Wed June 22 Lecture Notes: Coordinate Transformations and Nonlinear Dynamics

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Most of these notes kindasortasomewhat follow the treatment in the class text, Conte and MacKay, Chapter 10 on Resonances. Additional profitable recommended reading is Edwards and Syphers, Chapter 4, and S.Y. Lee's "Accelerator Physics", section 2.VII. Nonlinear dynamics are *intrinsic* to many large synchrotrons, where the natural chromaticity scales as the (strong focusing) tune and must be corrected with nonlinear sextupole fields. By the end of this lecture, you should have some exposure to the Hamiltonian approach to accelerator maps, canonical transformations and action-angle coordinates, the Floquet transformation, a couple of nonlinear resonances, Chirikov resonance overlap and conditions leading to chaotic motion in accelerators, why he's called Donkey Kong when he's not a donkey, and the answer to life, the universe, and everything. (In case we don't get there before the end of the lecture, the answer to the last part is 42.) I've also put up some other lecture notes on Lie operator approaches to accelerator dynamics on the class web site.

1 Another Quick Refresher

For the majority of this lecture, we'll consider only transverse particle motion. The **two-dimensional Hamiltonian for linear particle motion** can be written as

$$H(x, x', y, x'; s) = \frac{x'^2}{2} + K_x(s)\frac{x^2}{2} + \frac{y'^2}{2} + K_y(s)\frac{y^2}{2}. \tag{1.1}$$

Note that this is just the Hamiltonian for our simple harmonic oscillators with s-dependent focusing in both transverse coordinates (x, y). This is separable, so we'll treat only one dimension until we introduce coupling terms from nonlinearities. Hamilton's equations,

$$x' = \frac{\partial H}{\partial x'} \qquad (x')' = -\frac{\partial H}{\partial x} \tag{1.2}$$

immediately give the identity $x' = x'$ (so x' really *is* the conjugate momentum to x), and

$$x'' + K_x(s)x = 0 \tag{1.3}$$

This is a simple harmonic oscillator with a variable focusing strength, or Hill's equation. All the matrix work we did through last week to describe our lattice is really solving this second-order differential equation (hence 2×2 matrices!) in piecewise fashion through drifts, dipoles, and quadrupoles.

Our solution to the one-dimensional Hill's equation was parameterized as Conte and MacKay Eqn. (5.59):

$$x(s) = \sqrt{\mathcal{W}\beta(s)}\cos[\phi(s) + \phi_0]\ , \tag{1.4}$$

the phase advance $\phi(s)$ and tune Q are defined by integrals of the inverse beta function,

$$\phi(s) = \int_{s_0}^{s} \frac{dS}{\beta(S)} \qquad Q = \frac{1}{2\pi}\oint \frac{dS}{\beta(S)}\ , \tag{1.5}$$

and $\mathcal{W}$ is the Courant-Snyder invariant.

To propagate position and angle from one location $s = s_0$ to another s location, both with known Twiss parameters, we use MacKay and Conte (5.52):

$$M(s|s_0) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}[\cos\mu(s) + \alpha_0 \sin\mu(s)] & \sqrt{\beta_0\beta(s)}\sin\mu(s) \\ -\frac{[\alpha(s)-\alpha_0]\cos\mu(s)+[1+\alpha_0\alpha(s)]\sin\mu(s)}{\sqrt{\beta_0\beta(s)}} & \sqrt{\frac{\beta_0}{\beta(s)}}[\cos\mu(s) - \alpha(s)\sin\mu(s)] \end{pmatrix} \quad (1.6)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{pmatrix}^{-1} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix} \quad (1.7)$$

$$= T(s)^{-1}R(\mu)T(0) \quad (1.8)$$

Even though (1.6) looks horrible, it can be decomposed by local transformations $T(s)$ into **normalized phase space**, and a separate pure rotation matrix $R(\mu)$ that rotates through a phase advance. We've seen this before, but now we'll apply a few more canonical transformations so we can concentrate on resonances and nonlinear dynamics. The objective of these transformations is to get from what you know to the first equation in Chapter 10 of Conte and MacKay. Some of what we're covering here is known as **normal form** theory.

2 Normalized Coordinates

From the decomposition of $M(s|s_0)$, we have found a linear transformation $T(s)$ at a given lattice location s to new **normalized coordinates** (x_N, x'_N) (also seen in Conte and MacKay problem 5-4 in your homework) where

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = T(s)\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} \quad (2.1)$$

Is this transformation canonical? $\det T(s)$=1, and it turns out that *all* linear transformations of this type with unit determinant, or elements of the $SL_2(R)$ group, are canonical. This is a moderately complicated transformation in the sense that it involves $\beta(s)$ and $\alpha(s)$, but it does two nice things: the canonical coordinate and momenta (x_N, x'_N) are in the same units, and the transport matrix is only a phase rotation. So particle motion in this normalized phase space is along circles.

