# Thu June 16 Lecture Notes: Lattice Exercises I

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Most of these notes follow the treatment in the class text, Conte and MacKay, Chapter 6 on Lattice Exercises. The portions on orbit correction follow Edwards and Syphers, section 3.4. By the end of these lectures, you should understand a bit about orbit correction in a circular accelerator, how to calculate optics parameters for a FODO lattice (and other combinations of drifts, dipoles, and quads), FODO lattice stability, and FODO lattice dispersion.

## 1 Review

Rcall that the parameterization of the **one-dimensional periodic transfer matrix** is given by Conte/MacKay (5.21-22):

$$M_{\rm periodic} = I\cos\mu + J\sin\mu = e^{\mu J} \tag{1.1}$$

where

$$J \equiv \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \qquad J^2 = -I \tag{1.2}$$

$\alpha(s) \equiv -\beta'(s)/2$ and $\gamma(s) \equiv (1+\alpha(s)^2)/\beta(s)$. Here I've included the explicit $s$-dependence. $\beta(s)$ is the square of the envelope function $w(s)$, or amplitude scaling of particle motion, Conte/Mackay (5.69):

$$x(s) = \sqrt{\mathcal{W}\beta(s)}\cos[\psi(s)+\psi_0] \tag{1.3}$$

where

$$\psi'(s) = 1/\beta(s) \qquad \psi(s) = \int ds/\beta(s) \qquad \beta(s) = \beta(s+C) \tag{1.4}$$

The **beta function** has the periodicity of the lattice, as Waldo showed, but the **phase advance** $\psi(s)$ does NOT have this periodicity — integrating it around the accelerator gives us an extra phase advance of $\mu \equiv 2\pi Q$ where $Q$ is called the **betatron tune**. (See Conte/Mackay Eqns. (5.54–55).) There are beta functions, phase advances, and betatron tunes for each plane $(H, V)$. Note that some references use $\nu$ for tunes instead of $Q$ and that since the transfer matrix around one turn of the accelerator is periodic, we have

$$M_{\rm one\,turn} = I\cos 2\pi Q + J\sin 2\pi Q = e^{2\pi Q J} \tag{1.5}$$

In the weak focusing betatron case where the field index is $0 \le n \le 1$, $Q_x = \sqrt{1-n}$ and $Q_y = \sqrt{n}$, the tunes must also be in the range $0 \le Q_{x,y} \le 1$. Indeed, this is practically what is defined as **weak focusing**. In contrast, accelerators with alternating quadrupole focusing that lead to $Q > 1$ are termed **strong focusing**.

To propagate position and angle from one location $s = s_0$ to another $s$ location, both with known Twiss parameters, we use MacKay and Conte (5.52):

$$M(s|s_0) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}[\cos\mu(s) + \alpha_0 \sin\mu(s)] & \sqrt{\beta_0\beta(s)}\sin\mu(s) \\ -\frac{[\alpha(s)-\alpha_0]\cos\mu(s)+[1+\alpha_0\alpha(s)]\sin\mu(s)}{\sqrt{\beta_0\beta(s)}} & \sqrt{\frac{\beta_0}{\beta(s)}}[\cos\mu(s) - \alpha(s)\sin\mu(s)] \end{pmatrix} \tag{1.6}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{pmatrix}^{-1} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix} \tag{1.7}$$

$$= T(s)^{-1}R(\mu)T(0) \tag{1.8}$$

Even though (1.6) looks horrible, it can be decomposed into local transformations $T(s)$ into **normalized phase space**, and a separate pure rotation matrix $R(\mu)$ that rotates through the phase advance. You'll see this in problem 5-4 of tonight's homework.

Figure 1 shows particle motion through a FODO lattice, a series of alternating focusing and defocusing quadrupoles. The dark lines that are plotted are $\pm\sqrt{\beta(s)}$, which is directly proportional to the maximum amplitude of the particle oscillation. You can see why the beta function is also called the envelope function! Waldo's showed something like this and you've also seen it in the labs. Let's analyze the FODO lattice in more detail and calculate some things about it after a brief digression into a real application of what you've learned so far.

