UNCOUPLED ACHROMATIC TILTED S-BEND*

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Abstract

A particular section of the electron beam transport line, to be used in the e-cooling project [1] of the Relativistic Heavy Ion Collider (RHIC), is constrained to displace the trajectory with both horizontal and vertical offsets so that the outgoing beamline is parallel to the incoming beamline. We also require that section be achromatic in both planes. This mixed horizontal and vertical achromatic S-bend is accomplished by rotating the two dipoles and the quadrupoles of the line, about the longitudinal axis of the incoming beam. However such a rotation of the magnetic elements may couple the transported beam through the first order beam transfer matrix (linear coupling). In this paper we study a sufficient condition, that the first order transport matrix (R-matrix) can satisfy, so that this section of beam transfer line is both achromatic and linearly uncoupled. We provide a complete solution for the beam optics which satisfies both conditions.

TILTED S-BEND

A particular section of the electron beam transfer line of the e-RHIC project employs an S-bend, aiming to make a parallel displacement of the electron beam in both the horizontal and vertical planes simultaneously. Such a line which is shown in Fig. 1 can be made both, achromatic (see subsection *Achromatic Conditions*) and linearly uncoupled (see subsection *Uncoupled Conditions*).

Achromatic Conditions

The transport matrix of the first bend may be written in the form of 2×2 blocks as

$$\mathbf{B}_1 = \begin{pmatrix} \mathbf{M}_x & \mathbf{0} & \mathbf{D} \\ \mathbf{0} & \mathbf{M}_y & \mathbf{0} \\ -\tilde{\mathbf{D}} & \mathbf{0} & \mathbf{G} \end{pmatrix}, \quad \text{where} \tag{1}$$

$$\mathbf{M}_x = \begin{pmatrix} \cos\phi & \rho\sin\phi \\ -\frac{1}{\rho}\sin\phi & \cos\phi \end{pmatrix}, \quad \mathbf{M}_y = \begin{pmatrix} 1 & \rho\phi \\ 0 & 1 \end{pmatrix}, \tag{2}$$

$$\mathbf{D} = \begin{pmatrix} 0 & \rho(1-\cos\phi) \\ 0 & \sin\phi \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 1 & G \\ 0 & 1 \end{pmatrix} \tag{3}$$

$$\tilde{\mathbf{D}} = \sigma_y \mathbf{D}^{\mathrm{T}} \sigma_y = \begin{pmatrix} -\sin\phi & -\rho(1-\cos\phi) \\ 0 & 0 \end{pmatrix}, \tag{4}$$

with the Pauli matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

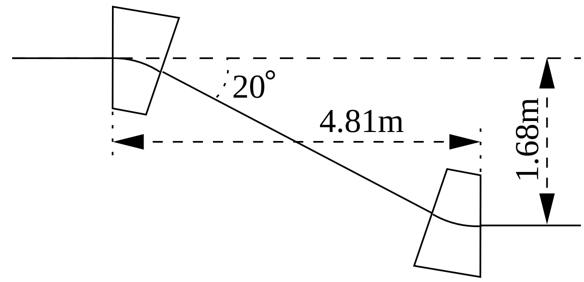

Figure 1: Two equal but opposite sector dipoles provide a parallel translation of the beam in a plane. The plane of the beamline may be rotated about the dashed line by an angle $\theta = 22.5°$ to allow for both horizontal and vertical shifts of the beam by 1.56 and 0.64 m, respectively.

The reversed second bend has the matrix

$$\mathbf{B}_2 = \begin{pmatrix} \mathbf{M}_x & \mathbf{0} & -\mathbf{D} \\ \mathbf{0} & \mathbf{M}_y & \mathbf{0} \\ \tilde{\mathbf{D}} & \mathbf{0} & \mathbf{G} \end{pmatrix}, \tag{5}$$

with only the upper right and lower left 2×2 blocks of opposite sign from $\mathbf{B}_1$. Both bends have the same diagonal blocks.