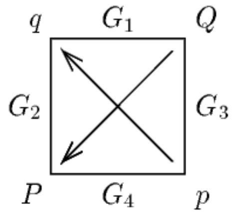
3 Action-Angle Coordinates

Since particle motion is now described by circles in normalized phase space as we proceed through our lattice (albeit in a coordinate system that depends on $\beta(s)$ and therefore depends on where we are in the lattice), the next natural step is to look for a canonical transformation that puts us in cylindrical coordinates, or **action-angle coordinates**. We expect that the radius (or action) is going to be a constant of the motion and therefore invariant. (Gee, do you think it's going to be related to another invariant we've seen?) We also expect that our phase will be $\phi(s)$ from the one-dimensional Hill's equation ansatz, Eqn. (1.4), where we've set $\phi_0 = 0$:

$$x(s) = \sqrt{\mathcal{W}\beta(s)}\cos[\phi(s)] \quad (3.1)$$

This is an equation that relates an old coordinate (x) to a new coordinate (ϕ). Let's take a derivative so we can use a generating function to find a proper canonical momentum to ϕ:

$$x'(s) = -\frac{x}{\beta(s)}\left[\alpha(s) + \tan\phi(s)\right] \quad (3.2)$$



$$G_1(q,Q): \quad p = \partial G_1/\partial q \quad P = -\partial G_1/\partial Q$$
$$G_2(q,P): \quad p = \partial G_2/\partial q \quad Q = \partial G_2/\partial P$$
$$G_3(p,Q): \quad q = -\partial G_3/\partial p \quad P = -\partial G_3/\partial Q$$
$$G_4(p,P): \quad q = -\partial G_4/\partial p \quad Q = \partial G_4/\partial P$$

Figure 1: The generating function mnemonic square. The generating functions G_i are functions of the coordinates that bracket them in the square, and partial derivatives in the direction of the arrows are positive; against the arrows they are negative. If the generating function is time-dependent, the Hamiltonian also changes by dG_i/ds. (Q as a coordinate is a convention from Goldstein, and is not to be regretfully confused with the tune.)

where we've used

$$\phi(s) = \int_{s_0}^{s} \frac{dS}{\beta(S)} \quad \Rightarrow \quad \phi'(s) = \frac{1}{\beta(s)} \tag{3.3}$$

Why was this piece of math useful? Let's take a brief detour into **generating functions** for canonical transformations.

3.1 A quick aside: Generating functions

Generating functions G_i are used to generate canonical transformations, which permit changes of coordinates while respecting Hamilton's equations. For each degree of freedom, they are functions of one old coordinate q or momentum p and one new coordinate Q or momentum P. There are four of these combinations, imaginatively called type 1 through 4 generating functions. A mnemonic diagram for coordinate transformations, similar to the thermodynamic square, is shown in Fig. 1.

Now let's look back at that equation for $x'(s)$. It's an original canonical momentum expressed as a function of both old and new coordinates. This suggests that we will get a type 1 generating function G_1 if we integrate our equation for $x'(s)$ over x:

$$G_1(x,\phi;s) = \int_0^x x'(s)dx = -\frac{x^2}{2\beta(s)}\left[\alpha(s) + \tan\phi(s)\right] \tag{3.4}$$

The new canonical momentum variable that is conjugate to the coordinate $\phi(s)$ is called the **action**, or J:

$$J = -\frac{\partial G_1}{\partial \phi} = \frac{x^2}{2\beta(s)}\sec^2\phi(s) = \frac{1}{2\beta(s)}\left(\frac{x}{\cos\phi(s)}\right)^2 = \frac{\mathcal{W}}{2} \tag{3.5}$$

This immediately draws a equivalence between the action J and adiabatic invariant $\mathcal{W}$ for a given particle. We generally speak of single particles having "actions" (since it's a coordinate, even if it's invariant here), while beam distributions have emittances. As we add nonlinearities to the system, we'll find that the action is no longer invariant and can be treated perturbatively in terms of nonlinear magnet strengths.

The generating function (3.4) produces the coordinate transformations:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta(s)}\cos\phi(s) \\ -\sqrt{1/\beta(s)}[\sin\phi(s) + \alpha(s)\cos\phi(s)] \end{pmatrix} \tag{3.6}$$

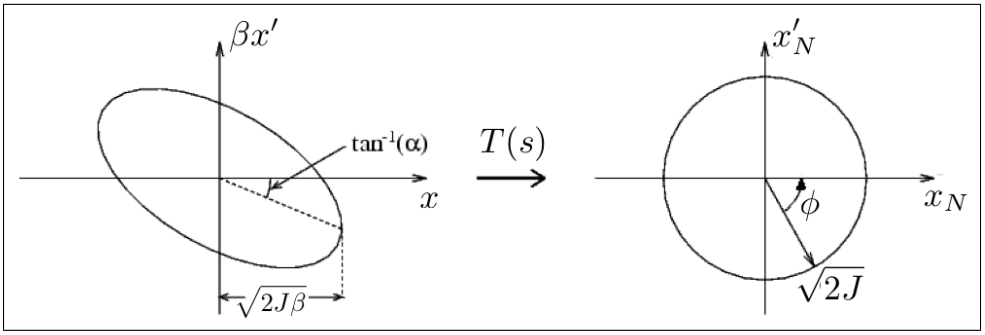


Figure 2: The phase space ellipse in physical and normalized coordinates. The one-turn matrix M "rotates" particles around a given phase space ellipse. With a suitable coordinate transformation T to normalized coordinates (x_N, x'_N), this ellipse becomes a circle. The portrayal of many iterations of the one-turn map within phase space is called a Poincaré surface of section. Adding perturbative nonlinearities will deform the otherwise perfect circles of the normalized coordinate system.

These equations are pretty familiar, but not so pretty. What about normalized coordinates?

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \cos\phi(s) \\ -\sin\phi(s) \end{pmatrix} \tag{3.7}$$

$$J = \frac{x_N^2 + x_N'^2}{2} \qquad \phi(s) = \arctan(x'_N/x_N) \tag{3.8}$$

Much prettier!