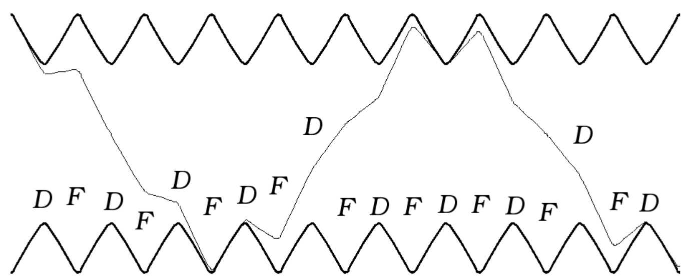


Figure 1: This figure shows a plot of betatron oscillations, equation (1.3) and Conte/MacKay Fig. 5.3, through a regular FODO lattice. Waldo showed this picture earlier. If you follow the trajectory, you can see that the kicks alternate between focusing and defocusing, and the betatron oscillation is slow compared to the periodicity of the FODO lattice. This example has about 6.5 FODO cells per betatron oscillation, so this FODO lattice has a phase advance per cell of about $2\pi/6.5 = 55$ degrees. What is the relationship between the focal length $f$ and the FODO cell length $L$ here?

## 2 Steering Errors and Orbit Correction

Suppose we have a single steering error in an otherwise perfect circular accelerator that gives a kick

$$\Delta x' = \frac{\Delta B\, l}{(B\rho)} \tag{2.1}$$

loated without loss of generality at $s = 0$. What is the new closed (1-turn periodic) orbit? Say that the orbit immediately downstream of this kick is $(x_0, x_0')$. If we propagate this orbit through one turn with a one-turn matrix $M_{\text{one turn}}$ and then proceed through the kick, we should get the same conditions. This the definition of closed orbit, the fixed point of the one-turn map. So

$$M_{\text{one turn}} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} . \tag{2.2}$$

Solving for $(x_0, x_0')$ gives

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M_{\rm one\,turn})^{-1} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} . \tag{2.3}$$

We can now use $M_{\rm one\,turn} = e^{2\pi QJ}$ to perform a little trick:

$$\begin{aligned} (I - M_{\rm one\,turn})^{-1} &= \left(I - e^{2\pi QJ}\right)^{-1} = \left[e^{\pi QJ}\left(e^{-\pi QJ} - e^{\pi QJ}\right)\right]^{-1} && (2.4) \\ &= -(2J\sin\pi Q)^{-1}\left(e^{\pi QJ}\right)^{-1} && (2.5) \\ &= \frac{1}{2\sin\pi Q} J e^{-\pi QJ} && (2.6) \\ &= \frac{1}{2\sin\pi Q}(J\cos\pi Q + I\sin\pi Q) && (2.7) \end{aligned}$$

The closed orbit solution is then

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M_{\rm one\,turn})^{-1} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \frac{\Delta x'}{2\sin\pi Q} \begin{pmatrix} \beta_0 \cos\pi Q \\ \sin\pi Q - \alpha_0\cos\pi Q \end{pmatrix} \tag{2.8}$$

where the twiss parameters $(\beta_0, \alpha_0)$ are at the position of the kick. We can then propagate this to any location $s$ in the ring with the general $M(s|s_0)$ matrix (1.6) to find

$$\begin{aligned} \Delta x(s) &= \frac{\Delta x' \beta^{1/2}(s)\beta_0^{1/2}}{2\sin\pi Q}\cos[(\psi(s) - \psi_0) - \pi Q] && (2.9) \\ &= \frac{\partial x(s)}{\partial x'(0)}\Delta x'(s=0) && (2.10) \end{aligned}$$

This result really represents a standing wave solution through the accelerator. Note some interesting points about (2.9):

- The sensitivity to the kick is proportional to the square root of the product of the beta functions at the kick location and position observation. Dipole errors have the worst effect at areas of high $\beta$.
- The orbit change is inversely proportional to $\sin\pi Q$. For integer $Q$, dipole error effects diverge. This is an example of a **resonance**. We will later see that quadrupole error effects are inversely proportional to $\sin 2\pi Q$ and so on to higher orders.
- Even though the kick is only an angle kick, there is a closed orbit change $\Delta x$ at the kick location.

