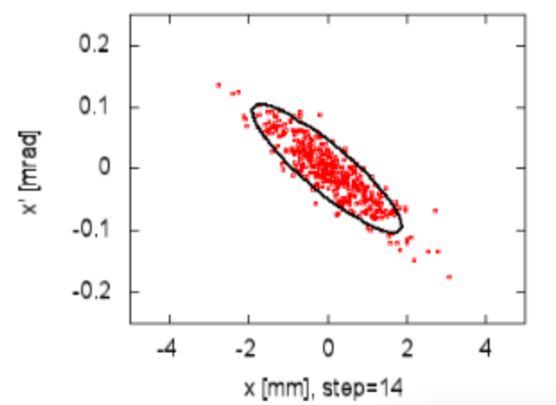
Emittance is often quoted as the area of the ellipse that would contain a certain fraction of all (Gaussian) beam particles

e.g. RMS emittance contains 39% of 2D beam particles

Related to RMS beam size $\sigma_{\text{RMS}}$

$$\sigma_{\text{RMS}} = \sqrt{\epsilon\beta(s)}$$

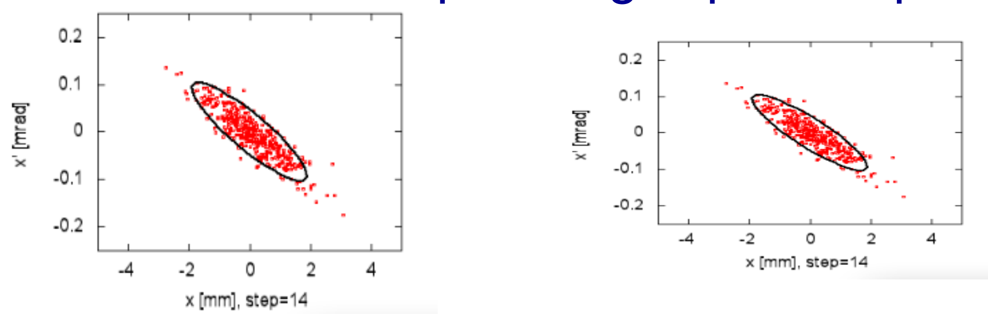
Yes, this RMS beam size depends on s!




RMS emittance convention is fairly standard for electron rings, with units of mm-mrad

# Adiabatic Damping and Normalized Emittance

- But assumption (A4) is violated when we accelerate!
  - When we accelerate, invariant emittance is not invariant!
  - We are defining areas in $(x, x')$ phase space
  - The definition of x doesn't change as we accelerate
  - But $x' \equiv dx/ds = p_x/p_0$ **does** since $p_0$ changes!
  - $p_0$ scales with relativistic beta, gamma: $p_0 \propto \beta_r \gamma_r$
  - This has the effect of compressing x' phase space by $\beta_r \gamma_r$

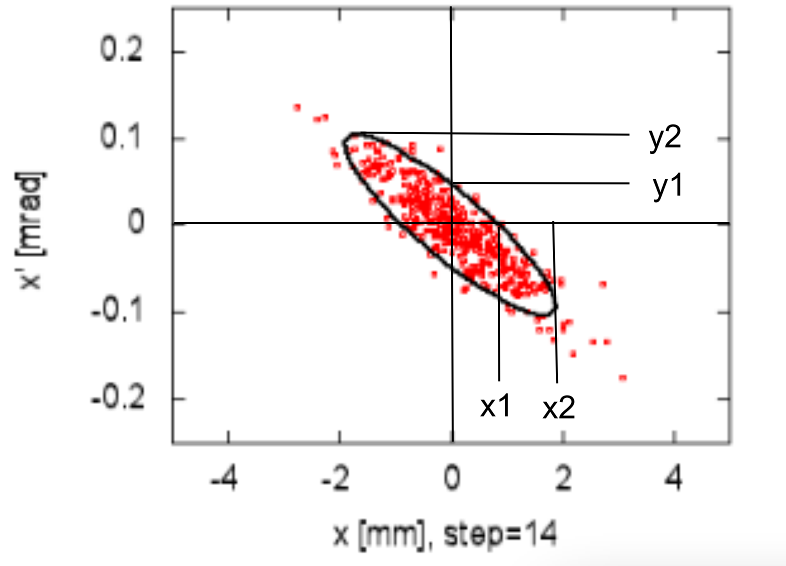



  - **Normalized emittance** is the invariant in this case $\epsilon_N = \beta_r \gamma_r \epsilon$