The new Hamiltonian is

$$\tilde{H}(\phi, J; s) = H + \frac{\partial G_1}{\partial s} = \frac{J}{\beta(s)} \tag{3.9}$$

Something magical has happened; $\tilde{H}$ no longer depends on $\phi(s)$, the new coordinate, so the new action J is a constant of the motion. This is nicely consistent with what we know so far. We will see later that introducing perturbative nonlinearities into the equations of motion includes terms here that have both J and ϕ components.

3.2 Floquet transformation

The simple action-angle Hamiltonian, $\tilde{H} = J/\beta(s)$, still has a bit of a problem — it depends on the time-like variable s, so it's not truly a constant of the motion. $\beta(s)$ is not constant, so the phase advance is modulated as we advance smoothly in s. To solve this problem, we can perform a change of coordinates known as the **Floquet transformation** to new coordinates (θ, I). This is often seen in nonlinear dynamics sections of texts (e.g. Edwards and Syphers pp. 137 onward or Lee pp. 57 onward) so ring-wide Fourier expansions of nonlinear driving terms can be calculated. Let's pick our a new phase variable that advances linearly in s with the unperturbed problem (which ϕ does not). The quantity

$$\int \frac{ds}{\beta(s)} - 2\pi Q \frac{s}{C} = \mu(s) - Q\frac{s}{R} \tag{3.10}$$

represents the difference or "flutter" between the the phase advance given by integrating the beta function and the new "smooth" phase advance that we want, with C being the accelerator design circumference, $R = C/2\pi$ being the average accelerator design radius, and $\mu(s) \equiv \int ds/\beta(s)$. We want our new smooth phase coordinate to be

$$\phi = \theta + \text{"flutter"} = \theta + \mu(s) - Q\frac{s}{R} \tag{3.11}$$

We can find a type 2 generating function that depends on the old coordinate ϕ and new coordinate I:

$$G_2(\phi, I; s) = I\left(\phi + Q\frac{s}{R} - \mu(s)\right) \tag{3.12}$$

The new conjugate coordinates $(\bar{\phi}, \bar{J})$ are

$$\theta = \phi - \mu(s) + Q\theta, \quad I = J \tag{3.13}$$

The new Hamiltonian becomes QI (a real constant!), and the angle coordinate θ ranges from $0 - 2\pi$ for every revolution around the accelerator. The equation of motion becomes that of a free simple harmonic oscillator with frequency Q.

A Hamiltonian like this is what is used for the left hand side of Conte and MacKay (10.1),

$$\frac{d^2x}{d\theta^2} + Q_H^2 x = 0 \quad \text{or} \quad (1-n)\Delta R \tag{3.14}$$

(The driving term here is for a radial misalignment ΔR of *all* magnets in a weak-focusing case.) The most important things to take away from all this is that transverse particle motion can be canonically transformed to that of a simple harmonic oscillator (albeit through tangled transforms). Accelerator nonlinear dynamicists treat nonlinearities as perturbations to this equation (or the one-turn Hamiltonian below) by adding extra driving terms to the right hand side as in the equation in Conte and MacKay.

4 One-turn maps

Note that the Hamiltonian $\tilde{H}$ in Eqn. (3.9) is **periodic** in s. If we integrate the motion over one turn, removing the s dependence, we find the **one-turn Hamiltonian** that is independent of the observation point!

$$\tilde{H}_{\text{one turn}} = J \oint \frac{ds}{\beta(s)} = 2\pi Q J \tag{4.1}$$

Just as nonlinearities are sometimes treated as driving terms of the simple harmonic oscillator above, they are also sometimes treated as perturbations (depending on higher orders of ϕ and J) that are added to this Hamiltonian.

We will also be concerning ourselves with phase space topology of Poincaré surfaces of section that are **one-turn maps**. We can apply a version of Hamilton's equations to the integrated one-turn Hamiltonian to get difference equations that describe our Poincaré surfaces:

$$\Delta\phi = \frac{\partial \tilde{H}_{\text{one turn}}}{\partial J} = 2\pi Q \qquad \Delta J = -\frac{\partial \tilde{H}_{\text{one turn}}}{\partial \phi} = 0 \tag{4.2}$$

This can be readily seen in Fig. 2, where the one-turn map is a rotation that adds the tune $2\pi Q$ and keeps the radius/action J fixed. From now on we will drop the cumbersome tilde and "one-turn" labels and just use an integrated Hamiltonian H to concentrate on the beauty of resonances and nonlinear dynamics.

5 Periodicity of Errors

We've gone through all this effort to establish simple periodicity in our phase coordinate θ because constant magnetic field errors in a synchrotron are also periodic, so it's natural to try to look at their Fourier decomposition. For example, when we looked at the orbit error from a single dipole, we got a new closed orbit that scaled with $\sqrt{\beta}$ at the dipole error:

$$
\begin{aligned}
x_{\rm co}(s) &= \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\oint\sqrt{\beta(S)}\frac{\Delta B(S)}{(B\rho)}\cos[\pi Q+\phi(s)-\phi(S)]dS && (5.1)\\
&= \frac{Q\sqrt{\beta(s)}}{2\sin\pi Q}\oint\left[\beta^{3/2}(\theta)\frac{\Delta B(\theta)}{(B\rho)}\right]\cos(\pi Q-\pi\theta)d\theta && (5.2)\\
&= \frac{Q\sqrt{\beta(s)}}{2\sin\pi Q}\oint f(\theta)\cos(\pi Q-\pi\theta)d\theta && (5.3)
\end{aligned}
$$