An example of a comparison of measurements and a calculation from (2.9) for a single dipole corrector change in RHIC is shown in Fig. 2.

(2.9) can be used as a fundamental equation of orbit correction since it relates a position change anywhere in the ring $x(s)$ to a change in a dipole corrector change $\Delta x'$ elsewhere in the ring. Because it is linear we can then write

$$\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \cdots \\ \Delta x_n \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_1}{\partial x_2'} & \cdots & \frac{\partial x_1}{\partial x_m'} \\ \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial x_2'} & \cdots & \frac{\partial x_2}{\partial x_m'} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial x_1'} & \frac{\partial x_n}{\partial x_n'} & \cdots & \frac{\partial x_n}{\partial x_m'} \end{pmatrix} \begin{pmatrix} \Delta x_1' \\ \Delta x_2' \\ \cdots \\ \Delta x_m' \end{pmatrix} \tag{2.11}$$

This (non-square!) matrix can be inverted optimally using, for example, singular value decomposition to find a solution to the changes in all $m$ dipole correctors $(\Delta x_1', \Delta x_2', \cdots, \Delta x_m')$ to effect an orbit change in all $n$ beam position monitor locations $(\Delta x_1, \Delta x_2, \cdots, \Delta x_n)$.

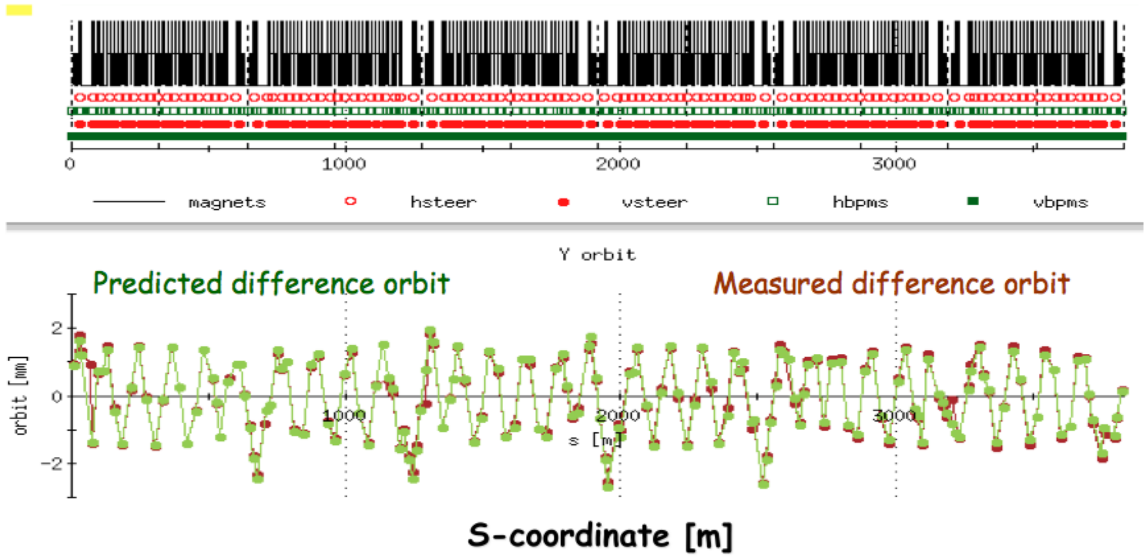


Figure 2: Lattice diagram and (vertical) "difference" orbit from RHIC, comparing measurement and modeled difference orbit from (2.9).