    unnormalized emittance goes down as we accelerate

    This is called **adiabatic damping**, important in, e.g., linacs

# Phase Space Ellipse Geography




- Now we can figure out some things from a phase space ellipse at a given s coordinate:

$$x_1 = \sqrt{\mathcal{W}/\gamma(s)} \qquad x_2 = \sqrt{\mathcal{W}\beta(s)}$$

$$y_1 = \sqrt{\mathcal{W}/\beta(s)} \qquad y_2 = \sqrt{\mathcal{W}\gamma(s)}$$

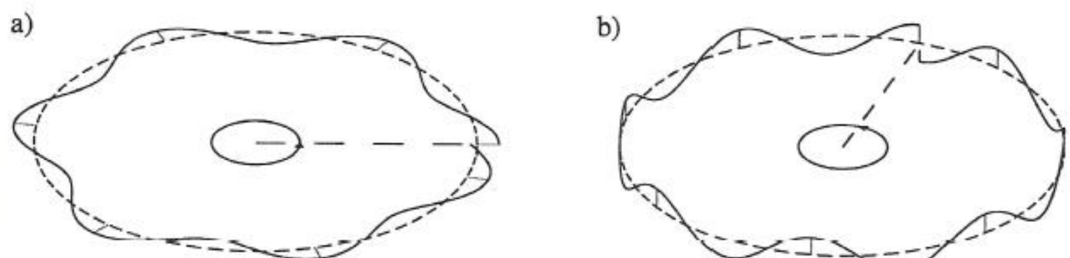
# Rings and Tunes

- A synchrotron is by definition a periodic focusing system
  - It is very likely made up of many smaller periodic regions too
  - We can write down a periodic **one-turn matrix** as before

$$M = I \cos \Delta\phi_C + J \sin \Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

  - Recall that we defined **tune** as the total betatron phase advance in one revolution around a ring divided by $2\pi$

$$Q_{x,y} = \frac{\Delta\phi_{x,y}}{\Delta\theta} = \frac{1}{2\pi} \oint \frac{ds}{\beta_{x,y}(s)}$$




Horizontal Betatron Oscillation with tune: $Q_h = 6.3$, i.e., 6.3 oscillations per turn.

Vertical Betatron Oscillation with tune: $Q_v = 7.5$, i.e., 7.5 oscillations per turn.

# Tunes

- There are horizontal and vertical tunes
  - turn by turn oscillation frequency

- Tunes are a direct indication of the amount of focusing in an accelerator
  - Higher tune implies tighter focusing, lower $\langle \beta_{x,y}(s) \rangle$

- Tunes are a critical parameter for accelerator performance
  - Linear stability depends greatly on phase advance
  - Resonant instabilities can occur when $nQ_x + mQ_y = k$
  - Often adjusted by changing groups of quadrupoles

$$M_{\text{one turn}} = I \cos(2\pi Q) + J \sin(2\pi Q)$$

# Chromaticity

- Just like bending depended on momentum (dispersion), focusing (and thus tunes) depend on momentum
  - The variation of tunes with $\delta$ is called **chromaticity**
  - Insert a momentum perturbation is like adding a small extra focusing to our one-turn matrix that depends on the unperturbed focusing $K_0$

$$M_{\text{one turn}}(\delta) = \begin{pmatrix} 1 & 0 \\ K_0 \delta ds & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi Q) + \alpha \sin(2\pi Q) & \beta \sin(2\pi Q) \\ -\gamma \sin(2\pi Q) & \cos(2\pi Q) - \alpha \sin(2\pi Q) \end{pmatrix}$$

$$M_{\text{one turn}}(\delta) = \begin{pmatrix} \cos(2\pi Q) + \alpha \sin(2\pi Q) & \beta \sin(2\pi Q) \\ -\gamma \sin(2\pi Q) + K_0 \delta [\cos(2\pi Q) + \alpha \sin(2\pi Q)] ds & \cos(2\pi Q) - \alpha \sin(2\pi Q) + K_0 \delta \beta \sin(2\pi Q) ds \end{pmatrix}$$