The bracketed term $f(\theta)$ is periodic in θ and isolates the dipole field and how we're sensitive to the dipole field. Since θ is smoothly periodic, though, we can write this as a Fourier decomposition:

$$f(\theta)\equiv\beta^{3/2}(\theta)\frac{\Delta B(\theta)}{(B\rho)}=\sum_{k=-\infty}\infty f_k e^{ik\theta} \tag{5.4}$$

where the Fourier components are calculated from

$$f_k=\frac{1}{2\pi}\oint\beta^{3/2}\frac{\Delta B}{(B\rho)}e^{-ik\theta}=\frac{1}{2\pi Q}\oint\sqrt{\beta}\frac{\Delta B}{(B\rho)}e^{-ik\theta} \tag{5.5}$$

to make all the terms explicit for the dipole case. For higher order multipole errors, these terms have different β dependence that arises from their order of coordinate expansion, so a quadrupole fourier component is

$$f_k=\frac{1}{2\pi}\oint\beta\frac{\Delta B'}{(B\rho)}e^{-ik\theta} \tag{5.6}$$

and so on.

6 Integer Resonances (from Conte/MacKay)

So let's consider the topic in Conte and MacKay section 10.1, integer resonances. Consider a single dipole error in the accelerator. We've discussed this before in two contexts, the linear orbit distortion from a dipole error (which we saw diverged near integer tune), and the case of the driven damped periodic oscillator where the driving is on resonance and there is no damping. The book evaluates the equation of motion in the presence of a single Fourier component of a periodic dipole error:

$$\frac{d^2x}{d\theta^2}+Q_H^2x=\epsilon\cos(m\theta) \tag{6.1}$$

All of that just to connect this (simple) equation to the complex periodic transverse accelerator motion that we've seen before!

As given in the book and extrapolated from the driven damped harmonic oscillator we saw earlier, the solutions of this equation of motion can be written as the sum of the homogeneous solution and a particular solution:

$$
\begin{aligned}
x_{\rm homogeneous} &= A\cos(Q_H\theta)+B\sin(Q_H\theta) && (6.2)\\
x_{\rm particular} &= \frac{\epsilon}{Q_H^2-m^2}[\cos(m\theta)-\cos(Q_H\theta)] && (6.3)
\end{aligned}
$$

The particular solution diverges where $Q_H=m$ in resonant, Lorentzian-like fashion. A single isolated error in the ring has Fourierr components at all integer m, and so we have divergent resonances at all integer Q_H driven by this dipole error. If the error is systematic in each of N periodic cells, then we have resonances at all $Q_H=Nk$ where k is an integer.

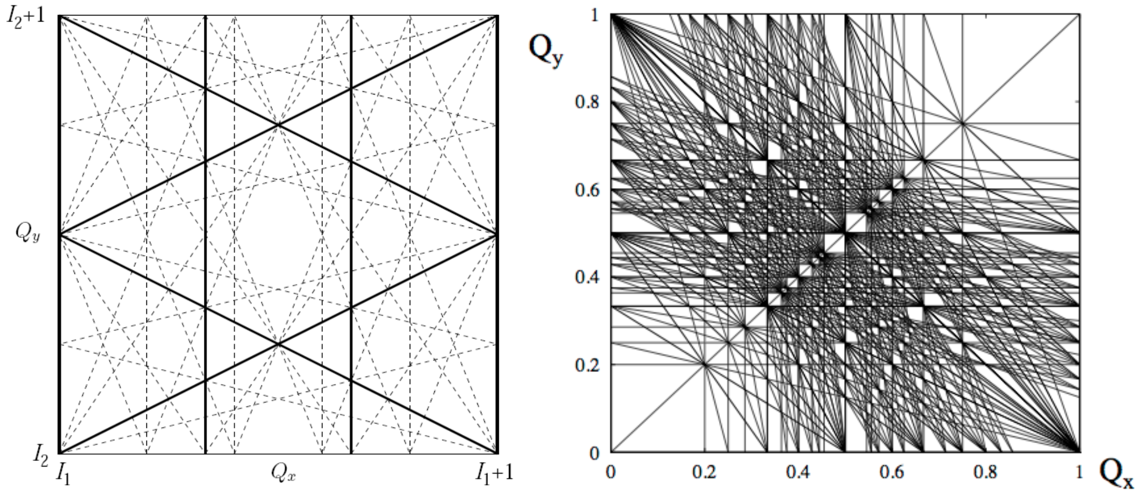


Figure 3: The tune diagram showing resonance lines driven by a normal sextupole (n=2, heavy lines) and normal decapole (n=4, dashed lines), from Conte/MacKay (left), and a tune diagram with many higher order resonances also shown (right).

7 Comments on Coupling Resonances

Section 10.2.1 of Conte and MacKay treats linear coupling resonances, where the resonant conditions are

$$Q_H + Q_V = m \qquad |Q_H - Q_V| = m \tag{7.1}$$

Coupling behavior is different for **sum resonances** and **difference resonances**. Near a difference resonance, energy is exchanged between the two planes and beating occurs at the difference frequency $|Q_H - Q_V|$, but particle motion is still overall stable. However, sum resonances add energy in phase to both planes, and tend to make the particle motion unstable.