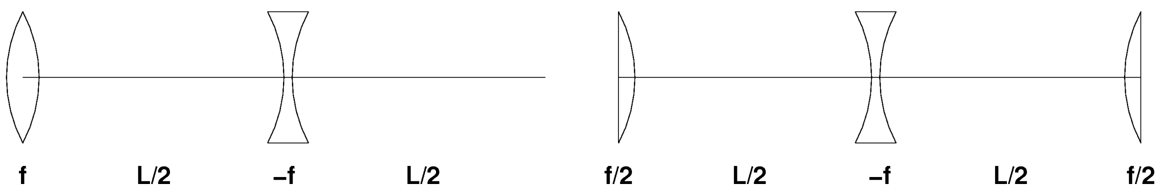


Figure 3: Two ways of calculating transfer for a regular FODO cell of length $L$ with two thin quadrupoles. The one on the left is not symmetric and gives optics at one side of the focusing quad. The one on the right is symmetric and gives optics in the center of the focusing quadrupole, where we expect to locate $\beta_{\rm max}$ even in the case of thick quadrupoles.

## 3 FODO Lattice

One of the most common lattice layouts in modern separated-function accelerators is the FODO lattice, a lattice with alternating focusing and defocusing quadrupoles separated by drift spaces or dipoles. We have started to see how this combination gives you a net focusing. Using the thin lens approximation for equal-strength quadrupoles (i.e. assuming their focal lengths are much larger than their lengths), separating them by $L/2$ length drift spaces so the FODO "cell length" is $\approx L$ , we find:

$$M_{\rm FODO} = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \tag{3.1}$$

$$= \begin{pmatrix} 1 - \frac{L}{2f} - \frac{L^2}{4f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 + \frac{L}{2f} \end{pmatrix} \tag{3.2}$$

Let's assume that we have a string of FODO cells. The system is periodic through every FODO cell, so the transfer matrix of each FODO cell must be expressible in the form of (1.1):

$$\begin{pmatrix} 1 - \frac{L}{2f} - \frac{L^2}{4f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 + \frac{L}{2f} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{3.3}$$

If these are equal, their traces must be equal:

$$2 - \frac{L^2}{4f^2} = 2\cos\mu = 2 - 4\sin^2\frac{\mu}{2} \tag{3.4}$$

$$\Rightarrow \quad \sin\frac{\mu}{2} = \pm\frac{L}{4f} \tag{3.5}$$

using the trig identity $\cos(2\mu) = \cos^2\mu - \sin^2\mu = 1 - 2\sin^2\mu$. (3.5) relates the **FODO cell phase advance** $\mu$ to the parameters $L$ and $f$ that we used to construct it.

For a FODO lattice, the maximum of the beta function $\beta = \beta_{\rm max}$ is located at the center of the focusing quadrupole. The transfer matrix from the center of one focusing quadrupole (with focal length $f$) to another in this lattice is found by splitting the focusing quadrupole in half (focal length $2f$) as shown in Fig. 3. This even works if we use the thick lens quadrupole transfer matrix.

$$\begin{aligned} M_{\rm f\ to\ f} &= \begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/2f & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) \\ \frac{L}{4f^2}\left(\frac{L}{4f} - 1\right) & 1 - \frac{L^2}{8f^2} \end{pmatrix} \end{aligned} \tag{3.6}$$

At the maximum of the beta function in the (split) focusing quadrupole, $\alpha(s) = -\beta'(s)/2 = 0$, so this matrix can also be written as

$$M_{\rm f\ to\ f} = \begin{pmatrix} \cos\mu & \beta_{\rm max}\sin\mu \\ -\sin\mu/\beta_{\rm max} & \cos\mu \end{pmatrix} \tag{3.7}$$

