# Introduction to Accelerator Physics Old Dominion University

# This Week: Dispersion, Dispersion Suppression, Transition Energy, and Longitudinal Motion

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**Select final presentation topic by Thu 3 Nov or one will be assigned to you!**

# Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi-periodic ansatz solution} \quad x(s) = \sqrt{\epsilon\beta(s)}\cos[\phi(s) + \phi_0]$$

$$\beta(s) = \beta(s+C) \quad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \quad \phi(s) = \int \frac{ds}{\beta(s)}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(s)\sin\Delta\phi_C & \beta(s)\sin\Delta\phi_C \\ -\gamma(s)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(s)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

betatron phase advance $\Delta\phi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$ $\quad \mathrm{Tr}\, M = 2\cos\Delta\phi_C$

$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

Courant – Snyder invariant $\mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$

# Dispersion

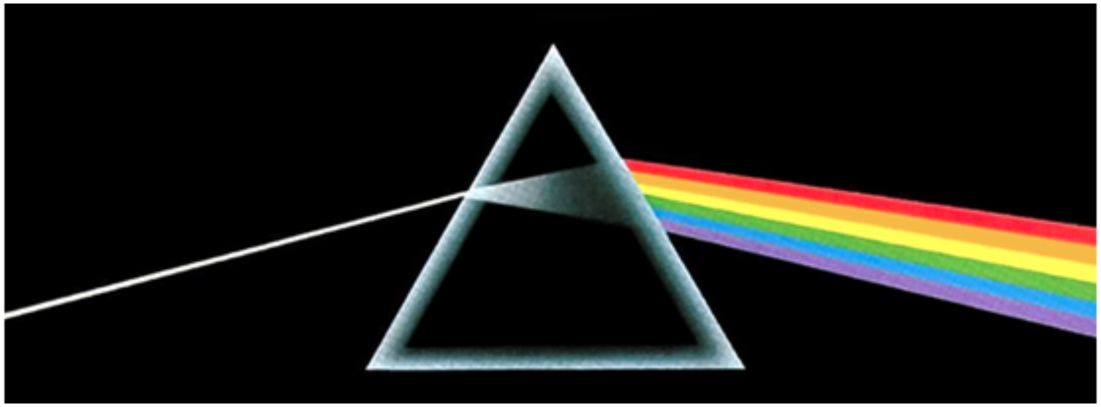
- There is one more important lattice parameter to discuss
- **Dispersion** $\eta(s)$ is defined as the change in particle position with fractional momentum offset $\delta \equiv \Delta p/p_0$

$$x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$$

Dispersion originates from momentum dependence of dipole bends

Equivalent to separation of optical wavelengths in prism

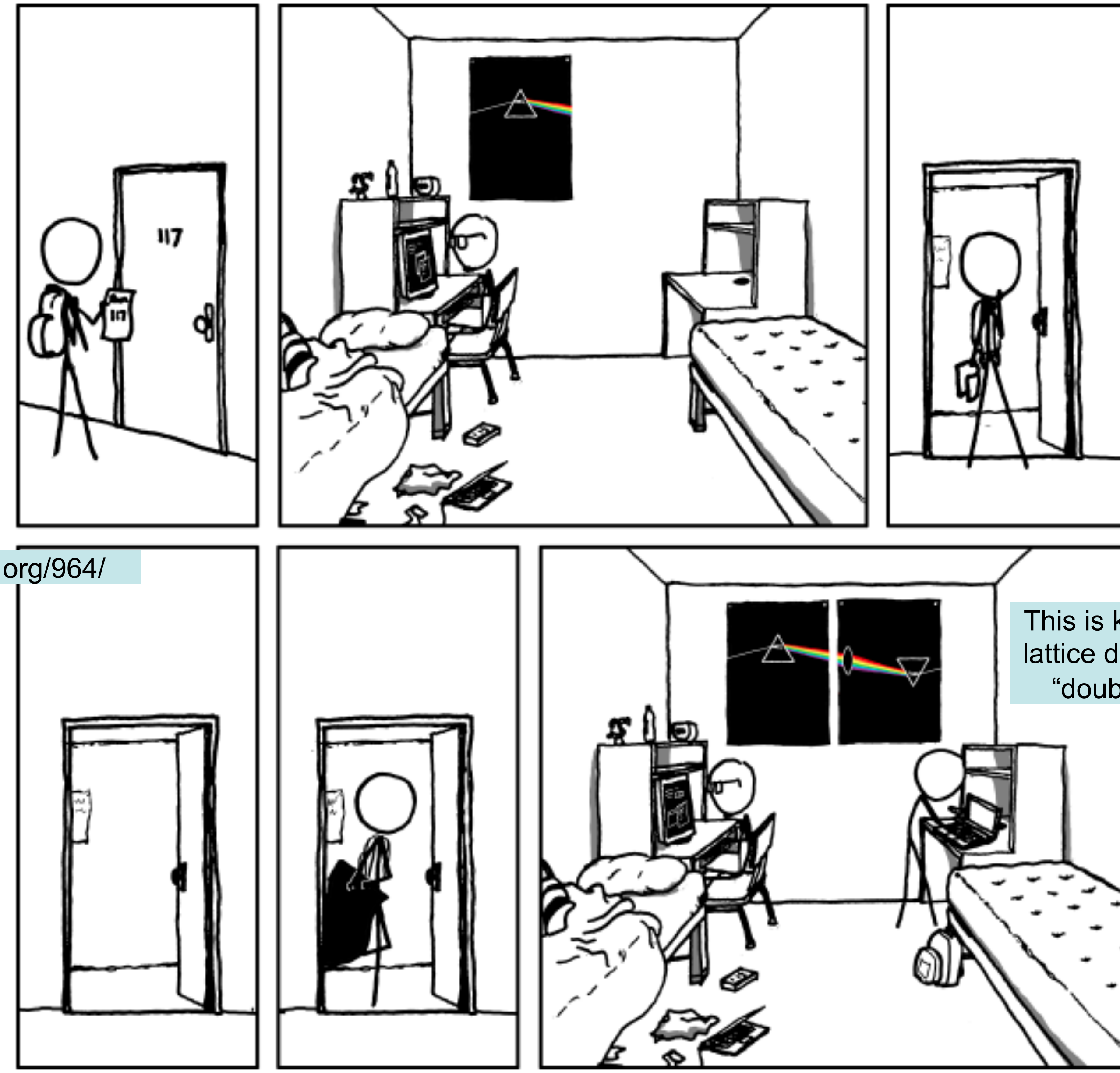
White light with many frequencies (momenta) enters, all with same initial trajectories (x,x')



Different positions due to different bend angles of different wavelengths (frequencies, momenta) of incoming light

# (xkcd interlude)

http://www.xkcd.org/964/




This is known in accelerator lattice design language as a "double bend achromat"