8 Comments on the Tune Diagram

In general, an n^{th} order multipole drives $(n+1)^{\text{st}}$ order resonances of the form

$$lQ_x + mQ_y = k \tag{8.1}$$

where $(l+m) = n+1$ and k, l, m, n are integers. These are lines in the tune plane of (Q_x, Q_y); some of these resonance lines can be seen in Fig. 3. Note that there are sextupole resonances at $3Q_x = k$; these are third-order resonances, and we'll consider them in the next section.

9 Example: Sextupoles and Third-Integer Resonant Extraction

Let's go back to action-angle coordinates and evaluate the effect of a sextupole perturbation near a low-order $Q_x = 3m$ resonance line seen in the above figure. The two-dimensional one-turn Hamiltonian now gains an additional term from the sextupole kick,

$$H = 2\pi Q_x J_x + 2\pi Q_y J_y + V_3(x, y; s) \tag{9.1}$$

where $V_3(x, y; s) = \frac{1}{6}S(s)(x^3 - 3xy^2)$ and $S(s) = -B''(s)/(B\rho)$. It must be noted that nonlinear resonances can be driven to higher order than just first order, and that this Hamiltonian is only a first-order approximate Hamiltonian. Particles circulating in the accelerator go through the sextupole magnets many times, after all!

Using Eqn. (3.6) to convert to action-angle coordinates and expanding powers of the cosine:

$$\begin{aligned} V_3 &= -\frac{\sqrt{2}}{4}(\beta_x J_x)^{1/2}(\beta_y J_y)S(s)[2\cos\theta_x + \cos(\theta_x + 2\theta_y) + \cos(\theta_x - 2\theta_y)] \\ &\quad + \frac{\sqrt{2}}{12}(\beta_x J_x)^{3/2}S(s)[\cos(3\theta_x) + 3\cos\theta_x]\,. \end{aligned} \tag{9.2}$$

The various θ terms in Eqn. (9.3) drive resonances including sum, difference, and parametric resonances. This is awfully messy, but we can concentrate on one particular resonance, $\cos(3\theta_x) = k$. The s-dependent terms are periodic, so we can expand V_3 in Fourier harmonics (see Lee's book), and the full Hamiltonian becomes:

$$\begin{aligned} H &= 2\pi Q_x J_x + 2\pi Q_y J_y + \sum_l G_{3,0,l} J_x^{3/2}\cos(3\theta_x + \xi_{3,0,l}) \\ &\quad + o(\cos(\theta_x + 2\theta_y)) + o(\cos(\theta_x - 2\theta_y)) + \ldots \end{aligned} \tag{9.3}$$

We'll concentrate on one particular resonance, the **one-dimensional third order resonance** produced by the $\cos 3\theta_x$ term. Near a resonance at $3Q_x = k$, the Fourier sums of other nonresonant terms tend to average out to zero. Here the Fourier amplitude $G_{3,0,k}$ and the phase $\xi_{3,0,k}$ can be calculated for any sextupole distribution:

$$G_{3,0,k}e^{j\xi_{3,0,k}} = \frac{\sqrt{2}}{24\pi}\oint \beta_x^{3/2}(s)S(s)e^{j[3\theta_x(s)]}ds \tag{9.4}$$

We now have a simple (!!) one-dimensional one-turn Hamiltonian:

$$H = 2\pi Q_x J_x + G_{3,0,k}J_x^{3/2}\cos(3\theta_x + \xi_{3,0,k}) \tag{9.5}$$

If we are near the $3Q_x = k$ resonance, or $\delta \equiv Q_x - k/3 << 1$, then every third turn (or iteration of this map) will come back to near its starting point. We want to examine this small motion, so we can integrate the Hamiltonian again to produce not a one-turn Hamiltonian, but a three-turn Hamiltonian:

$$H_3 = 2\pi\delta J_x + 3G_{3,0,k}J_x^{3/2}\cos(3\theta_x + \xi_{3,0,k}) \tag{9.6}$$

Now we can finally solve the Hamiltonian! One way to "solve" the Hamiltonian is by drawing contours of constant H; another is to solve Hamilton's equations directly.

Hamilton's equations give

$$\Delta\theta_x = 2\pi\delta + \frac{9}{2}G_{3,0,k}J_x^{1/2}\cos 3\theta_x + \xi_{3,0,k}, \quad \Delta J_x = 9G_{3,0,k}J_x^{3/2}\sin 3\theta_x + \xi_{3,0,k} \tag{9.7}$$

where the time step of the Δ is three turns. Fixed points are located where $\Delta\theta_x = \Delta J_x = 0$. The latter immediately gives $\theta_{x\text{FP}} = m(\pi - \xi_{3,0,k})/3$ for integer m; there are three unique solutions in the range $0 < \theta_{x\text{FP}} < 2\pi$. $\Delta\theta_x = 0$ only gives real solutions for even m if $\delta/G_{3,0,k} < 0$, or odd m if $\delta/G_{3,0,k} > 0$. These fixed points can be seen by inspection in Fig. 4.

In either case the action at the fixed point is given by:

$$J_{x\text{FP}} = \left(\frac{4\pi\delta}{9G_{3,0,k}}\right)^2 \tag{9.8}$$

which is positive definite. We can show that these fixed points are all unstable fixed points by examining the equation of motion infinitesimally far away from the fixed point.