While the transport between the two dipoles might contain quadrupoles of various rotations about the beamline, it is perhaps conceptually simpler to start with a transversely decoupled solution relative to the plane of the two bends. We want to find a configuration of quadrupoles and drifts which will zero the dispersion terms R_{16} and R_{26} of the whole section of beamline.

$$\mathbf{R} = \mathbf{B}_2\mathbf{N}\mathbf{B}_1 = \begin{pmatrix} \mathbf{M}_x & \mathbf{0} & -\mathbf{D} \\ \mathbf{0} & \mathbf{M}_y & \mathbf{0} \\ \tilde{\mathbf{D}} & \mathbf{0} & \mathbf{G} \end{pmatrix} \begin{pmatrix} \mathbf{N}_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \times \begin{pmatrix} \mathbf{M}_x & \mathbf{0} & \mathbf{D} \\ \mathbf{0} & \mathbf{M}_y & \mathbf{0} \\ -\tilde{\mathbf{D}} & \mathbf{0} & \mathbf{G} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{M}_x\mathbf{N}_x\mathbf{M}_x & \mathbf{0} & \mathbf{M}_x\mathbf{N}_x\mathbf{D} - \mathbf{D} \\ \mathbf{0} & \mathbf{M}_y\mathbf{N}_y\mathbf{M}_y & \mathbf{0} \\ \tilde{\mathbf{D}}\mathbf{N}_x\mathbf{M}_x - \tilde{\mathbf{D}} & \mathbf{0} & \tilde{\mathbf{D}}\mathbf{N}_x\mathbf{D} + \mathbf{G}^2 \end{pmatrix}, \tag{6}$$

since $\mathbf{D}\tilde{\mathbf{D}} = \mathbf{0}$, $\mathbf{D}\mathbf{G} = \mathbf{D}$, and $\mathbf{G}\tilde{\mathbf{D}} = \tilde{\mathbf{D}}$. In order to cancel the dispersion, we must have $\mathbf{M}_x\mathbf{N}_x\mathbf{D} = \mathbf{D}$, i. e., the second column of $\mathbf{D}$ must be an eigenvector of $\mathbf{M}_x\mathbf{N}_x$ with eigenvalue of 1. Rearranging our achromatic condition gives the pair of equations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \rho(1-\cos\phi) \\ \sin\phi \end{pmatrix} = \begin{pmatrix} -\rho(1-\cos\phi) \\ \sin\phi \end{pmatrix}, \tag{7}$$

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with explicit elements a, b, c, and d for $\mathbf{N}_x$. These two equations, together with requirement $\det(\mathbf{N}_x) = 1$, yield three equations in four unknowns. Eliminating three of the variables, we get

$$\mathbf{N}_x = \begin{pmatrix} a & -(1+a)\rho\tan\frac{\phi}{2} \\ \frac{1-a}{\rho}\cot\frac{\phi}{2} & a \end{pmatrix} \tag{8}$$

which has identical values on the diagonal. Given values of ρ and θ for the bends, then there is only one degree of freedom left in $\mathbf{N}_x$ for $\mathbf{R}$ to be achromatic, with only 2×2 blocks along the diagonal of $\mathbf{R}$ and blocks of zeros away from the diagonal.

Uncoupled Conditions

Pivoting the section of beamline about the incoming beam (dashed line of Fig. 1) by an angle θ would tend to produce xy-coupling when θ is not a multiple of $90°$:

$$\mathbf{R} = \begin{pmatrix} \mathbf{I}\cos\theta & \mathbf{I}\sin\theta \\ -\mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{R}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_y \end{pmatrix} \begin{pmatrix} \mathbf{I}\cos\theta & -\mathbf{I}\sin\theta \\ \mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{R}_x\cos^2\theta + \mathbf{R}_y\sin^2\theta & \frac{1}{2}(\mathbf{R}_y - \mathbf{R}_x)\sin(2\theta) \\ \frac{1}{2}(\mathbf{R}_y - \mathbf{R}_x)\sin(2\theta) & \mathbf{R}_x\sin^2\theta + \mathbf{R}_y\cos^2\theta \end{pmatrix}, \tag{9}$$

unless we have $\mathbf{R}_x = \mathbf{R}_y$, in which case $\mathbf{R}$ is independent of the rotation θ. This means that the transport between the bends must have

$$\mathbf{N}_y = \mathbf{M}_y^{-1}\mathbf{M}_x\mathbf{N}_x\mathbf{M}_x\mathbf{M}_y^{-1}. \tag{10}$$

When both the achromaticity and uncoupled conditions are satisfied the $\mathbf{R}$ transfer matrix is independent of the rotation angle θ $\mathbf{R}(\theta) = \mathbf{R}(\mathbf{0^o})$.

Mirror Symmetry

Given a beamline $\mathbf{R} = \mathbf{E}_n\cdots\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1$, its mirror image $\breve{\mathbf{R}} = \mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\cdots\mathbf{E}_n$ with the order of the elements reversed can be calculated [2] from $\mathbf{R}^{-1}$ as

$$\breve{\mathbf{R}} = \mathbf{S}_t\mathbf{R}^{-1}\mathbf{S}_t, \tag{11}$$

with the help of the time reversal operator

$$\mathbf{S}_t = \begin{pmatrix} \sigma_z & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_z & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\sigma_z \end{pmatrix} \text{ with } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{12}$$

where the longitudinal 2×2-block has a minus sign since the time-like coordinate z is the fifth component of the vector rather than the sixth.