# Dispersion

- Add explicit momentum dependence to equation of motion again

$$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

Assume our ansatz solution and use initial conditions to find

$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\delta_0$$
$$x'(s) = C'(s)x_0 + S'(s)x'_0 + D'(s)\delta_0$$

$$D(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau$$

Particular solution of inhomogeneous differential equation with periodic ρ(s)

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

The trajectory has two parts:

$$x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$$

# Dispersion Continued

- Substituting and noting dispersion is periodic, $\eta_x(s+C) = \eta_x(s)$

$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \\ \delta_0 \end{pmatrix} \qquad \text{achromat} : D = D' = 0$$

  - If we take $\delta_0 = 1$ we can solve this in a clever way

$$\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(S) \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

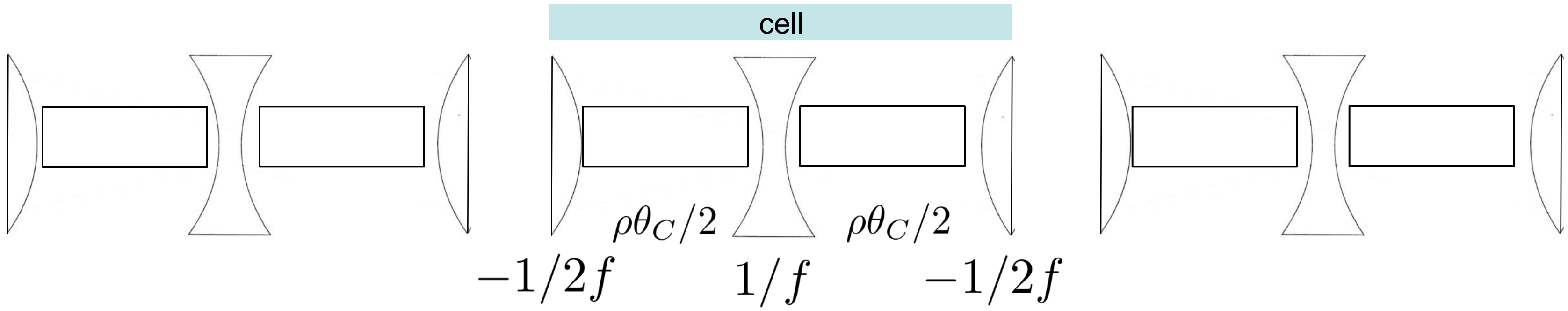
$$(I - M) \begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} \quad \Rightarrow \quad \boxed{\begin{pmatrix} \eta_x(s) \\ \eta'_x(s) \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}}$$

  - Solving gives

$$\eta(s) = \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\Delta\phi)}$$

$$\eta'(s) = \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\Delta\phi)}$$

# FODO Cell Dispersion




- A periodic lattice without dipoles has no **intrinsic** dispersion
- Consider FODO with long dipoles and thin quadrupoles
  - Each dipole has total length $\rho\theta_C/2$ so each cell is of length $L = \rho\theta_C$
  - Assume a large accelerator with many FODO cells so $\theta_C \ll 1$

$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\text{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_C}{8} \\ 0 & 1 & \frac{\theta_C}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{\text{FODO}} = M_{-2f} M_{\text{dipole}} M_f M_{\text{dipole}} M_{-2f}$$

$$M_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_C \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_C \\ 0 & 0 & 1 \end{pmatrix}$$

