Introduction to Accelerator Physics
Old Dominion University

Nonlinear Dynamics
Examples in Accelerator Physics

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http://www.toddsatogata.net/2011-ODU

Tuesday Nov 22-**Tuesday Nov 29** 2011

Class Schedule

- We have only a few more weeks of class left
- Here is a modified version of the syllabus for the remaining weeks including **where we are**

Thursday 17 Nov	Synchrotron Radiation II, Cooling
Tuesday 22 Nov	Nonlinear Dynamics I
Thursday 24 Nov	*No class (Thanksgiving!)*
Tuesday 29 Nov	**Nonlinear Dynamics II**
Thursday 1 Dec	Survey of Accelerator Instrumentation
Tuesday 6 Dec	Oral Presentation Finals (10 min ea)
Thursday 8 Dec	*No class (time to study for exams!)*
Exam week	*Do well on your other exams!*

- Fill in your class feedback survey!

Transverse Motion Review

$$\text{Hill's equation} \quad x'' + K(s)x = 0$$

$$\text{quasi-periodic ansatz solution} \quad x(s) = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0]$$

amplitude function

$$\beta(s) = \beta(s+C) \qquad \gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

derivative

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s) \qquad \phi(s) = \int \frac{ds}{\beta(s)}$$

phase advance

$$\text{Matrix formulation of one-period motion}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = \begin{pmatrix} \cos\Delta\phi_C + \alpha(0)\sin\Delta\phi_C & \beta(0)\sin\Delta\phi_C \\ -\gamma(0)\sin\Delta\phi_C & \cos\Delta\phi_C - \alpha(0)\sin\Delta\phi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

betatron tune Q

$$Q \equiv \frac{\Delta\phi_C}{2\pi} = \frac{1}{2\pi}\int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

$$\text{Tr } M = 2\cos(2\pi Q)$$

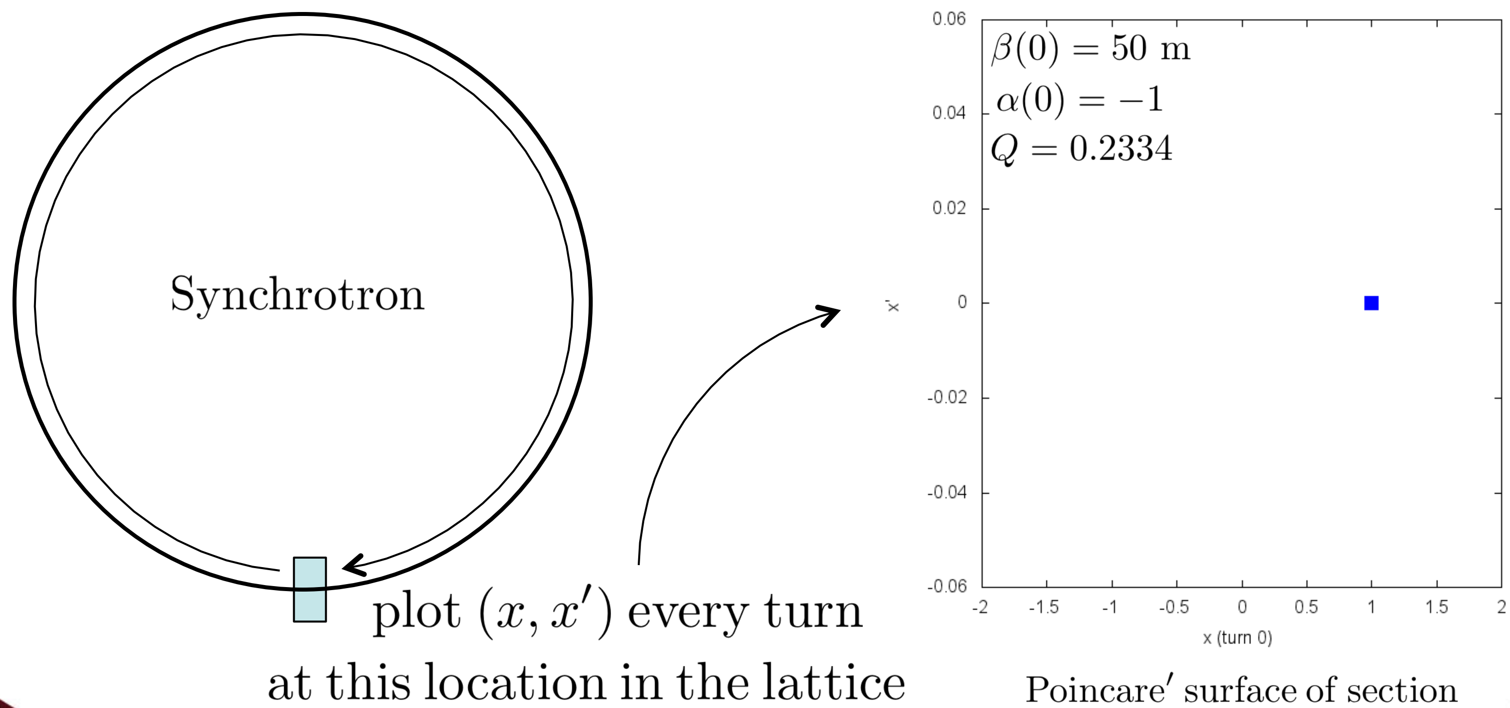
$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}$$