A palindromic beamline is formed when the second half of the beamline contains the elements of the first half placed in reversed order. Given half the beamline for $\mathbf{N}_j$ for the j^{th} 2×2 block on the diagonal as

$$\mathbf{A} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{13}$$

and the mirror image for the other half, then

$$\begin{aligned} \mathbf{N}_j &= \breve{\mathbf{A}}\mathbf{A} = \sigma_z\mathbf{A}^{-1}\sigma_z\mathbf{A} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u & -s \\ -t & r \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} \\ &= \begin{pmatrix} ru+st & -2su \\ 2rt & ru+st \end{pmatrix}. \end{aligned} \tag{14}$$

If we build $\mathbf{N}$ from a set of quadrupoles mirrored about the midpoint between the dipoles, then we are guaranteed to have $N_{11} = N_{22}$ as in Eq. (8) and $N_{33} = N_{44}$ and hence $R_{11} = R_{22}$ and $R_{33} = R_{44}$.

DESCRIPTION OF THE BEAMLINE

This line uses two dipoles each bending the beam by $20°$ but in opposite directions, and 2×3 quadrupoles all symmetrically placed about the center of the line. The whole S-bend is then rotated by an angle $\theta = 22.5°$ about the beam axis.

In order to find the minimum number of quadrupoles required to generate such an achromatic matrix which also satisfies the condition $\mathbf{R}_x = \mathbf{R}_y$, we note that an achromatic R-matrix which describes a horizontal S-bend requires a minimum of two quadrupoles symmetrically placed about the center of the line and excited at the same strength. The strength and the location of the quadrupoles is determined by the achromaticity condition ($R_{16} = 0$ and $R_{26} = 0$), and the mirror symmetry of the beamline requires that ($R_{11} = R_{22}$) and $R_{33} = R_{44}$). Any additional quadrupoles that have to be placed in the line to satisfy the decoupling condition $\mathbf{R}_x = \mathbf{R}_y$, should come in pairs to preserve the symmetry required by the achromaticity condition. Therefore the first additional pair of quadrupoles will satisfy the equality ($R_{11} = R_{22}$) and the second pair will satisfy the equality ($R_{12} = R_{34}$). The equality ($R_{21} = R_{43}$) is automatically satisfied from the symplecticity conditions ($R_{11}R_{22} = R_{12}R_{21}$) and ($R_{33}R_{44} = R_{34}R_{43}$). Thus the minimum number of quadrupoles required to generate an achromatic matrix of the tilted S-bend line that also satisfies the condition $\mathbf{R}_x = \mathbf{R}_y$ is six. In the following two subsections we present two examples of achromatic and uncoupled lines that make use of six and seven quadrupoles respectively. Although it is possible to find a solution in a closed form, for a line that can satisfies the conditions mentioned in the previous section, we have chosen to use an optimization beamline code like TRANSPORT [3] or MAD [4] to find the optimum location and strength of the quadrupoles that satisfy the conditions for achromaticity and no coupling.

Line with Six Quadrupoles

As discussed earlier, the achromaticity condition requires symmetric placement of quadrupoles with respect to the center of the line. As a result the dispersion function will appear as an antisymmetric function since it transforms like the six dimensional vector **x** of the particle's coordinate, namely $\mathbf{x}_{\text{out}} = \mathbf{R}\mathbf{x}_{\text{in}}$. Thus $\eta_{\text{out}} = (\eta_x, \eta_x^{'}, \eta_y, \eta_y^{'}, 0, 1)^{\text{T}}_{\text{out}} = \mathbf{R}\eta_{\text{in}}$. The use of six quadrupoles however symmetrically placed, can only satisfy the achromaticity and decoupling conditions. Thus when using six quadrupoles, in order to control the values of the β_x and β_y functions along the line and keep them as low as possible, we impose the conditions $\alpha_x = \alpha_y = 0$ at the center of the line. This is done by varying the values of β_x, β_y, α_x, and α_y at the entrance of the line.