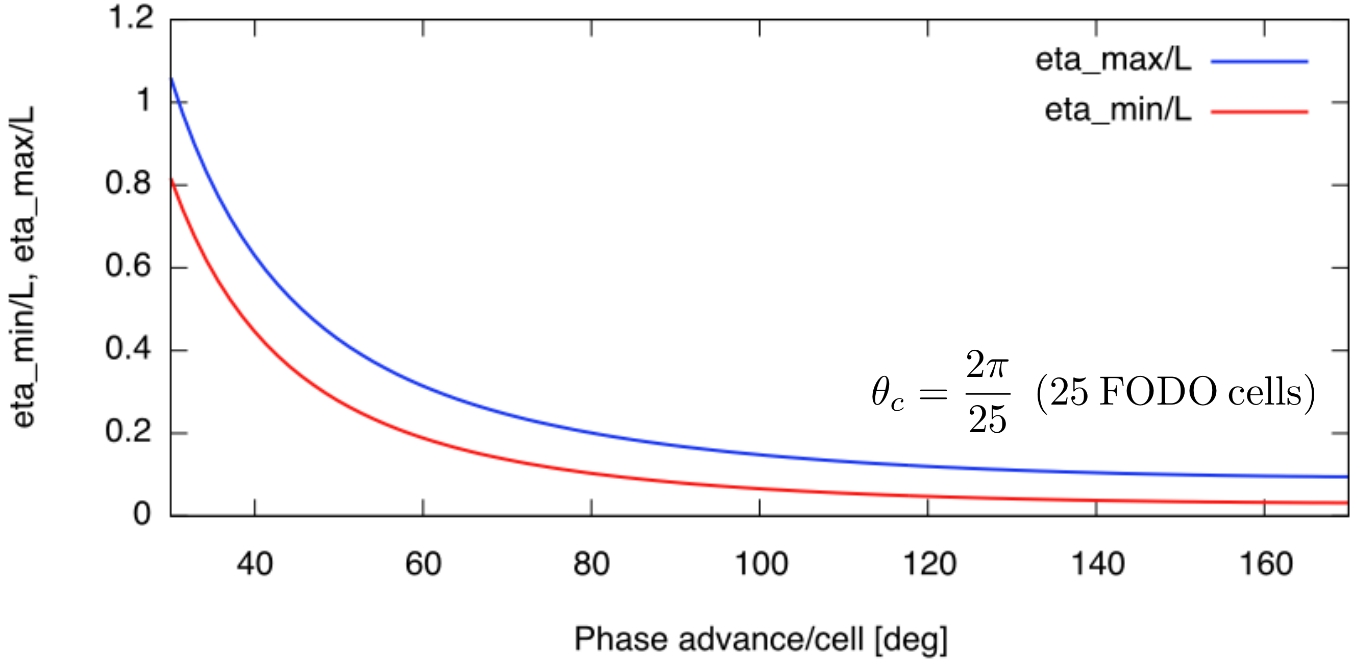
# FODO Cell Dispersion

- Like $\hat{\beta}$ before, this choice of periodicity gives us $\hat{\eta}_x$

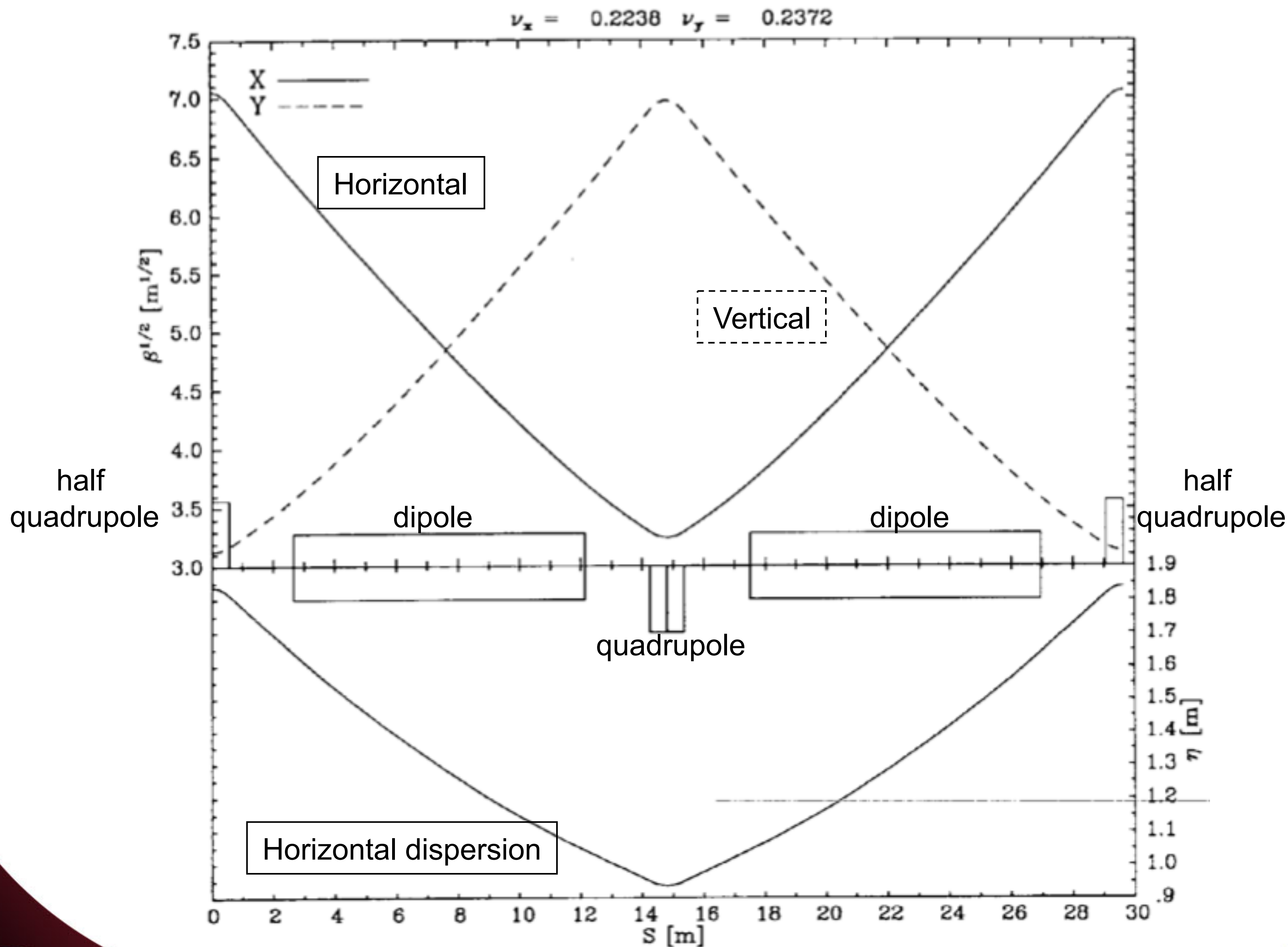
$$\hat{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right] \qquad \eta_x' = 0 \text{ at max}$$

- Changing periodicity to defocusing quad centers gives $\check{\eta}_x$

$$\check{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1-\frac{1}{2}\sin\frac{\Delta\phi}{2}}{\sin^2\frac{\Delta\phi}{2}}\right]$$

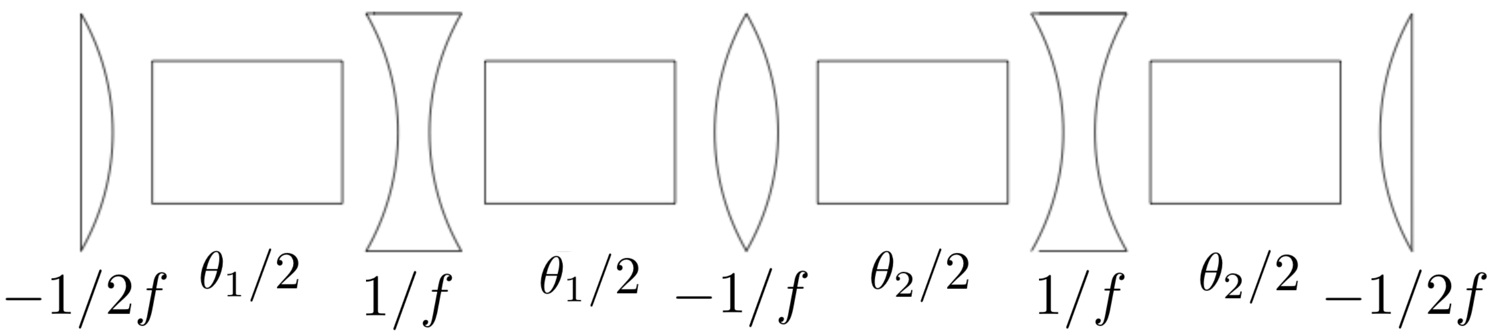


# RHIC FODO Cell



# Dispersion Suppressor

- The FODO dispersion solution is non-zero everywhere
  - But in straight sections we often want $\eta_x = \eta'_x = 0$
    - e.g. to keep beam small in wigglers/undulators in a light source
  - We can "match" between these two conditions with with a **dispersion suppressor**, a **non-periodic** set of magnets that transforms FODO $(\eta_x, \eta'_x)$ to zero.




  - Consider two FODO cells with different total bend angles $\theta_1, \theta_2$
    - Same quadrupole focusing to not disturb $\beta_x, \Delta\phi_x$ much
    - We want this to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0, 0)$
    - $\alpha_x = 0$ at ends to simplify periodic matrix

# FODO Dispersion Suppressor

Zero dispersion area

slope η'=0

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\Delta\phi_x & \beta_x \sin 2\Delta\phi_x & D(s) \\ -\frac{\sin 2\Delta\phi_x}{\beta_x} & \cos 2\Delta\phi_x & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_x \\ 0 \\ 1 \end{pmatrix}$$