$$J^2 = -I \quad \Rightarrow \quad M = e^{2\pi Q J(s)}$$

Maps

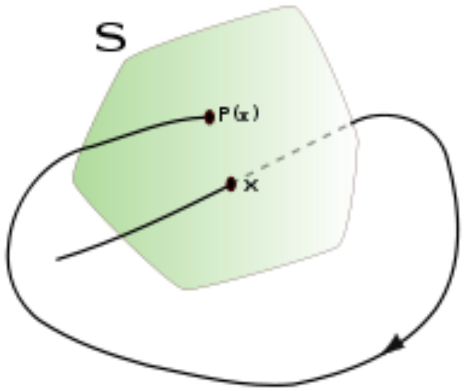
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos(2\pi Q) + \alpha(0)\sin(2\pi Q) & \beta(0)\sin(2\pi Q) \\ -\gamma(0)\sin(2\pi Q) & \cos(2\pi Q) - \alpha(0)\sin(2\pi Q) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

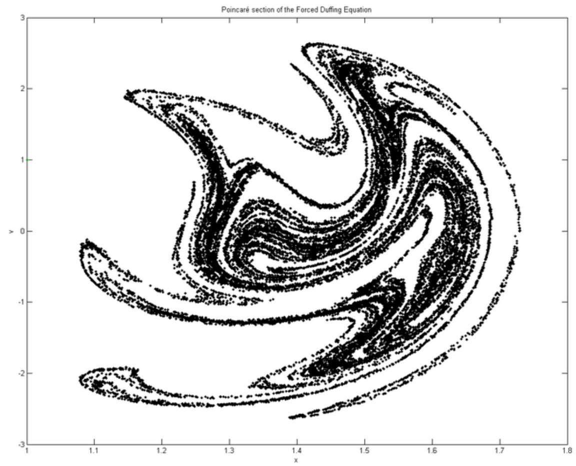
- This “one-turn” matrix M can also be viewed as a **map**
  - Phase space (x,x’) of this 2nd order ordinary diff eq is $\Re \times \Re = \Re^2$
  - M is an isomorphism of this phase space: $M : \Re^2 \rightarrow \Re^2$



Poincare' Surfaces of Section

Henri Poincare$'$





Poincare$'$ section of forced Duffing equation

- Poincare' surfaces are used to visualize complicated "orbits" in phase space
  - Intersections of motion in phase space with a subspace
  - Also called "stroboscopic" surfaces
    - Like taking a periodic measurement of (x,x') and plotting them
    - Here our natural period is one accelerator revolution or turn
  - Transforms a continuous system into a discrete one!
  - Originally used by Poincare' to study celestial dynamic stability

Linear and Nonlinear Maps

- M here is a **discrete linear transformation**
  - Derived from the forces of explicitly linear magnetic fields
  - Derived under conditions where energy is conserved
  - M is also expressible as a product of scalings and a rotation

$$M = V^{-1}RV = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

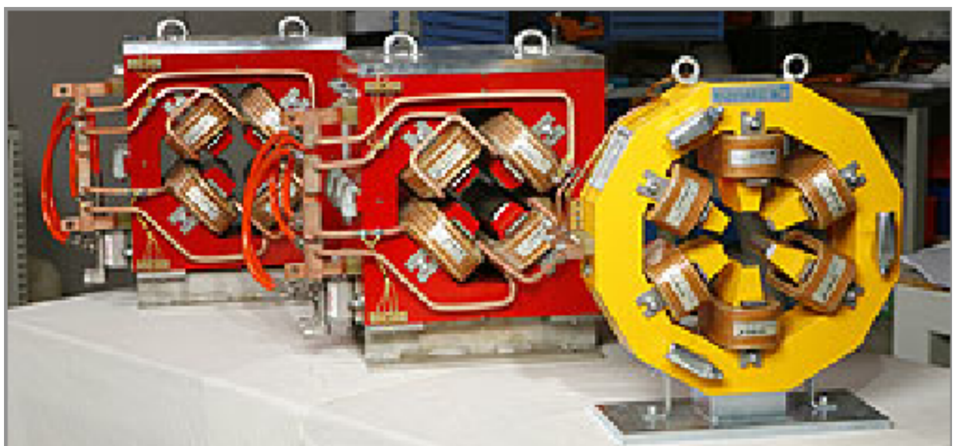
  - Here V is the scaling that transforms the phase space ellipse into a circle (**normalized coordinates)**
- Well-built accelerators are some of the most linear man-made systems in the world
  - Particles circulate up to tens of billions of turns (astronomical!)
  - Negligibly small energy dissipation, nonlinear magnetic fields
    - Perturbations of $o(10^{-6})$ or even smaller