Fig. 2 shows the β_x, β_y, η_x, and η_y functions, plotted along the line as calculated using the computer code MAD in the "coupled" mode. The elements of the R-matrix of the six-quadrupole line are:

$$\begin{pmatrix} 1.8268 & 0.2492 & 0.0000 & 0.0000 & 0 & 0.0000 \\ 9.3782 & 1.8268 & 0.0000 & 0.0000 & 0 & 0.0000 \\ 0.0000 & 0.0000 & 1.8268 & 0.2492 & 0 & 0.0000 \\ 0.0000 & 0.0000 & 9.3782 & 1.8268 & 0 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1 & -0.0071 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0 & 1.0000 \end{pmatrix}.$$

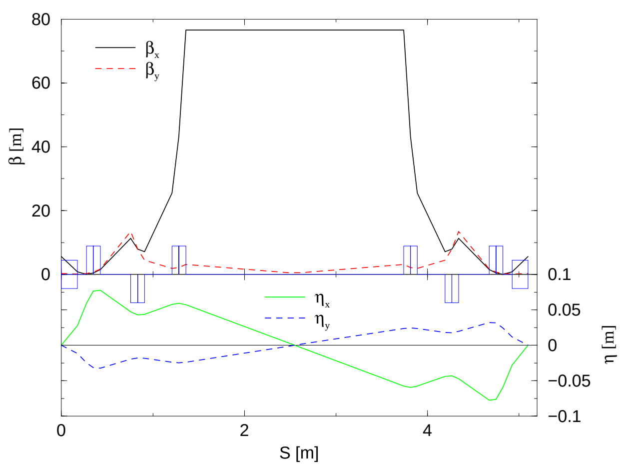

Figure 2: Values of β_x, β_y, η_x, and η_y along the line with six quads. The first and last blue rectangles represent the dipoles; the others are quadrupoles.

Line with Seven Quadrupoles

In order to provide additional control on the β_x β_y functions we introduced an additional quadrupole which was placed at the center of the line to preserve the symmetry. Fig. 3 shows the β_x, β_y,and η_x, η_y functions, plotted along the line as calculated using the computer code MAD in the "coupled" mode. Here the $\mathbf{R}$-matrix is:

$$\begin{pmatrix} 0.1882 & 0.1048 & 0.0000 & 0.0000 & 0 & 0.0000 \\ -9.2079 & 0.1882 & 0.0000 & 0.0000 & 0 & 0.0000 \\ 0.0000 & 0.0000 & 0.1882 & 0.1048 & 0 & 0.0000 \\ 0.0000 & 0.0000 & -9.2079 & 0.1882 & 0 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1 & -0.0071 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0 & 1.0000 \end{pmatrix}.$$

Table 1 contains the parameters of the magnets and the drift spaces for the first half of both the six-quadrupole and seven-quadrupole lines.

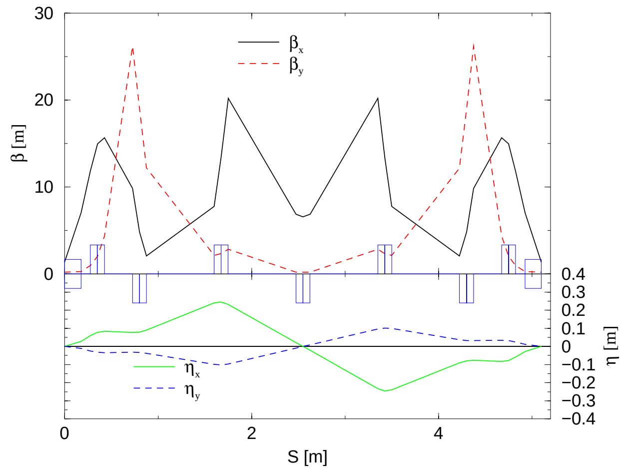

Figure 3: Values of the β_x, β_y, η_x, and η_y along the line with seven quadrupoles. The first and last blue rectangles represent the dipoles; the others are quadrupoles.

Table 1: Parameters of the elements for half of the six-quadrupole and seven-quadrupole lines

	Six Quads		**Seven Quads**	
Element	ρ [m]	θ	ρ [m]	θ
DIPOLE	0.5	20°	0.5	20°
Element	L [m]	k [m^{-2}]	L [m]	k [m^{-2}]
DRIFT	0.1	—	0.1	—
QUAD	0.15	36.849	0.15	29.7744
DRIFT	0.3334	—	0.3	—
QUAD	0.15	-20.8498	0.15	-18.8317
DRIFT	0.30106	—	0.724	—
QUAD	0.15	10.670	0.15	13.7085
DRIFT	1.1905	—	0.72574	—
$\frac{1}{2}$QUAD	—	—	0.075	-27.4169

CONCLUSION

We discussed sufficient conditions that the R matrix for a rotated S-bend should satisfy for the line to be achromatic and uncoupled. Such a line can be built with six quadrupoles. Although we only require a fixed rotation, this solution is invariant to the rotation and could be used as a gantry.

REFERENCES

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