The Hamiltonian at the fixed point is

$$H(\theta_{x\text{FP}}, J_{x\text{FP}}) = 2\pi\delta\left(\frac{4\pi\delta}{9G_{3,0,k}}\right) + 3G_{3,0,k}\left(\frac{4\pi\delta}{9G_{3,0,k}}\right) = \frac{160}{243}\frac{\pi^3\delta^3}{G_{3,0,k}^2} \tag{9.9}$$

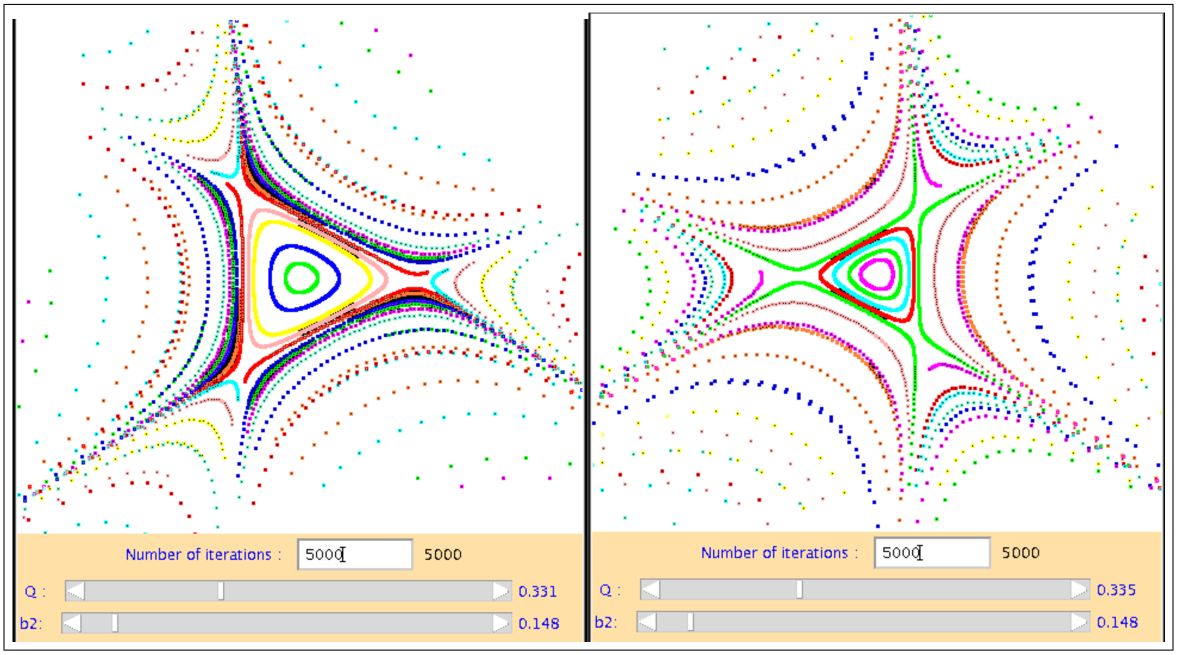


Figure 4: Two examples of Poincaré surfaces of section of the sextupole map from the URL http://www.rhichome.bnl.gov/AP/Java/henon3.html. Note that existence of a separatrix; particles outside the separatrix escape to infinity quickly, while particles inside the separatrix are stable. One example is for $\delta < 0$, while the other is for $\delta > 0$. What happens when δ crosses zero?

Setting the full Hamiltonian to this value produces an equation that is third order in ${J_x}^{1/2}$. We can rescale to new coordinates to more easily find the shape of the separatrix:

$$X \equiv \sqrt{\frac{J_x}{J_{x\mathrm{FP}}}} \cos\theta_x \qquad P \equiv -\sqrt{\frac{J_x}{J_{x\mathrm{FP}}}} \sin\theta_x \tag{9.10}$$

then the equation for the separatrix orbit becomes

$$[2X - 1]\left[P - \frac{1}{\sqrt{3}}(X+1)\right]\left[P + \frac{1}{\sqrt{3}}(X+1)\right] = 0 \tag{9.11}$$

These are three straight lines in this (X, P) phase space. An example of the phase space of this map, also known as the Henon map, is shown in Fig. 4, and you will explore this map in the afternoon lab.

Third order resonances can be used to extract particles slowly from a synchrotron. This is done by creating strong sextupole fields and slowly ramping the tune through the $3Q_x$ resonance; particles crossing the separatrix are quickly launched to large amplitudes, and can be extracted through a septum magnet.

Nonlinearities don't just drive phase-dependent terms, but they can act to produce detuning, a change of tune with particle amplitude. For example, an additional term is created to first order in octupole strength when driving the fourth-order one-dimensional resonance $4Q = l$:

$$H = 2\pi\delta J_x + \frac{1}{2}\alpha {J_x}^2 + G_{4,0,l}{J_x}^2 \cos 4\theta_x \tag{9.12}$$

Here this detuning term α serves to stabilize and close the resonance islands. The tune is not constant with amplitude J_x but varies as αJ_x. The difference in this behavior can be seen in Fig. 4, which also shows various canonical transformations to analyze resonant behavior.

Note that we have only investigated resonances to first order here. An entire framework exists to analyze resonances to higher order, making use of Lie algebras, which we will briefly cover tomorrow. Using this approach, one can show (for example) that sextupoles drive detuning to second order in sextupole strength.