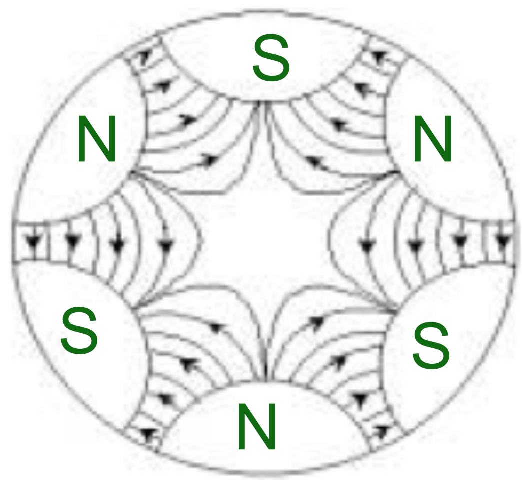
Nonlinear Magnets

- But we can also have **nonlinear magnets** (e.g. sextupoles)
  - These are still conservative but add extra **nonlinear kicks**

$$\text{sextupole kick} \quad \Delta x' = \frac{1}{2}\frac{B''L}{(B\rho)}(x^2+y^2) \equiv b_2(x^2+y^2)$$

  - These nonlinear terms are not easily expressible as matrices
  - Strong sextupoles are necessary to **correct chromaticity**
    - Variation of focusing (tune) over distribution of particle momenta
    - These and other forces make our linear accelerator quite **nonlinear**

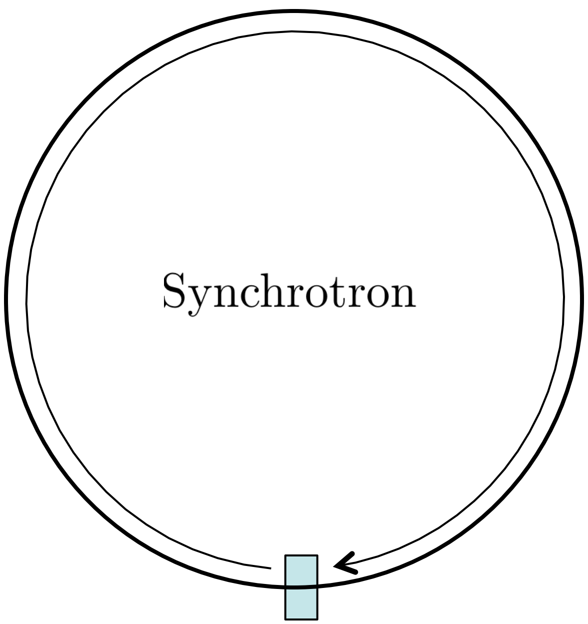




Nonlinear Maps

- Consider a simplified nonlinear accelerator map
  - A linear synchrotron with a single “thin” sextupole

$$M = \begin{pmatrix} \cos(2\pi Q) + \alpha(0)\sin(2\pi Q) & \beta(0)\sin(2\pi Q) \\ -\gamma(0)\sin(2\pi Q) & \cos(2\pi Q) - \alpha(0)\sin(2\pi Q) \end{pmatrix}$$



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$x'_{s_0+C} = x'_{s_0+C} + b_2 x^2_{s_0+C}$$

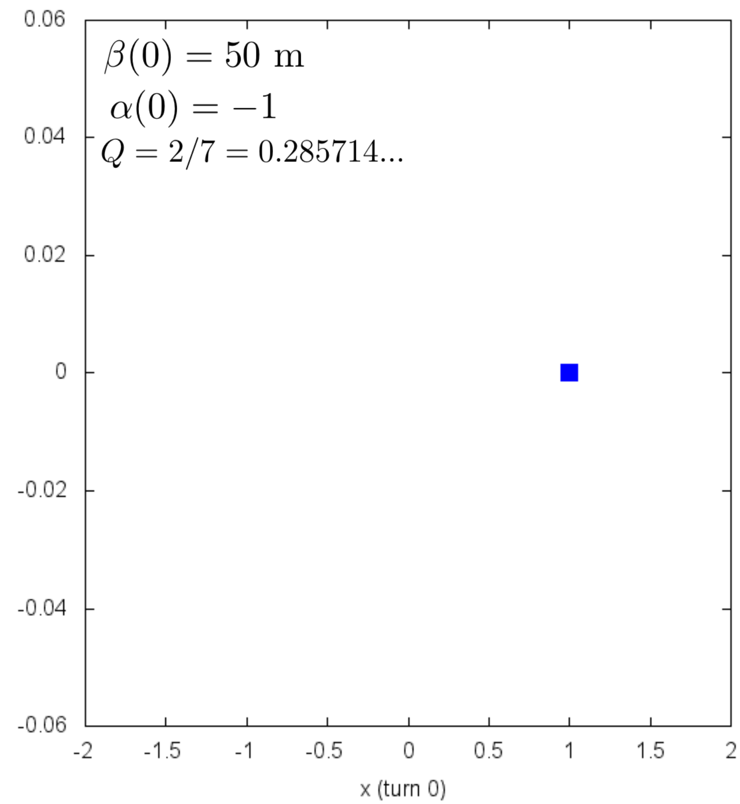
Henon map

- One iteration of this nonlinear map is one turn around the accelerator
  - Thin element: only x’ changes, not x
- Interactive Java applet at http://www.toddsatogata.net/2011-USPAS/Java/