FODO peak dispersion,

slope η'=0

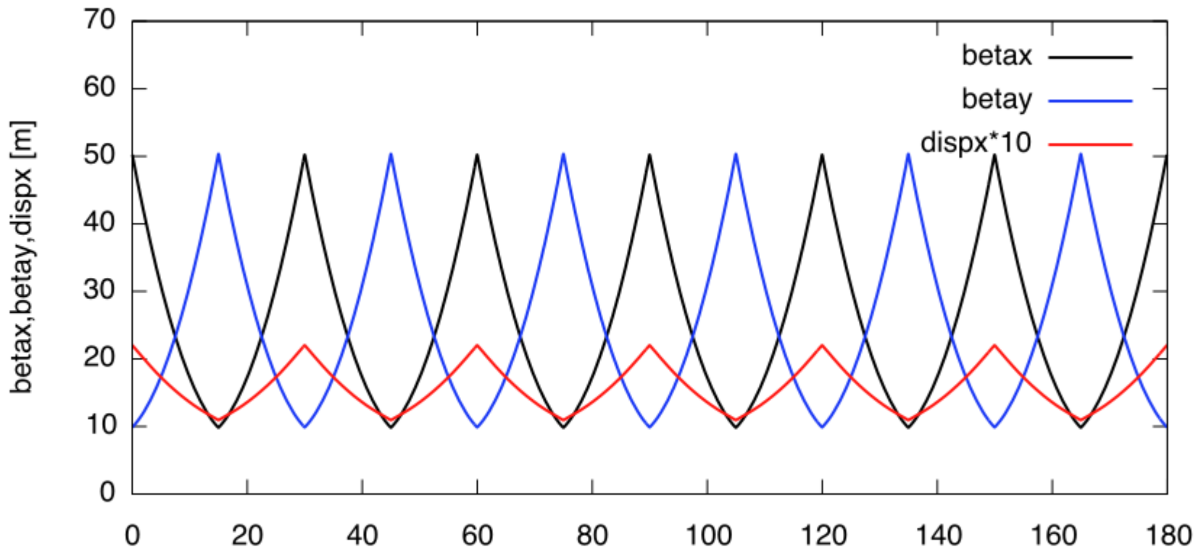
$$\text{multiply matrices} \quad \Rightarrow \quad \begin{aligned} D(s) &= \frac{L}{2}\left(1+\frac{L}{8f}\right)\left[\left(3-\frac{L^2}{4f^2}\right)\theta_1+\theta_2\right] \\ D'(s) &= \left(1-\frac{L}{8f}-\frac{L^2}{32f^2}\right)\left[\left(1-\frac{L^2}{4f^2}\right)\theta_1+\theta_2\right] \end{aligned}$$

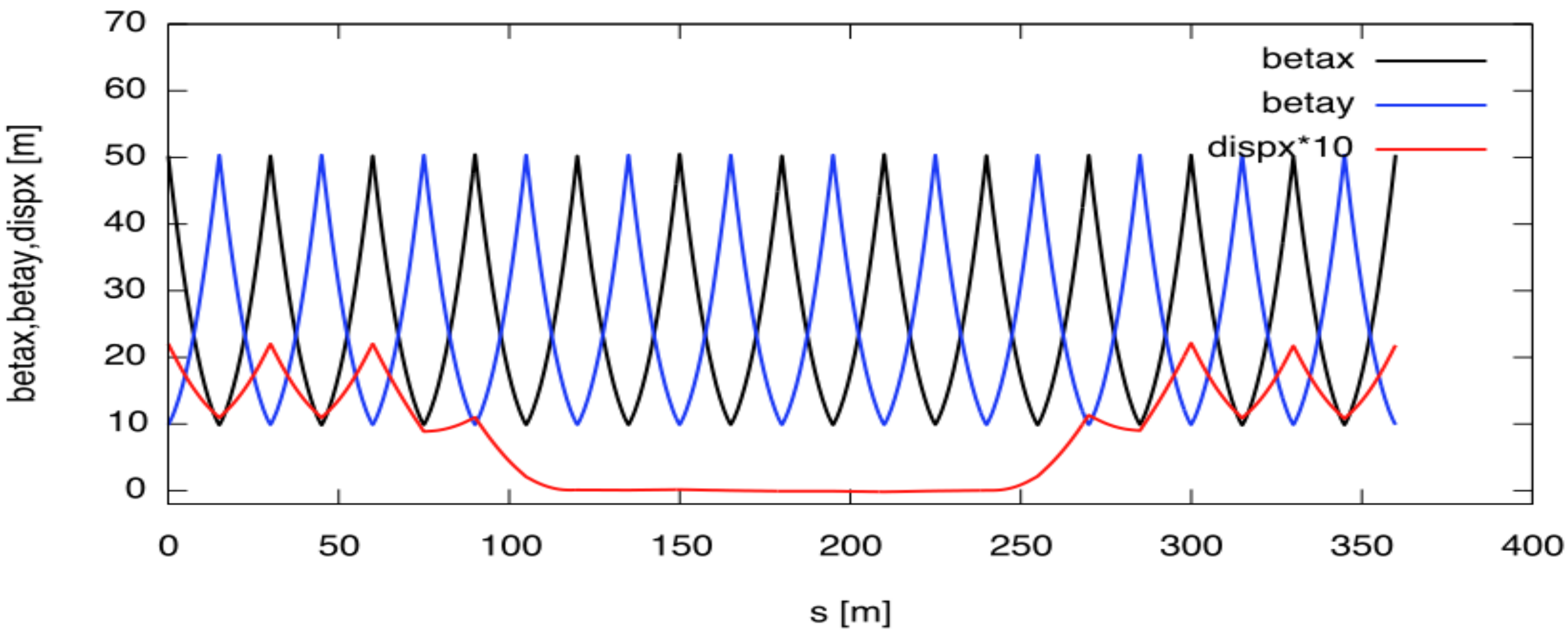
$$\hat{\eta}_x = \frac{4f^2}{L}\left(1+\frac{L}{8f}\right)(\theta_1+\theta_2)$$

$$\theta_1 = \left(1-\frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta \qquad \theta_2 = \left(\frac{1}{4\sin^2\frac{\Delta\phi_x}{2}}\right)\theta$$

$$\theta = \theta_1+\theta_2 \qquad \text{two cells, one FODO bend angle} \quad \rightarrow \quad \text{reduced bending}$$

# FODO Cell Dispersion and Suppressor



# Chapter 7: Synchrotron Oscillations

- Recall something called **momentum compaction**
  - (Section 1.6 of the book, way back…)
  - How does a particle's path length relative to the design particle change with its momentum relative to design particle?

$$\text{Momentum compaction } \alpha_P \equiv \left(\frac{dL}{L}\right) / \left(\frac{dp}{p_0}\right) = \frac{p_0}{L}\frac{dL}{dp}$$

  - Example: circular motion in a constant magnetic field B

$$\frac{p}{q} = B\rho \quad \Rightarrow \quad \alpha_P = \left(\frac{Bq}{2\pi}\right)\left(\frac{2\pi}{Bq}\right) = 1$$

  - Example: gravitational circular motion

$$F = \frac{GMm}{r^2} = \frac{p^2}{mr} \qquad \frac{dr}{dp} = -\frac{2pr^2}{gMm^2} \qquad \alpha_P = \frac{p}{r}\frac{dr}{dp} = -2$$

  - In general $\alpha_p$ really depends on the magnet layout
    - In particular, the dispersion! (difference of path with momentum)

# Transition Energy

- Relativistic particle motion in a periodic accelerator (like a synchrotron) creates some weird effects
  - For particles moving around with frequency $\omega$ in circumference $C$

