Introduction To Accelerator Physics: Lecture Notes 4 Addendum

Tuesday September 13, 2011

1 Promised Derivation #1

I promised that I would clarify the "magic step" in moving to the rotating frame of reference in

$$F = \frac{d}{dt}(\gamma m \dot{x}) \simeq \gamma m \left(\ddot{x} - \frac{v^2}{R} \right)$$

when deriving the equations of motion for a particle moving near a design circular orbit in a transverse magnetic field.

1.1 Invoking first year physics approach

Consider a particle moving along the design trajectory. It has $x = 0$ *by definition* since x is a (small) horizontal deviation from this design trajectory. The design particle experiences a magnetic Lorentz force:

$$F = -qvB$$

which is constant and always perpendicular to the direction of motion. This force produces circular motion of constant radius R; the sign has been chosen to be consistent with positive being "out" of the circle. The design particle is always undergoing a centripetal acceleration and force of

$$a = -\frac{v^2}{R} \qquad \Rightarrow \qquad F = -\frac{mv^2}{R}$$

The centripetal force is entirely supplied by the Lorentz magnetic force, so we can equate these two forces, giving the now-familiar

$$qB = \frac{mv}{R} \qquad \Rightarrow \qquad BR = \frac{p}{q} \tag{1.1}$$

In the "magic step" case, we have also expanded the fields around the design trajectory particle, with $B(x = 0) = B_0$. We therefore need to include the centripetal acceleration term in the force equation to make the force equation hold even when $x = 0$, which implies that the $x = 0$ case must reduce to Eqn. (1.1).

1.2 Constructive approach

This derivation follows that described in the Wikipedia article on "Rotating reference frame", `http://en.wikipedia.org/wiki/Rotating_reference_frame`. Consider a rotating reference frame with unit basis vectors $(\hat{i}, \hat{j}, \hat{k})$ in the rotating frame that correspond to our (also rotating frame) coordinates $(\hat{x}, \hat{s}, \hat{y})$. The rotating basis can be expressed as

$$\begin{aligned} \vec{i}(t) &= (\cos \Omega t, \sin \Omega t) \\ \vec{j}(t) &= (-\sin \Omega t, \cos \Omega t) \end{aligned}$$

where Ω is the rotation frequency and $\vec{\Omega}$ points in the right hand direction of the rotation or $\hat{\Omega} = \hat{s} \times \hat{x} = \hat{j} \times \vec{i}$. Time derivatives of these basis vectors are

$$\frac{d}{dt}\hat{u} = \vec{\Omega} \times \hat{u}$$

where $\hat{u}$ is $\hat{i}$ or $\hat{j}$.

The time derivative of a vector function $\vec{f}$ has contributions that are from the inertial reference frame, *and* from the time-dependence of the coordinates in the rotating frame:

$$\frac{d}{dt}\vec{f} = \left(\frac{d\vec{f}}{dt}\right)_r + \vec{\Omega} \times \vec{f}$$

where the r subscript indicates the time derivative in the rotating reference frame. Extending this to acceleration, a second time derivative of a vector quantity, we have

$$F/m = \vec{a}_r = (\vec{a}_i) - (2\vec{\Omega} \times \vec{v}_r) - (\vec{\Omega} \times (\vec{\Omega} \times \vec{r})) - \left(\frac{d\vec{\Omega}}{dt} \times \vec{r}\right)$$

This equation relates the observed accelerations in the inertial and rotating reference frames ($\vec{a}_i$ and $\vec{a}_r$), and includes three additional terms. In order from left to right, they are **the Coriolis Force**, **the Centrifugal Force**, and **the Euler Force**.

The Euler force is zero because the angular frequency is $\vec{\Omega}$ is constant. This leaves the Coriolis and Centrifugal forces. We have assumed the paraxial approximation, so particle velocity in the rotating frame of reference (i.e. relative to the design particle, $\vec{v}_r$) is much smaller than the overall velocity of rotation $\vec{\Omega} \times \vec{r}$. So we assume that the centrifugal force dominates. Note that this is a "fictitious" force — this is needed to balance the kinematics but there is no actual force accelerating the particle outwards. It's the **centripetal force** (towards the center of the circle) that is real, and that keeps the particle moving in a rotating reference frame.

Substituting $\Omega = v/r$ and noting that all quantities are perpendicular gives

$$F_{\text{centrifugal}} = -mv^2/r$$