The "Boring" Case

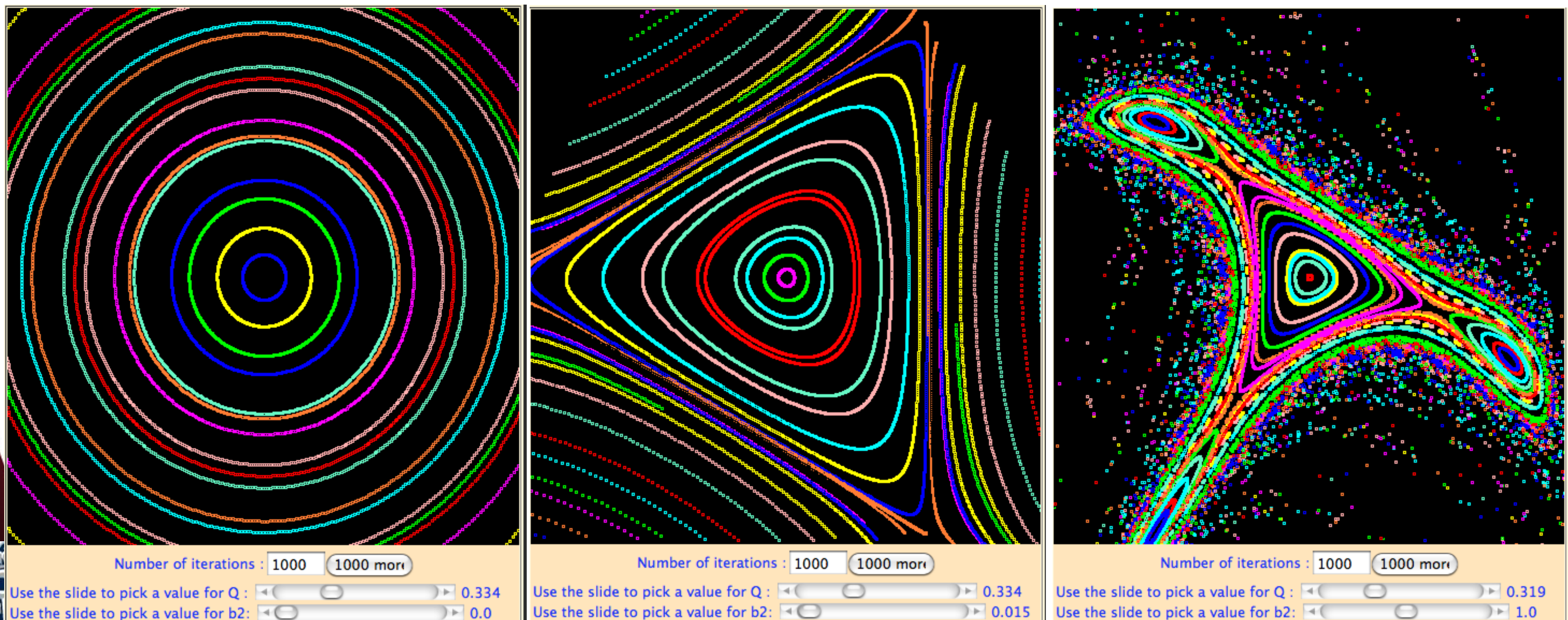
- When $b_2 = 0$ we just have our usual linear motion
  - However, rational tunes where Q=m/n exhibit intriguing behavior
  - The motion repeats after a small number of turns
  - This is known as a **resonance condition**
  - Perturbations of the beam at this resonant frequency can change the character of particle motion quite a lot

    $\Rightarrow$ **Nonlinearities!**



Not so Boring: Example from the Java Applet

- http://www.toddsatogata.net/2011-USPAS/Java/
- The phase space looks very different for Q=0.334 even when turning on one small sextupole…
- And monsters can appear when b2 gets quite large…



What is Happening Here?

- Even the simplest of nonlinear maps exhibits complex behavior
  - Frequency (tune) is no longer independent of amplitude
    - Recall tune also depends on particle momentum (chromaticity)
  - Nonlinear kicks sometimes push “with” the direction of motion and sometimes “against” it

  - (Todd sketches something hastily on the board here ☺)