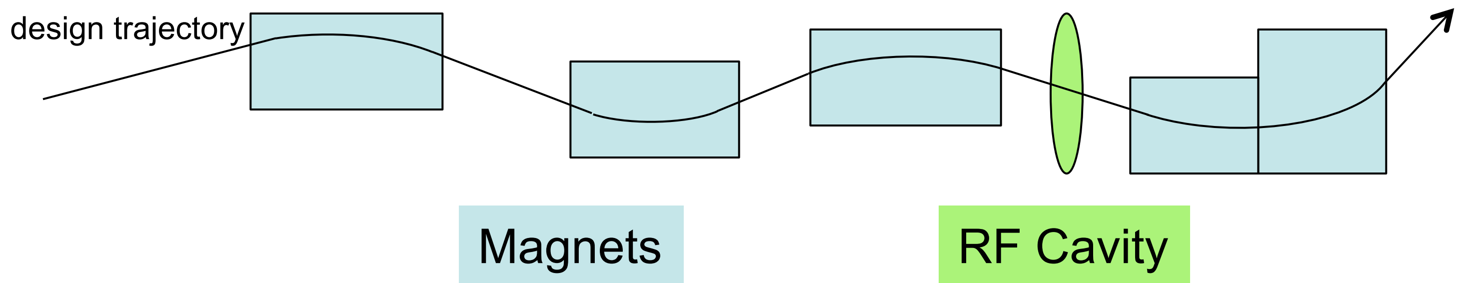
$$\omega = \frac{2\pi\beta_r c}{C} \quad \Rightarrow \quad \frac{d\omega}{\omega} = \frac{d\beta_r}{\beta_r} - \frac{dC}{C} = \left(\frac{1}{\gamma_r^2} - \frac{1}{\gamma_{\rm tr}^2}\right)\frac{dp}{p_0}$$

$$\text{momentum compaction } \alpha_P \equiv \frac{dC}{C}/\delta = \frac{p_0}{C}\frac{dC}{dp} \qquad \text{transition gamma } \gamma_{\rm tr} \equiv \frac{1}{\sqrt{\alpha_P}}$$

  - At "transition", $\gamma_r = \gamma_{\rm tr}$ and **particle revolution frequency does not depend on its momentum**
    - Reminiscent of a cyclotron but now we're strong focusing and at constant radius!

| | |
|---|---|
| electron ring | At $\gamma_r > \gamma_{\rm tr}$ higher momentum gives **lower** revolution frequency |
| electron linac | At $\gamma_r < \gamma_{\rm tr}$ higher momentum gives **higher** revolution frequency |

# Changing Pace: Longitudinal Motion and Energy



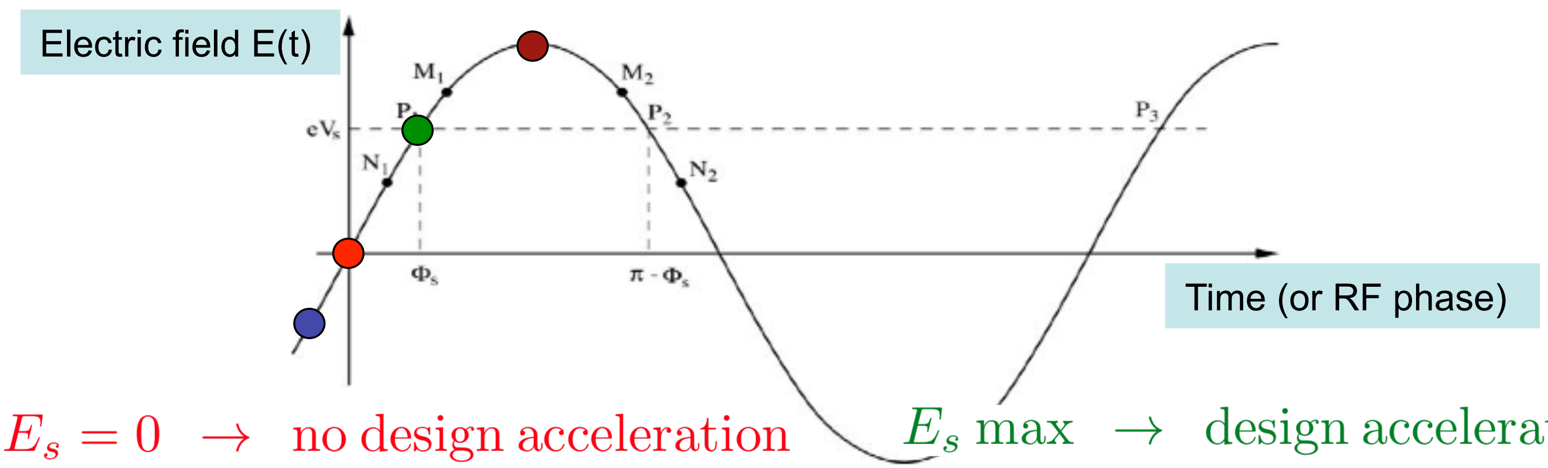

- Up to now we have considered **transverse motion** in our accelerator, mostly in systems with **periodic transverse focusing**
- But what about **longitudinal motion**? If we don't provide some longitudinal focusing, particles different than design momentum will move away from the design particle over time
  - Momentum spread corresponds to a velocity spread

$$\delta \equiv \frac{dp}{p_0} = \gamma^2 \frac{d\beta_r}{\beta_{r,0}}$$

  - For typical numbers $\delta \approx 10^{-3}, \gamma_r \approx 10^4 \Rightarrow d\beta_r \approx 10^{-11} c = 3\ \mathrm{mm/s}$
  - Our bunch spreads and loses energy to synchrotron radiation

# Synchronous Particle

- We'll be using periodic electric "RF fields"
  - Commonly in the MHz to GHz frequency range
    - Manageable wavelengths of EM waves: e.g. 100 MHz/c = 33 cm
  - Design trajectory now includes longitudinal location and time
    - **Time is equivalent to phase of arrival** in our oscillating RF field
    - The **design particle arrives** at an RF phase defined as the **synchronous phase** $\phi_s$ at synchronous electric field value $E_s$




$E_s = 0 \quad \rightarrow \quad \text{no design acceleration}$

$E_s < 0 \quad \rightarrow \quad \text{design deceleration}$

$E_s \text{ max} \quad \rightarrow \quad \text{design acceleration}$

$E_s \text{ max} \quad \rightarrow \quad \text{max design acceleration}$

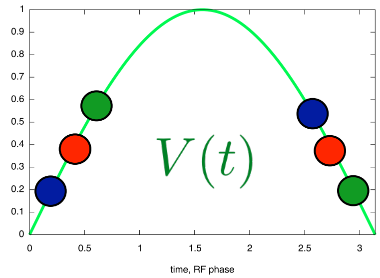
# RF Fields

- We need to accomplish two things
  - Add longitudinal energy to the beam to keep $p_0$ constant
  - Add longitudinal focusing
- RF is also used in accelerating systems to not just balance losses from synchrotron radiation, but
  - Accelerate the beam as a whole: $E_s \neq 0$
  - Keep the beam bunched (focusing, **phase stability)**: $\frac{dE_s}{ds} \neq 0$
- Use sinusoidally varying RF voltage in **short** RF cavities
  - Run at **harmonic number** $h$ of revolution frequency, $\omega_{\rm rf} = h\omega_{\rm rev}$