Equating first diagonal, then upper-right off-diagonal terms gives

$$\begin{aligned} \cos\mu = 1 - L^2/8f^2 \quad &\Rightarrow \quad \sin(\mu/2) = \sqrt{\frac{1-\cos\mu}{2}} = \frac{L}{4f} \\ \beta_{\rm max} &= L\left(\frac{1+\sin(\mu/2)}{\sin\mu}\right) \end{aligned} \tag{3.8}$$

The minimum of the beta function, $\beta_{\rm min}$, occurs at the center of the defocusing quadrupole, so it may be found by following the above analysis replacing $f$ with $-f$, producing:

$$\beta_{\rm min} = L\left(\frac{1-\sin(\mu/2)}{\sin\mu}\right) \tag{3.9}$$

For example, the phase advance per FODO cell at RHIC is about 78 degrees and the length of the RHIC FODO cell is about 30 m, so we can calculate that $\beta_{\rm max}$=50 m and $\beta_{\rm min}$=10.7 m. Figure 4 shows one sixth of the RHIC injection optics lattice, and you can readily see that the formulas for $\beta_{\rm min,max}$ work very well.

Note that we can calculate the Twiss parameters at any point in the FODO cell by changing the cut point for our lattice transport. Alternatively you can use the result of tonight's homework (problem 5-5) to propagate these Twiss functions through the FODO cell. Propagating through an entire FODO cell should give an identity matrix regardless of the cut point because the optics are periodic. We can test this for our example above using $M_{\rm f\ to\ f}$ from (3.6). This is thankfully NOT part of your homework. But it works:

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} \tag{3.10}$$

$$M_{\rm f\ to\ f} = \begin{pmatrix} 1 - L^2/8f^2 & L(1 + L/4f) \\ L^2/16f^3 - L/4f^2 & 1 - L^2/8f^2 \end{pmatrix} \tag{3.11}$$

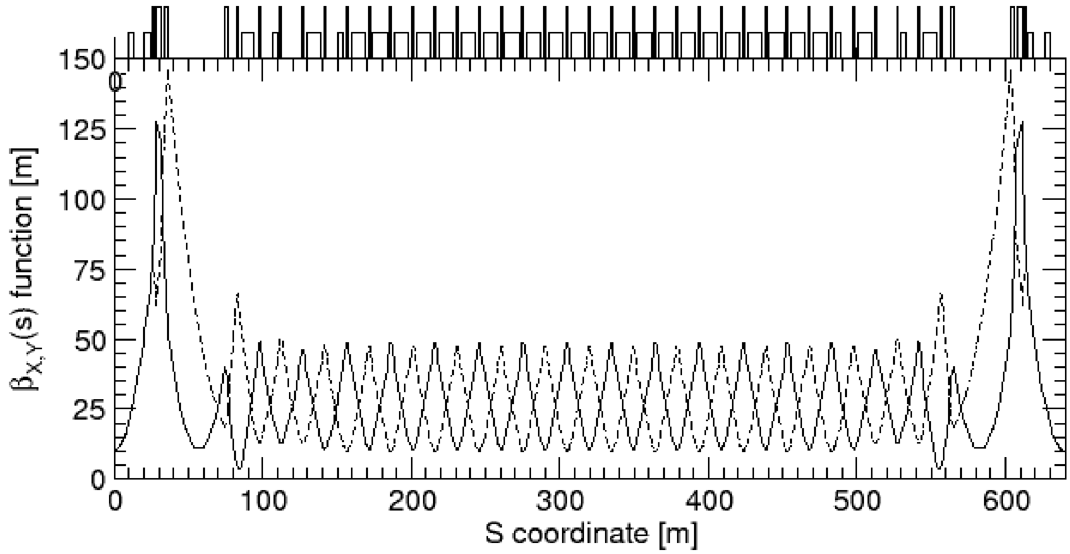


Figure 4: The RHIC injection lattice and beta functions. This shows one sixth of the RHIC lattice as Waldo has shown earlier; apply mirror antisymmetry and repeat the lattice three times to get the full lattice. Solid lines are $\beta_x(s)$; dotted lines are $\beta_y(s)$. The magnet lattice is shown at the top of the graph; short blocks are bending dipoles; taller blocks are quadrupoles. Low-beta insertions are clearly visible. Is the quadrupole at s=300m horizontally focusing or defocusing? How many FODO cells are in the regular section? RHIC has a total horizontal tune of about 28.23; does this number of FODO cells make sense?