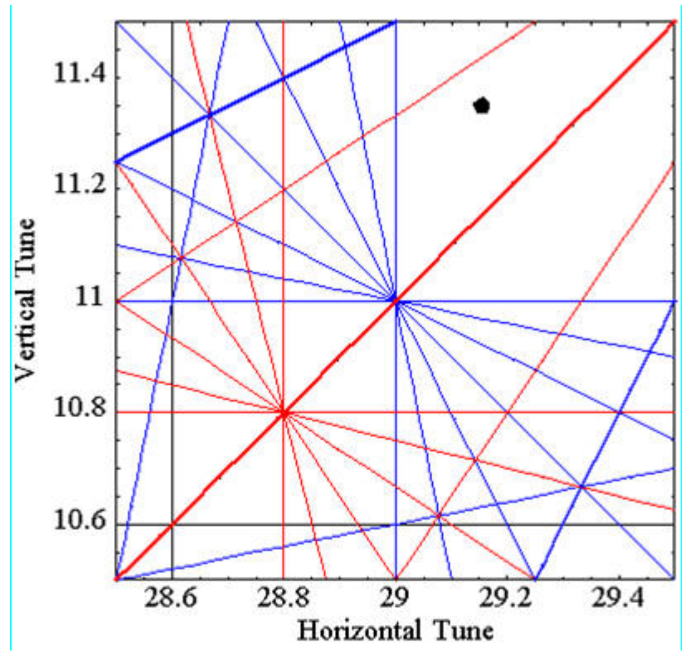
Resonance Conditions Revisited

- In general, we have to worry about all tune conditions where

$$mQ_x \pm nQ_y = l$$

in an accelerator, where m, n, l are integers. Any of these resonances results in repetitive particle motion

In general we really only have to worry when m,n are “small”, up to about 8-10



Discrete Hamiltonians

- Let's recast our original linear map in terms of a discrete Hamiltonian
  - Normalized circular motion begs for use of polar coordinates
  - In dynamical terms these are "action-angle" coordinates (J, ϕ)
    - action J: corresponds to particle dynamical energy

$$J = (x_n^2 + x_n'^2)/2 \qquad \begin{aligned} x_n &= \sqrt{2J}\cos\phi \\ x_n' &= \sqrt{2J}\sin\phi \end{aligned}$$

  - Linear Hamiltonian: $H = 2\pi Q J$
  - Hamilton's equations:

$$\Delta\phi = \frac{\partial H}{\partial J} = 2\pi Q \qquad \text{(Sensible!)}$$

$$\Delta J = -\frac{\partial H}{\partial \phi} = 0$$

  - J (radius, action) does not change
    - Action is an **invariant of the linear equations of motion**

Discrete Hamiltonians II

- Hamiltonians terms correspond to potential/kinetic energy
  - Adding (small) nonlinear potentials for nonlinear magnets
  - Assume one dimension for now (y=0)
  - We can then write down the sextupole potential as

$$V_{\text{sext}} = \frac{b_2}{3} x_n^3 = \frac{b_2}{3} (2J)^{3/2} \cos^3 \phi$$

(riff on trig calculations)

$$\cos^3 \phi = \frac{1}{4} (\cos 3\phi + 3 \cos \phi)$$

$$V_{\text{sext}} = \frac{b_2 \sqrt{2}}{6} J^{3/2} (3 \cos \phi + \cos 3\phi)$$

  - And the new perturbed Hamiltonian becomes

$$H(J, \phi) = 2\pi Q J + \frac{b_2 \sqrt{2}}{6} J^{3/2} (3 \cos \phi + \cos 3\phi)$$

  - The sextupole drives the 3Q=k resonance
  - Next time we'll use this to calculate where those triangular sides are, including **fixed points** and **resonance islands**

Review

- One-turn one-dimensional linear accelerator motion

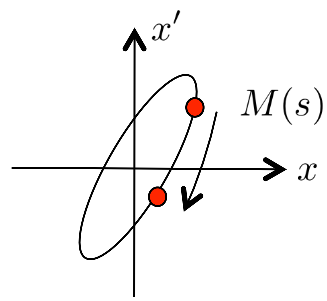
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = M(s) \begin{pmatrix} x \\ x' \end{pmatrix}_s \qquad M(s) = I\cos(2\pi Q) + J(s)\sin(2\pi Q) = e^{2\pi Q J(s)}$$

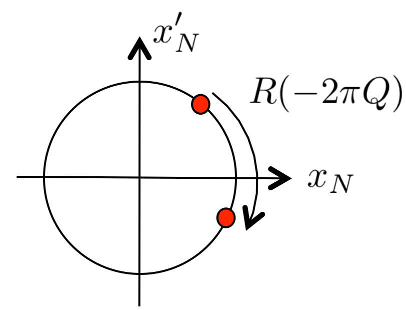
$$J(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \qquad J^2(s) = -I \quad \text{(independent of s)}$$

- Normalized coordinates (motion becomes circular rotations)

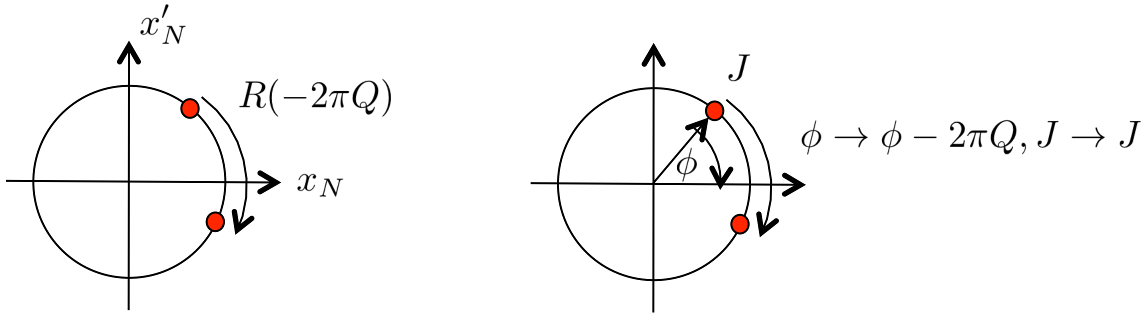
$$M(s) = \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{pmatrix} \begin{pmatrix} \cos(2\pi Q) & -\sin(2\pi Q) \\ \sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} = V(s)^{-1} R(-2\pi Q) V(s)$$