  - This looks painful, but remember the trace is related to the new tune

$$\cos(2\pi Q_{\text{new}}) = \frac{1}{2} \text{Tr}\, M = \cos(2\pi Q) + \frac{K_0 \delta}{2} \beta \sin(2\pi Q) ds$$

# Chromaticity Continued

$$\cos(2\pi Q_{\text{new}}) = \frac{1}{2}\text{Tr } M = \cos(2\pi Q) + \frac{K_0 \delta}{2}\beta \sin(2\pi Q) ds$$

$$\cos(2\pi Q_{\text{new}}) = \cos(2\pi(Q + dQ)) \approx \cos(2\pi Q) - 2\pi \sin(2\pi Q) dQ$$

- These last two terms must be equal, which gives

$$dQ = -\frac{K(s)\delta}{4\pi}\beta(s) ds$$

Integrate around the ring to find the total tune change

$$\Delta Q = -\frac{\delta}{4\pi}\oint K(s)\beta(s)\, ds$$

**Natural Chromaticity** is defined as

$$\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q}\oint K(s)\beta(s)\, ds$$

The tune Q invariably has some spread from momentum spread

# Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi} - \text{periodic ansatz solution} \quad x(s) = \sqrt{\epsilon\beta(s)}\cos[\phi(s) + \phi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \quad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(s)\sin\Delta\phi_C & \beta(s)\sin\Delta\phi_C \\ -\gamma(s)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(s)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

betatron phase advance $\Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$ $\quad \mathrm{Tr}\, M = 2\cos\Delta\phi_C$

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

Courant – Snyder invariant $\mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$

# Dispersion

- There is one more important lattice parameter to discuss
- **Dispersion** $\eta(s)$ is defined as the change in particle position with fractional momentum offset $\delta \equiv \Delta p/p_0$

  Dispersion originates from momentum dependence of dipole bends
- Add explicit momentum dependence to equation of motion again

$$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

Particular solution of inhomogeneous differential equation with periodic ρ(s)

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\delta_0$$

$$x'(s) = C'(s)x_0 + S'(s)x_0' + D'(s)\delta_0$$

$$D(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau$$

Use initial conditions etc to find

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix}$$

The trajectory has two parts: $x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$

# Dispersion Continued

- Substituting and noting dispersion is periodic, $\eta_x(s+C) = \eta_x(s)$

$$\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \\ \delta_0 \end{pmatrix} \qquad \text{achromat} : D = D' = 0$$

  - If we take $\delta_0 = 1$ we can solve this in a clever way

$$\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(S) \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

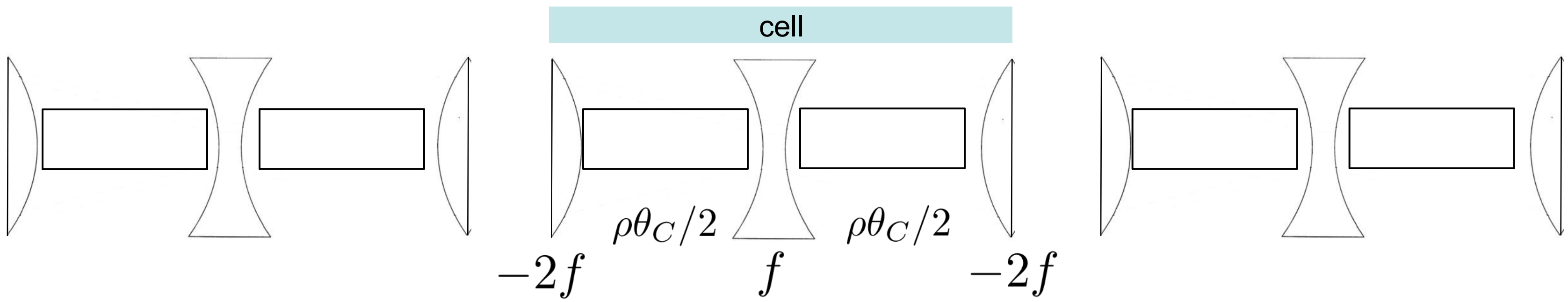
$$(I-M) \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = (I-M)^{-1} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