## 4 FODO Lattice Stability

Sometimes we do not want horizontal and vertical quadrupoles in a FODO lattice to have the same strengths, such as when we are matching the beam size into different optics. If we re-parameterize the quadrupole strengths as

$$-F \equiv -\frac{L}{2f_F} \qquad D \equiv \frac{L}{2f_D} \tag{4.1}$$

Calculating the trace of the FODO lattice matrix now gives something more general than (3.5):

$$\cos\mu = 1 + D - F - \frac{FD}{2} \qquad\qquad \sin^2\frac{\mu}{2} = \frac{FD}{4} + \frac{F-D}{2} \tag{4.2}$$

Recall from Waldo's lecture that the stability requirement is that the phase advance $\mu$ is real (since the eigenvalues of the transport matrix are $e^{\pm i\mu}$), or

$$-1 < \cos\mu < 1 \quad \text{or} \quad 0 < \sin^2\frac{\mu}{2} < 1 \tag{4.3}$$

What are the boundaries of real $\mu$ in terms of $F$ and $D$? One is where $\sin^2\frac{\mu}{2} = 0$, and we have

$$F = \frac{2D}{2+D} \tag{4.4}$$

The second case, $\sin^2\frac{\mu}{2} = 1$, requires that $F < 2$ for stability for positive $D$. The reverse conditions where we reverse the roles of F and D also provide boundaries, and the final stable area in the $(F, D)$ parameter space is shown in Fig. 5.

## 5 FODO Cell Dispersion

A FODO cell is a linear system in more than one way — the design orbit is straight! We want to steer the beam and eventually wrap the beamline around back on itself so we can

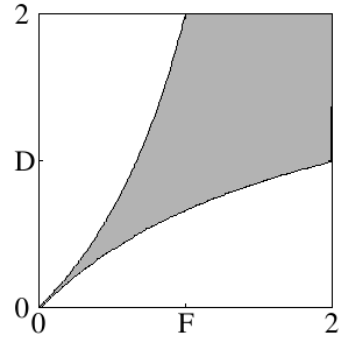


Figure 5: Stability or "necktie" diagram for an alternate focusing lattice, including the FODO lattice where the dimensionless parameterization is $F \equiv \frac{L}{2f_F}$ and $D \equiv \frac{L}{2f_D}$. The shaded area is the region of stability.

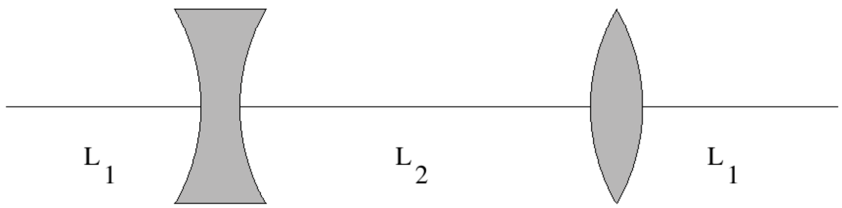


Figure 6: The $\frac{\pi}{2}$ lattice insertion.

make a **storage ring**, so we also must consider cases where the space between the FODO quadrupoles is not a simple drift, but filled with the bending field of a dipole. To make the math simpler (and to save my sanity), we'll assume that the length of the FODO cell is $L = \rho\theta_c$ and there are no drifts. The total bending angle of the cell is $\theta_c$. We will use the symmetric layout of the FODO cell in Fig. 2 to make some math simpler for dispersion suppressors, which we'll discuss later.