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix}_s = V(s) \begin{pmatrix} x \\ x' \end{pmatrix}_s$$





Action-Angle (Polar) Coordinates

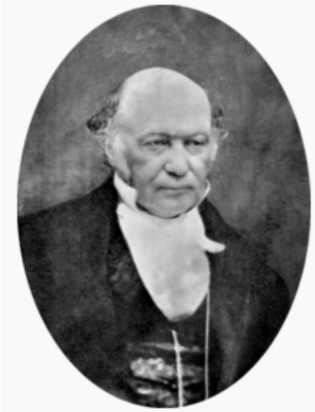


- One more coordinate transformation: action-angle coordinates

$$x_N = \sqrt{2J}\cos\phi \qquad J = \frac{x_N^2 + x_N'^2}{2}$$
$$x'_N = -\sqrt{2J}\sin\phi$$

- These are simply polar coordinates
  - But J acts as an "action" or total energy of the system
  - Note that rotations do not change J: this action is an **invariant**
- Many dynamical systems reduce to action-angle coordinates
  - Simple harmonic oscillators, coupled pendula, crystal lattices…
  - Any dynamical system with a natural scale of periodicity

Discrete Hamiltonian


William Rowan Hamilton (1805–1865)

- Hamiltonian: dynamics from total energy of system
  - Relates differential equations for a coordinate (position) x and a related ("canonical") momentum p_x

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{p_x} = -\frac{\partial H}{\partial x}$$

  - Example: Simple (spring) harmonic oscillator

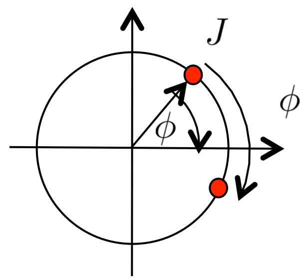
$$H = \text{P.E.} + \text{K.E.} = \frac{1}{2}kx^2 + \frac{p_x^2}{2m}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \dot{p_x} = -\frac{\partial H}{\partial x} = -kx$$

$$m\ddot{x} = \dot{p_x} = -kx \qquad \Rightarrow \qquad m\ddot{x} + kx = 0$$

  - The Hamiltonian energy of the system gives the dynamics (equation of motion)

Discrete Hamiltonian II



- In our linear accelerator system, the Hamiltonian is simple
  - But it's also important to notice that it's discrete!
  - ϕ is a position and J is the corresponding canonical momentum

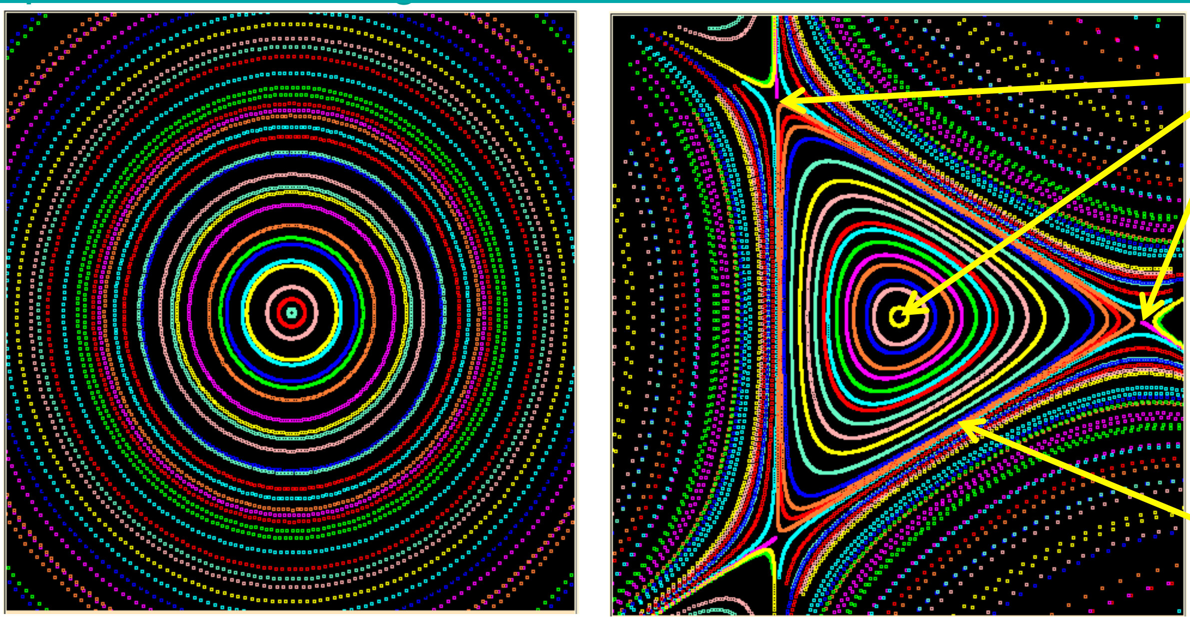
$$H(\phi, J) = 2\pi Q J$$

$$\Delta\phi = \frac{\partial H}{\partial J} = 2\pi Q \qquad \Delta J = -\frac{\partial H}{\partial \phi} = 0 \ \ (\text{conserved})$$