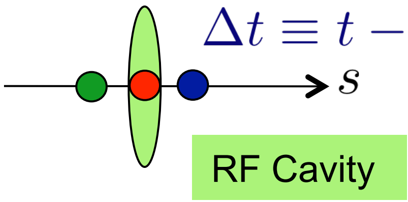
$$\vec{E}(s,t) = \hat{s}E(s,t) = \hat{s}V\sin(\omega_{\rm rf}t + \phi_s)\sum_{n=-\infty}^{\infty}\delta(s-nL)$$

**energy gain/turn**

$$\Delta U = qV\sin(\omega_{\rm rf}\Delta t + \phi_s)$$

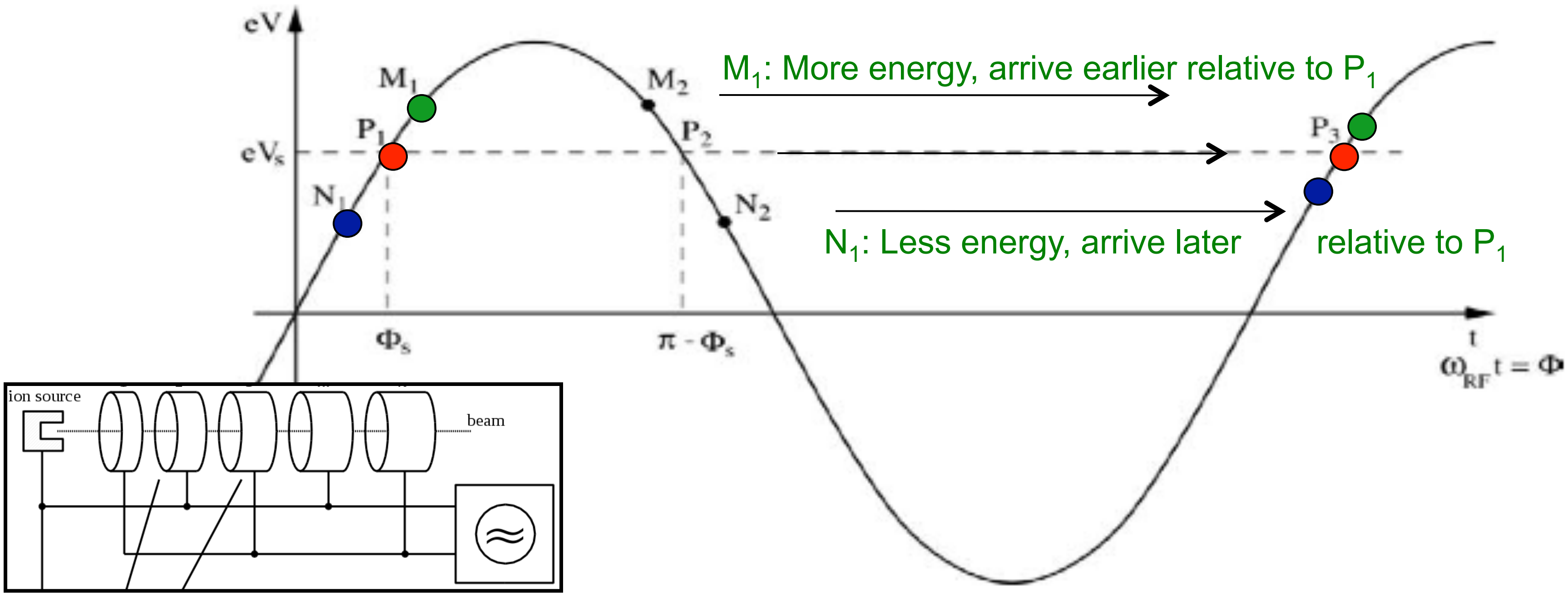



$$\Delta t \equiv t - t_{\rm synchronous}$$



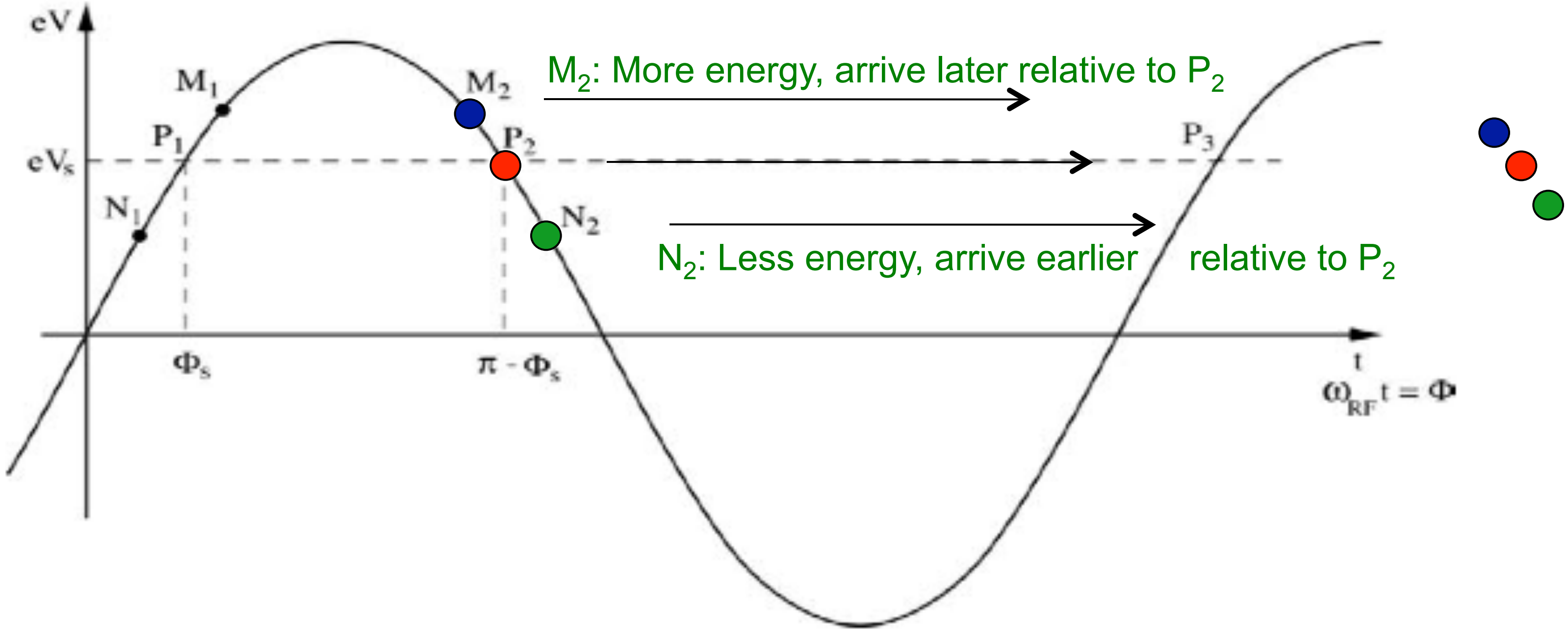

Earlier in time is earlier in phase!