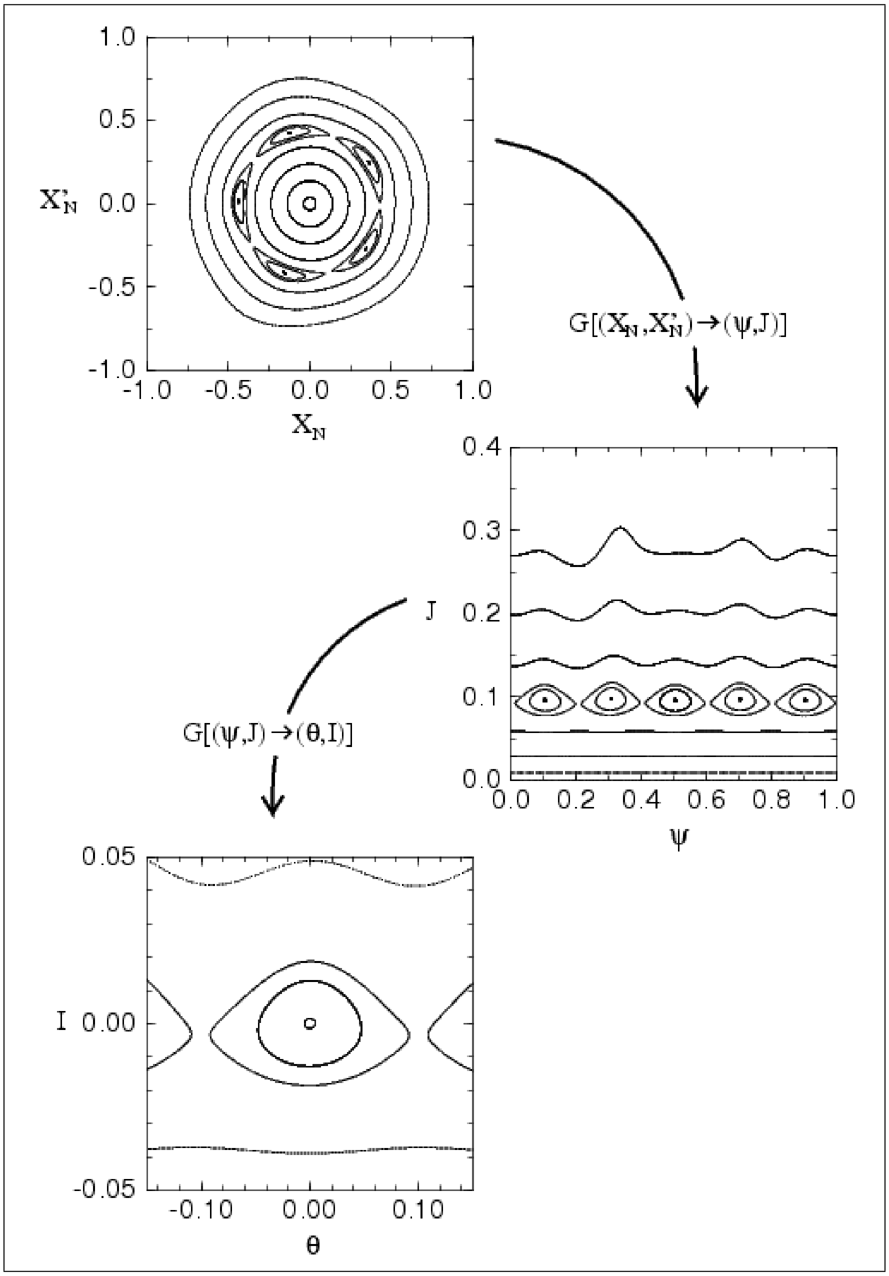


Figure 5: Isolated $5Q_x$ resonance islands in various phase space coordinate systems. Arrows indicate canonical transformations used to transform to action-angle coordinates, then into the rotating frame of the resonance itself.

10 Example: The Beam-Beam Force and Resonances

Many modern accelerators are colliders, and beams of charged particles produce their own electromagnetic fields. This means that the colliding beam effectively acts like an additional lens and provides additional linear and nonlinear focusing. Usually beams have a transverse Gaussian distribution and are equal size at the crossing point to maximize luminosity, so we will consider two Gaussian round beams with transverse charge distribution

$$\rho(r) = \frac{Ne}{2\pi\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right] \tag{10.1}$$

where Ne is the charge per unit length and σ^2 is the RMS beam width. The force acting on a particle displaced by r from the center of the opposing bunch is

$$F = \frac{\gamma N e^2(1+\beta^2)}{4\pi \mathcal{W}_0 r} \left(1 - \exp\left[-\frac{r^2}{2\sigma^2}\right]\right) \hat{r} \tag{10.2}$$

where $\beta = v/c$ and γ is the relativistic term.

Though this looks complicated, the exponential can be expanded in terms of r for small r. The first nontrivial term scales with r; the next varies with r^3, etc. In general the force (or kick) only has odd components. The first-order term that scales with r looks like a linear focusing term in both transverse directions, in other words, like a quadrupole. This produces a linear beam-beam tune shift of

$$\xi = \delta\nu = \frac{N_B r_0 \beta^\star}{4\pi\gamma\sigma^2} \tag{10.3}$$

where $r_0 = e^2/(4\pi\mathcal{W}_0 mc^2)$ is the classical particle radius, $\beta^\star$ is the optics function at the interaction point, and N_B is the total number of particles in the opposing bunch.

The beam-beam kick has only odd terms, which means it derives from a potential with only even power terms in r. However, the exponential expansion implies that it has *all* even powers of r, so the beam-beam force is also a source for rich nonlinearities, and it drives all even-order nonlinear resonances. This will be seen in the lab, where motion near the $3\nu_x = l$ beam-beam resonance produces not three, but six resonance islands.