- Hamiltonians are very useful when systems are "nearly" linear
  - We add small nonlinear perturbations to the above regular motion
  - This approach often gives the good insights into the dynamics

The Henon Map and Sextupoles

- Sextupoles are the most common magnet nonlinearities in accelerators
  - Necessary for correcting chromaticity (dependence of focusing strength of quadrupoles with particle momentum)
- Our linear system with one sextupole kick is also a classic dynamics problem: the Henon Map
  - http://www.toddsatogata.net/2011-USPAS/Java/henon.html



Sextupole Hamiltonian

- Let's calculate the locations of the nontrivial fixed points
  - Here note that the **linear tune** Q must be **close** to 1/3 since the tune of particles at the fixed points is **exactly** 1/3
- The sextupole potential is like the magnetic potential
  - Recall that the sextupole field is quadratic
  - The field is the derivative of the potential, so the potential is **cubic**

$$V_{\mathrm{sext}} = \frac{b_2}{3} x_n^3 = \frac{b_2}{3} (2J)^{3/2} \cos^3 \phi$$

$$x_N = \sqrt{2J} \cos \phi$$
$$x'_N = -\sqrt{2J} \sin \phi$$

$$V_{\mathrm{sext}} = \frac{b_2 \sqrt{2}}{6} J^{3/2} (3 \cos \phi + \cos 3\phi)$$

$$H(J, \phi) = 2\pi Q J + \frac{b_2 \sqrt{2}}{6} J^{3/2} (3 \cos \phi + \cos 3\phi) \qquad Q = \frac{1}{3} + \epsilon$$

  - We also assume that b_2 and ϵ are small (perturbation theory)

More Sextupole Hamiltonian

$$H(J,\phi) = 2\pi Q J + \frac{b_2\sqrt{2}}{6} J^{3/2}(3\cos\phi + \cos 3\phi)$$

$$\Delta\phi = \frac{\partial H}{\partial J} = 2\pi Q + \frac{b_2\sqrt{2}}{4} J^{1/2}(3\cos\phi + \cos 3\phi)$$

$$\Delta J = -\frac{\partial H}{\partial \phi} = \frac{b_2\sqrt{2}}{2} J^{3/2}(\sin\phi + \sin 3\phi)$$

- We want to find points where $\Delta\phi = 2\pi$ and $\Delta J = 0$ after three iterations of this map (three turns)
- Every turn, $\Delta\phi \approx 2\pi Q = 2\pi/3 + 2\pi\epsilon$
- Every turn, $\Delta J \approx 0$
- The $\sin\phi$ and $\cos\phi$ "average out" over all three turns
- But the $\sin(3\phi)$ and $\cos(3\phi)$ do not since their arguments advance by $3\Delta\phi \approx 2\pi + 6\pi\epsilon \approx 2\pi$ every turn
-

Three-Turn Map

- We can now approximate the three-turn map as three applications of the (approximately constant) one-turn map

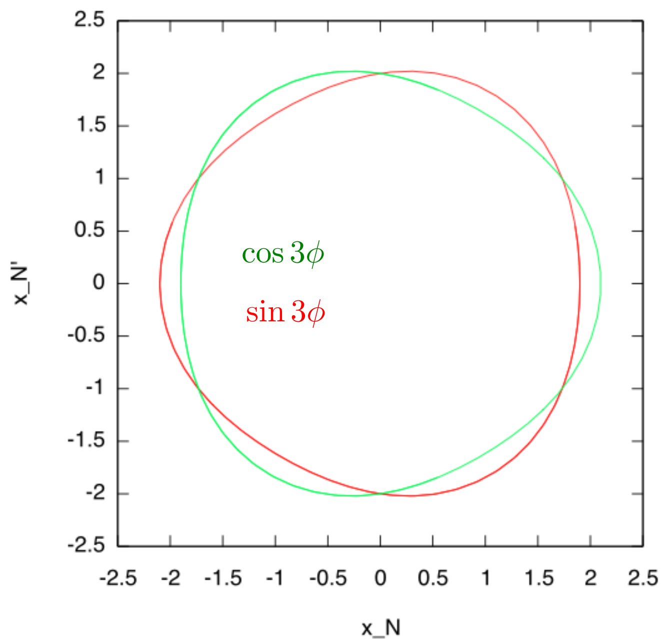
times 3 for
3-turn map!