The transfer matrix for the single FODO cell with the dispersive component included is

$$M_{\text{f to f}} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_c}{8} \\ 0 & 1 & \frac{\theta_c}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_c}{8} \\ 0 & 1 & \frac{\theta_c}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

$$= \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_c \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_c \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2}$$

where we have kept only first order in $\theta_c$ since usually $\theta_c << 1$.

Waldo talked a bit about the dispersion function $\eta(s)$ earlier. The periodic dispersion function of the FODO cell can be obtained from

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} \tag{5.3}$$

The bottom row of $M$ in (5.2) is trivial, so we can also write the dispersion as a nonhomogeneous equation:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = M' \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix} \tag{5.4}$$

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = (I - M') \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix} \tag{5.5}$$

where $M'$ is the upper left 2x2 block of (5.2). This is calculable and gives the $\eta_{\max}$, just like we had $\beta_{\max}$:

$$\eta_{\max} = \frac{L\theta_c}{4}\left(\frac{1+\frac{L}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right) \tag{5.6}$$

$$\eta' = 0 \tag{5.7}$$

This is rather unsurprisingly a lot like the formula for $\beta_{\max}$, except now the maximum dispersion also scales linearly with the total length of the FODO cell, $L\theta_c$. The minimum of the dispersion is also easy to find:

$$\eta_{\min} = \frac{L\theta_c}{4}\left(\frac{1-\frac{L}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right) \tag{5.8}$$

$$\eta' = 0 \tag{5.9}$$

## 6 A Simple Lattice Insertion

In any FODO cell one or both quadrupole magnets can be moved away leaving room for actual straight sections, slightly modifying the lattice dispersion without altering the optics of the lattice, as thoroughly discussed in the previous sections. Obviously, such free spaces may not be sufficient to host long items like septum magnets for injection and extraction, chains of rf cavities, etc. Longer straight sections have to be provided, leaving the optics of the machine unaffected. How do we make the longest straight section possible?

An elegant solution is the so-called $\frac{\pi}{2}$ insertion. This insertion has a long straight section of length $L_2$, encompassed by two quadrupoles, one focusing and one defocusing, and further symmetric straight sections of length $L_1$. This is schematically shown in figure 6.

Assuming the quadrupoles have equal focal length $f$, we calculate the transfer matrix again:

$$M = \begin{pmatrix} 1-\frac{L_1L_2}{f^2}+\frac{L_2}{f} & 2L_1+L_2-\frac{L_1^2L_2}{f^2} \\ -\frac{L_2}{f^2} & 1-\frac{L_1L_2}{f^2}-\frac{L_2}{f} \end{pmatrix} \tag{6.1}$$

We want this insertion to match into the periodicity of the lattice, so it must match to our *general* periodic lattice description of (1.1):

$$M = I\cos\mu + J\sin\mu = \begin{pmatrix} \cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu \end{pmatrix} \tag{6.2}$$

Taking the difference of the diagonal terms gives

$$L_2 = \alpha f \sin\mu \tag{6.3}$$

and $L_2$ is maximized when $\mu = \frac{\pi}{2}$. (It's now obvious why this is called a $\frac{\pi}{2}$ insertion.) Then $\cos\mu = 0$ and we can get other parameters in terms of the lengths $L_1$ and $L_2$:

$$f^2 = L_1L_2 \qquad \alpha = \frac{L_2}{f} \qquad \gamma = \frac{L_2}{f^2} \qquad \beta = L_1 + L_2 \tag{6.4}$$

and we can design our insertion to match into our FODO lattice.

One surprising observation is that the transfer matrix now reduces to

$$M = J \tag{6.5}$$

and recall that $M^2 = -I$! So we can now design an interesting insertion by putting two of these insertions back to back. This is a simple case of a **low-$\beta$ insertion**.