# Phase Stability in a Linac




- Consider a series of accelerating gaps (or a ring with one gap)
  - **By design** synchronous phase $\Phi_s$ gains just enough energy to balance radiation losses and hit same phase $\Phi_s$ in the next gap
  - $P_1$ are our design particles: they "ride the wave" exactly in phase
- If increased energy means increased frequency ("below transition", e.g. linac)
  - $M_1$,$N_1$ will move towards $P_1$ (local stability) => **phase stability**
  - $M_2$,$N_2$ will move away from $P_2$ (local instability)

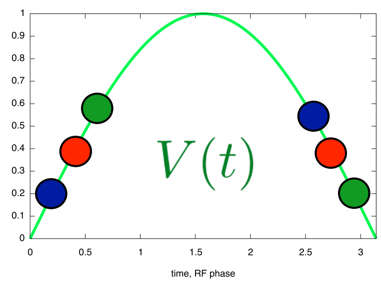
# Phase Stability in an Electron Synchrotron




- If increased energy means decreased frequency ("above transition")
  - $P_2$ are our design particles: they "ride the wave" exactly in phase
  - $M_1$,$N_1$ will move away from $P_1$ (local instability)
  - $M_2$,$N_2$ will move towards $P_2$ (local stability) => **phase stability**
  - All synchrotron light sources run in this regime ( $\gamma_r \gg 1$ )
  - Note $\phi_s$ is given by maximum RF voltage and required energy gain per turn

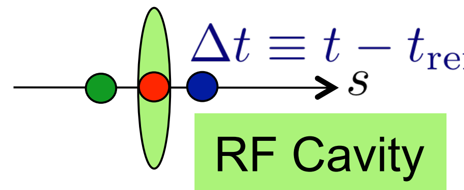
# Synchrotron Oscillations

$$\vec{E}(s,t) = \hat{s}E(s,t) = \hat{s}V\sin(\omega_{\rm rf}t+\phi_s)\sum_{n=-\infty}^{\infty}\delta(s-nL)$$




$$\Delta U = qV\sin(\omega_{\rm rf}\Delta t+\phi_s)$$

$$\Delta t \equiv t - t_{\rm reference}$$




- The electric force is sinusoidal so we expect particle motion to look something like a pendulum
  - Define coordinate **synchrotron phase** of a particle $\varphi \equiv \phi - \phi_s$
  - We can go through tedious relativistic mathematics (book pages 144-146) to find a biased pendulum equation

$$\ddot{\varphi} + \frac{h\omega_{\rm ref}^2\eta_{\rm tr}qV}{2\pi\beta_r^2U_{\rm ref}}[\sin(\phi_s+\varphi)-\sin(\phi_s)] = 0$$

where $\omega_{\rm rf} = h\omega_{\rm ref}$ $\qquad \eta_{\rm tr} \equiv \left(\frac{1}{\gamma_r^2}-\frac{1}{\gamma_{\rm tr}^2}\right)$ $\qquad \omega_{\rm ref}$ : revolution frequency of synchronous particle

# Linearized Synchrotron Oscillations

$$\ddot{\varphi} + \frac{h\omega_{\rm ref}^2 \eta_{\rm tr} qV}{2\pi\beta_r^2 U_{\rm ref}} [\sin(\phi_s + \varphi) - \sin(\phi_s)] = 0$$

- If these synchrotron phase oscillations are small, this motion looks more like (surprise!) a simple harmonic oscillator

$$\sin(\phi_s + \varphi) \approx \varphi\cos(\phi_s) + \sin(\phi_s)$$

$$\ddot{\varphi} + \Omega_s^2 \varphi = 0$$

$$\Omega_s \equiv \omega_{\rm ref} \sqrt{\frac{h\eta_{\rm tr}\cos(\phi_s)}{2\pi\beta_r^2\gamma_r} \frac{qV}{mc^2}}$$ **synchrotron frequency**

$$Q_s \equiv \frac{\Omega_s}{\omega_{\rm ref}} = \sqrt{\frac{h\eta_{\rm tr}\cos(\phi_s)}{2\pi\beta_r^2\gamma_r} \frac{qV}{mc^2}}$$ **synchrotron tune**

Note that $\eta_{\rm tr}\cos(\phi_s) > 0$ is required for phase stability.

Example: ALS synchrotron frequency on order of few $10^{-3}$

$(\varphi, \dot{\varphi} \equiv d\varphi/dt)$ are natural phase space coordinates

# Large Synchrotron Oscillations

- Sometimes particles achieve large momentum offset $\delta$ and therefore get a large phase offset $\varphi$ relative to design
  - For example, particle-particle scattering (IBS or Touschek)
  - Then our longitudinal motion equation becomes

$$\ddot{\varphi} + \frac{\Omega_s^2}{\cos\phi_s}[\sin(\varphi+\phi_s) - \sin(\phi_s)] = 0 \qquad \varphi \equiv \phi - \phi_s$$

$$\frac{d(\dot{\phi}^2)}{dt} = 2\ddot{\phi}\frac{d\phi}{dt} \quad \Rightarrow \quad d\left(\dot{\phi}^2\right) = \frac{2\Omega_s^2}{\cos\phi_s}(-\sin\phi\, d\phi) + 2\Omega_s^2 \tan\phi_s\, d\phi$$