  - Solving gives

$$\eta(s) = \frac{[1-S'(s)]D(s) + S(s)D'(s)}{2(1-\cos\Delta\phi)}$$

$$\eta'(s) = \frac{[1-C(s)]D'(s) + C'(s)D(s)}{2(1-\cos\Delta\phi)}$$

# FODO Cell Dispersion




- A periodic lattice without dipoles has no **intrinsic** dispersion
- Consider FODO with long dipoles and thin quadrupoles
  - Each dipole has total length $\rho\theta_C/2$ so each cell is of length $L = \rho\theta_C$
  - Assume a large accelerator with many FODO cells so $\theta_C \ll 1$

$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\mathrm{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_C}{8} \\ 0 & 1 & \frac{\theta_C}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{\mathrm{FODO}} = M_{-2f} M_{\mathrm{dipole}} M_f M_{\mathrm{dipole}} M_{-2f}$$

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_C \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_C \\ 0 & 0 & 1 \end{pmatrix}$$

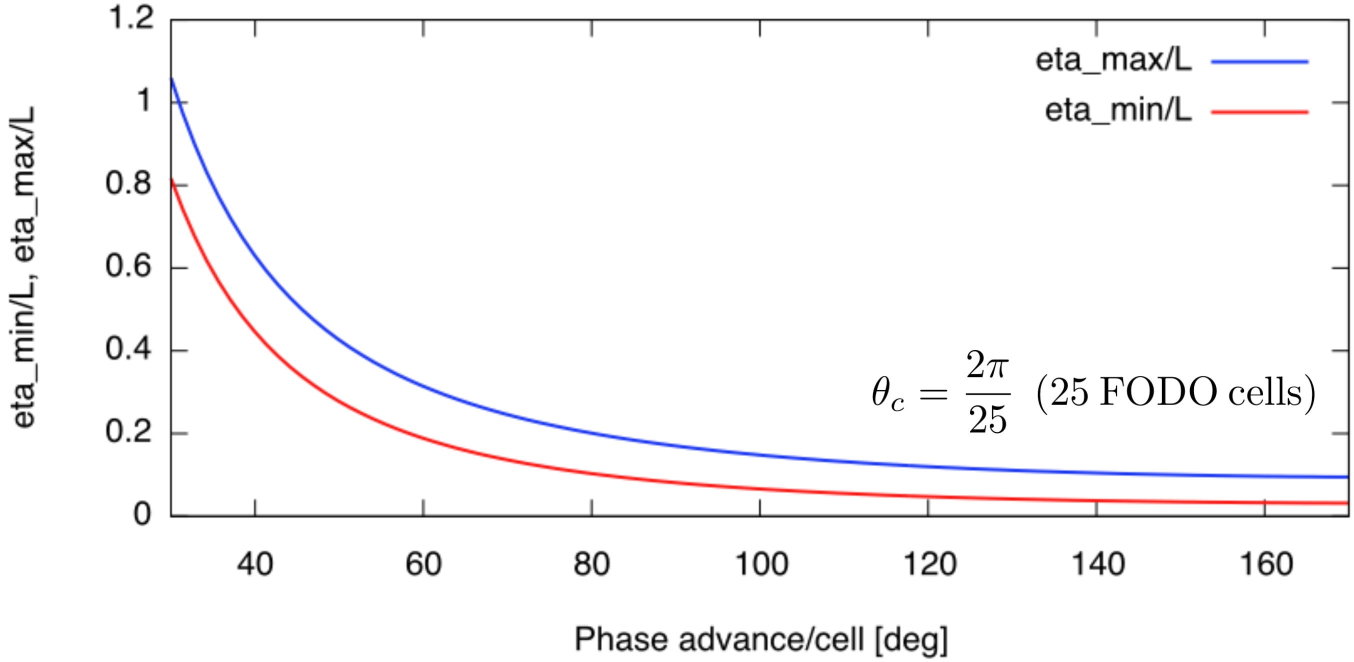
# FODO Cell Dispersion

- Like $\hat{\beta}$ before, this choice of periodicity gives us $\hat{\eta}_x$

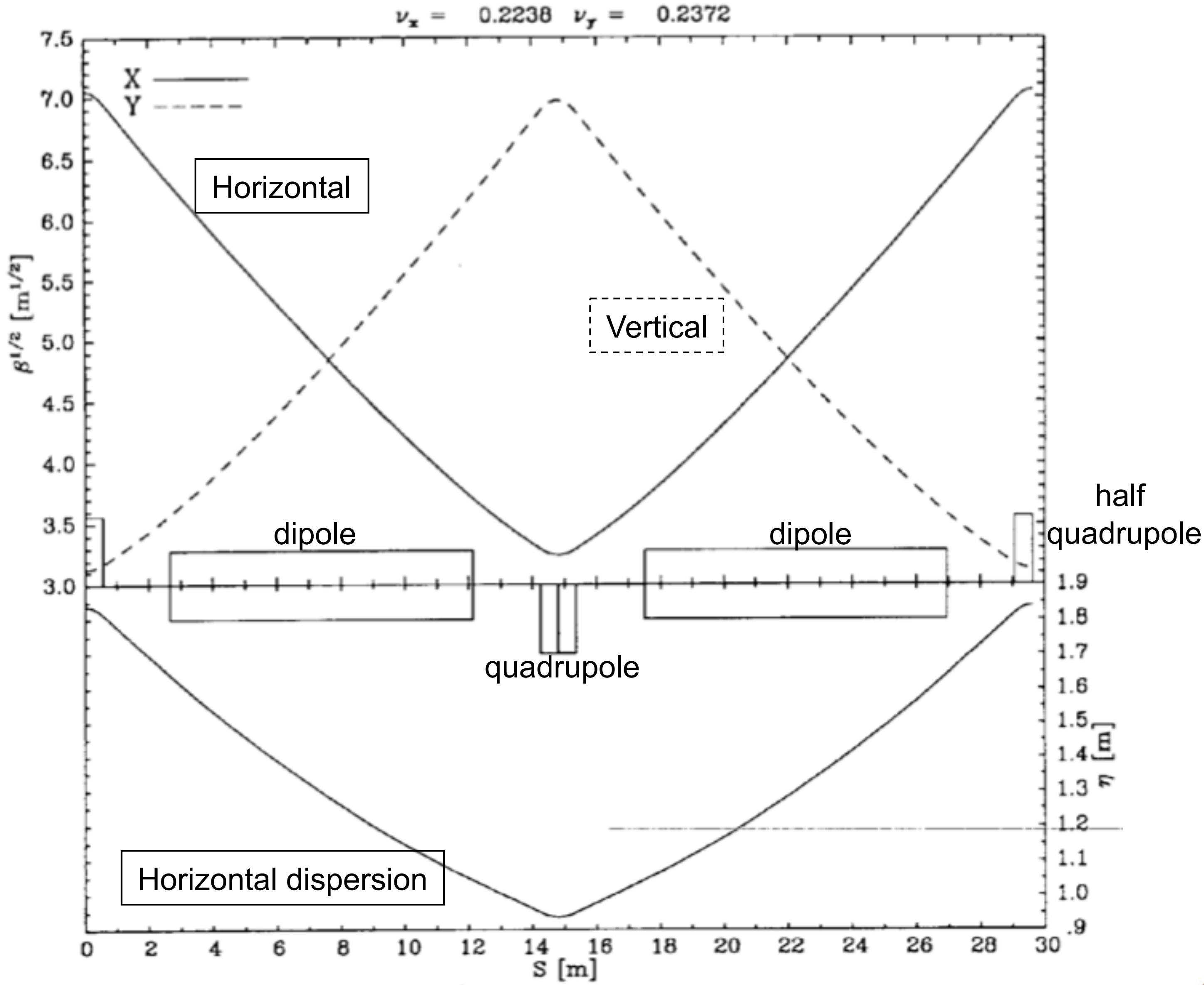
$$\hat{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right] \qquad \eta'_x = 0 \text{ at max}$$