$$\Delta\phi = \frac{\partial H}{\partial J} = 2\pi Q + \frac{b_2\sqrt{2}}{4} J^{1/2}(3\cos\phi + \cos 3\phi)$$

$$\Delta J = -\frac{\partial H}{\partial \phi} = \frac{b_2\sqrt{2}}{2} J^{3/2}(\sin\phi + \sin 3\phi)$$

$$\Delta\phi(3\,\mathrm{turns}) = 2\pi + 6\pi\epsilon + \frac{3b_2\sqrt{2}}{4} J^{1/2}\cos 3\phi$$

$$\Delta J(3\,\mathrm{turns}) = \frac{3b_2\sqrt{2}}{2} J^{3/2}\sin 3\phi$$



Fixed Points

$$\Delta\phi(3\,\text{turns}) = 2\pi + 6\pi\epsilon + \frac{3b_2\sqrt{2}}{4} J^{1/2}\cos 3\phi$$

$$\Delta J(3\,\text{turns}) = \frac{3b_2\sqrt{2}}{2} J^{3/2}\sin 3\phi$$

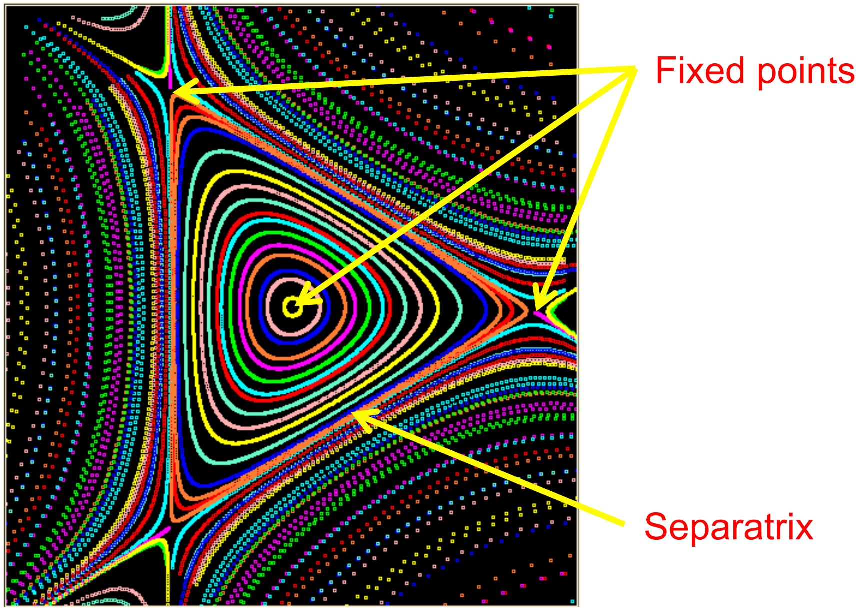
- The fixed points are located where $\Delta\phi(3\,\text{turns}) = 2\pi$ and $\Delta J(3\,\text{turns}) = 0$
- The fixed point phases are found where $\Delta J(3\,\text{turns}) = 0$

$$\sin 3\phi_{\text{FP}} = 0 \qquad \phi_{\text{FP}} = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \;\; (\text{six!})$$

- The fixed point actions are found where $\Delta\phi(3\,\text{turns}) = 2\pi$

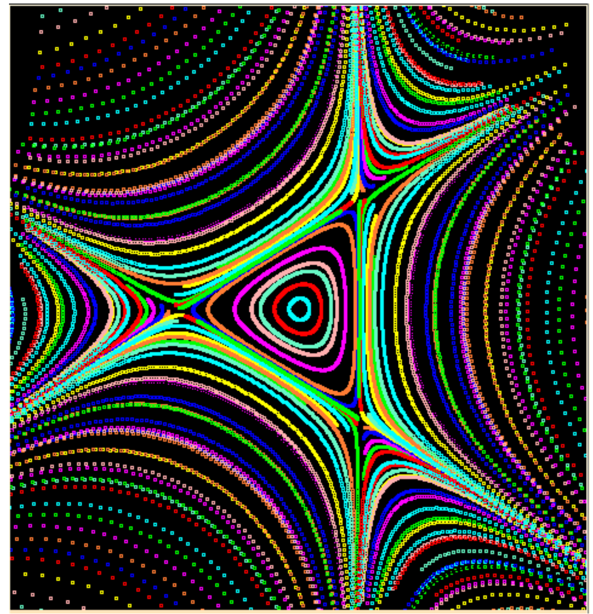
$$6\pi\epsilon + \frac{3b_2\sqrt{2}}{4} J_{\text{FP}}^{1/2} = 0 \qquad J_{\text{FP}} = \left(\frac{-8\pi\epsilon}{b_2\sqrt{2}}\right)^2$$

Wow, it seems to work!



Q=0.332
b2=0.04

$$\epsilon = -0.00133$$



Q=0.334
b2=0.04

$$\epsilon = 0.00667$$

$$\phi_{\mathrm{FP}} = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$J_{\mathrm{FP}} = \left(\frac{-8\pi\epsilon}{b_2\sqrt{2}} \right)^2$$

J_{FP} scales with ϵ^2

$x_{N,\mathrm{FP}}$ scales with ϵ