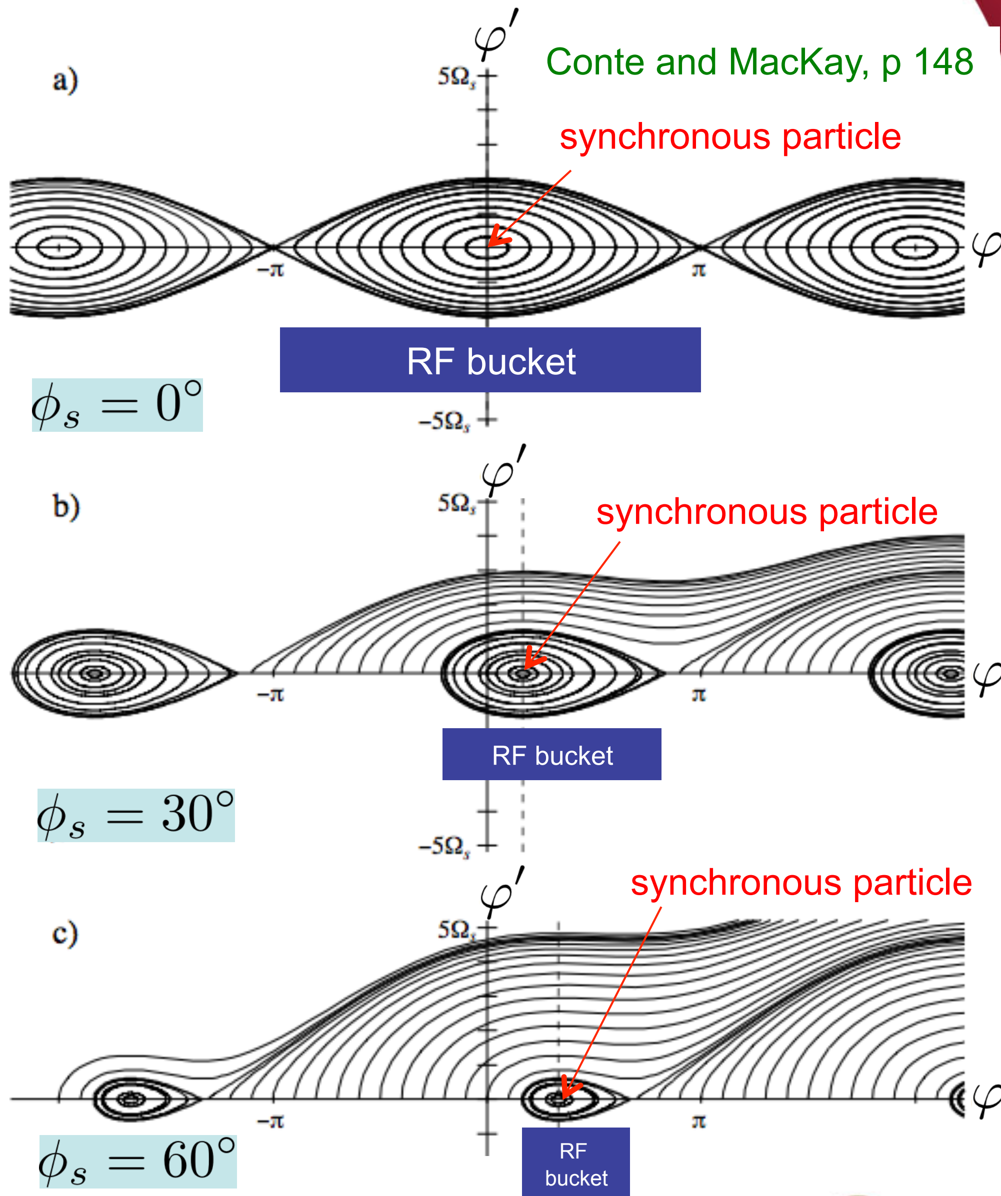
  - Integrate with a constant $\phi_0 \equiv \phi(t=0)$

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\cos\phi_s} + 2(\phi-\phi_0)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}$$

  - This is not closed-form integrable but you can write a computer program to iterate initial conditions to find $(\varphi(t), \varphi'(t))$

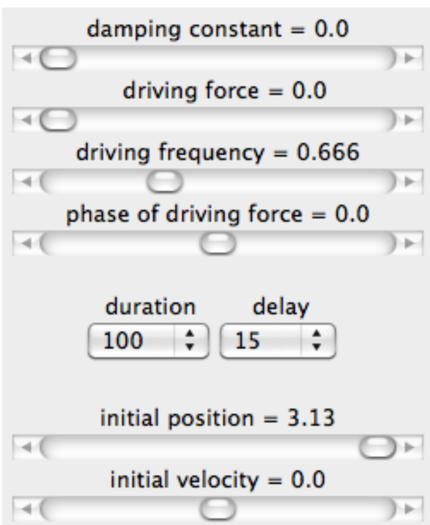
# Synchrotron Oscillation Phase Space

- Start particles at $\varphi \neq 0$ and $\varphi' = 0$
- $\varphi'$ is how phase moves
  - Related to momentum offset $\delta$
- Area of locally stable motion is called **RF bucket**
  - Move like stable biased pendula
- Synchronous particle and nearby particles are stable
  - But some particles "spin" through phases like unstable biased pendula
  - $\Rightarrow$ $\varphi', \delta$ grow, particle is lost at momentum aperture



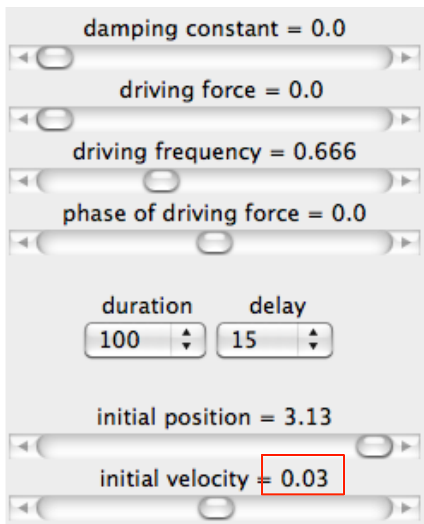
# Pendulum Motion and Nonlinear Dynamics

- Time variations of the RF fields (particularly voltage or phase modulation) can cause very complicated dynamics
  - Driven pendula are classic examples in nonlinear dynamics
  - See http://www.physics.orst.edu/~rubin/nacphy/JAVA_pend
  - Some class demos:



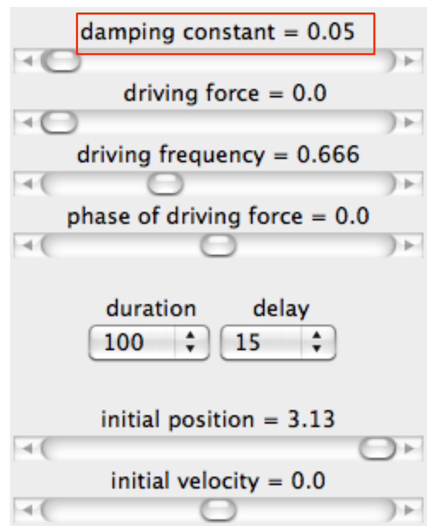

no damping, no drive

pendulum separatrix



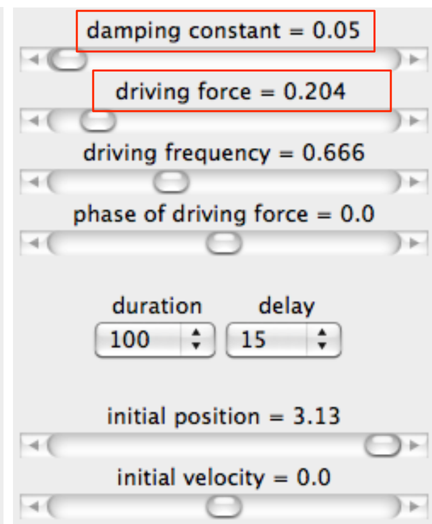

no damping, no drive

precessing pendulum




damping, no drive

damped pendulum




damping, driven

**chaotic** pendulum