11 Chaos: The Chirikov Resonance Overlap Criterion

A Famous Paper (tm) in 1979 by Boris Chirikov, "A Universal Instability Of Many-Dimensional Oscillator Systems", presented a criterion for the onsent of chaos in systems where multiple resonances are present. This 90-page paper is not for the faint of heart, but the criterion is easy to state: chaos or stochasticity is observed when the nonlinearity is strong enough that resonance islands from separate resonances start to overlap.

Quantitatively, given resonance island action half-widths of ΔJ_1 and ΔJ_2, and a distance between the resonances of δJ, the Chirikov overlap criterion is

$$\delta J \leq \Delta J_1 + \Delta J_2 \tag{11.1}$$

In actuality, this slightly overstates the matter as motion near the separatrix has infinite period and is therefore highly susceptible to any perturbation. Including this effect, as well as the effects of smaller, higher-order resonances, Chirikov calculated a correction:

$$\frac{2}{3}\delta J \leq \Delta J_1 + \Delta J_2 \tag{11.2}$$

How does one calculate resonance widths? Recall that the Hamiltonian is a constant of the motion on the separatrix. The unstable fixed points are known, so the Hamiltonian must have the same value at the stable fixed points on the separatrix.

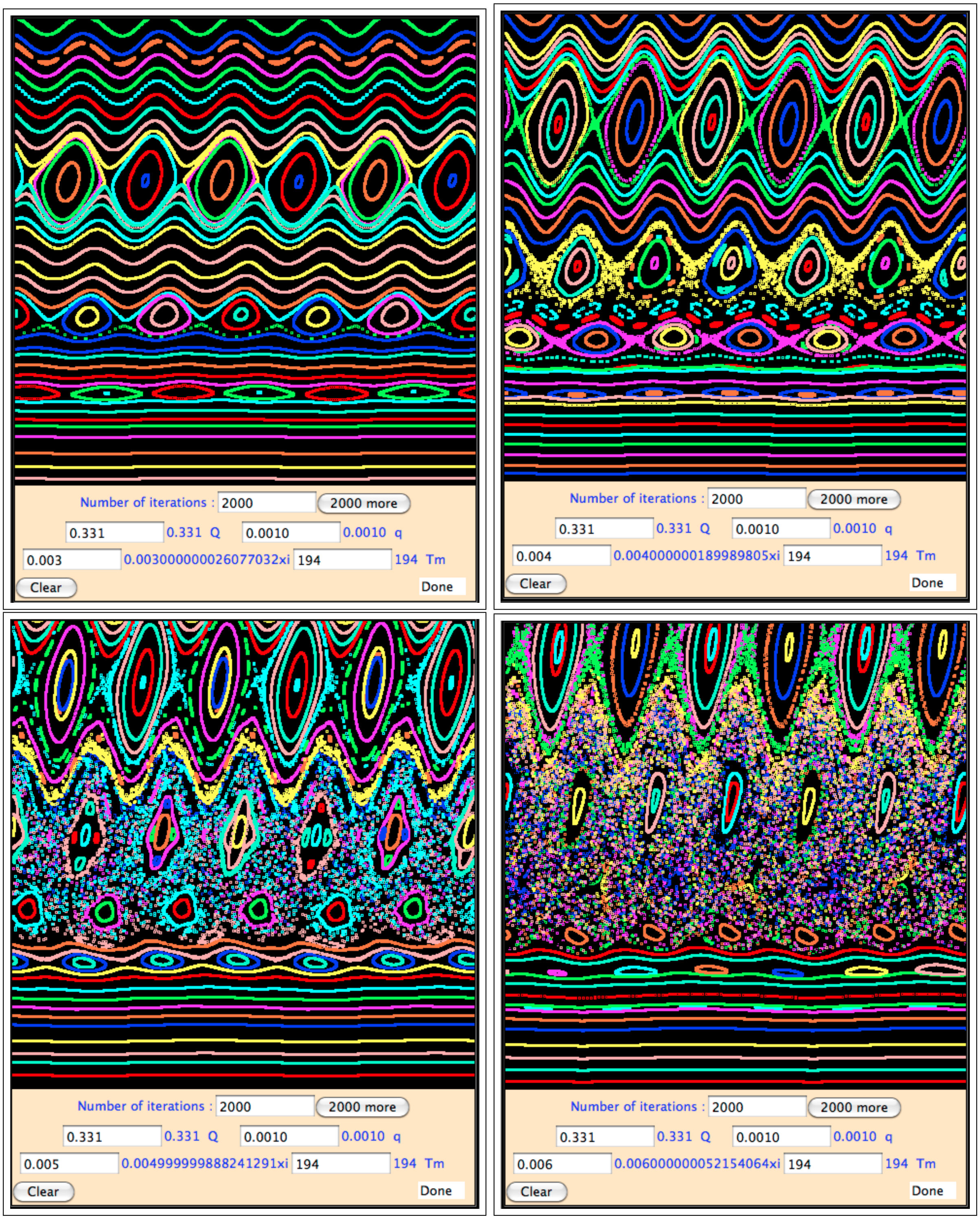


Figure 6: The Chirikov overlap criterion for the development of bounded chaotic areas in phase space. Here the horizontal axes are phase, and the vertical axes are action. At low amplitudes or actions, particles have roughly constant action — particle motion is circular. Resonances appear at higher amplitudes. The beam-beam tuneshift is increased from upper left to lower right, showing the development of chains of resonance islands, their eventual overlap, and ensuing development of areas of chaotic motion.