- Changing periodicity to defocusing quad centers gives $\check{\eta}_x$

$$\check{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1-\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right]$$

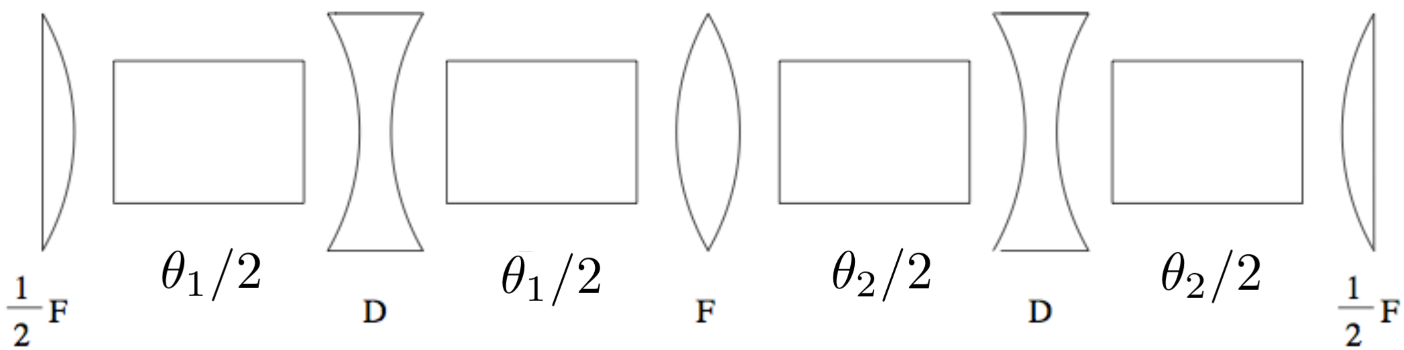


# RHIC FODO Cell



# Dispersion Suppressor

- The FODO dispersion solution is non-zero everywhere
  - But in straight sections we often want $\eta_x = \eta'_x = 0$
    - e.g. to keep beam small in wigglers/undulators in light source
  - We can "match" between these two conditions with with a **dispersion suppressor**, a **non-periodic** set of magnets that transforms FODO $(\eta_x, \eta'_x)$ to zero.




  - Consider two FODO cells with different total bend angles $\theta_1, \theta_2$
    - Same quadrupole focusing to not disturb $\beta_x, \Delta\phi_x$ much
    - We want this to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0, 0)$
    - $\alpha_x = 0$ at ends to simplify periodic matrix

# (FODO Dispersion Suppressor)

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\Delta\phi_x & \beta_x \sin 2\Delta\phi_x & D(s) \\ -\frac{\sin 2\Delta\phi_x}{\beta_x} & \cos 2\Delta\phi_x & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_x \\ 0 \\ 1 \end{pmatrix}$$

multiply matrices $\Rightarrow$

$$D(s) = \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

$$D'(s) = \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

$$\hat{\eta}_x = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)(\theta_1 + \theta_2)$$

$$\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta \qquad \theta_2 = \left(\frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta$$

$\theta = \theta_1 + \theta_2$ two cells, one FODO bend angle $\rightarrow$ reduced bending