7.4 Longitudinal coordinates and synchrobetarton coupling

For a charged particle of charge q in an external electromagnetic field, we may write the relativistic Hamiltonian as

$$\mathcal{H}(x, P_x, y, P_y, z, P_z; t) = U = \sqrt{(\vec{P} - \vec{A})^2 + m^2c^4} + q\Phi, \tag{7.58}$$

with vector potential $\vec{A}$, and electric potential Φ, canonical momentum $\vec{P} = \vec{p} + q\vec{A}$, and total energy U. Here the kinetic momentum $\vec{p} = \gamma\vec{\beta}mc$. In the usual cylindrical coordinates of accelerator physics with radius of curvature ρ, the Hamiltonian may be written as

$$\begin{aligned}\mathcal{H}(x, P_x, y, P_y, s, P_s; t) &= U \\ &= c\sqrt{(P_x - qA_x)^2 + (P_y - qA_y)^2 + \left(\frac{P_s - qA_s}{1 + x/\rho}\right)^2 + m^2c^2} + q\Phi.\end{aligned} \tag{7.59}$$

Recalling that a canonical transformation from variables $(\vec{q}, \vec{p})$ to variables $(\vec{Q}, \vec{P})$ preserves the Poincaré-Cartan integral invariant

$$\vec{p} \cdot d\vec{q} - H\,dt = \vec{P} \cdot d\vec{Q} - K\,dt, \tag{7.60}$$

we can interchange one canonical pair (q_j, p_j) with the time-energy pair $(t, -H)$ by writing the invariant as

$$\left(\sum_{i \neq j} p_i dq_i + (-H)dt\right) - (-p_j)dq_j. \tag{7.61}$$

This transformation gives the new Hamiltonian

$$\begin{aligned}\mathrm{H}(x, P_x, y, P_y, t, -U; s) &= -P_s \\ &= -qA_s - \left(1 + \frac{x}{\rho}\right) \\ &\quad\times \sqrt{\left(\frac{U - q\Phi}{c}\right)^2 - (mc)^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2}.\end{aligned} \tag{7.62}$$

If there are no electrostatic fields then we may write $\Phi = 0$; the fields in rf cavities may be obtained from the time derivative of $\vec{A}$. Ignoring solenoids for now, with only transverse magnetic guide fields and the longitudinal electric fields of the cavities may be described by A_s, so we may take

$$A_x = 0, \quad A_y = 0, \quad \text{and} \quad \Phi = 0. \tag{7.63}$$

To model dipoles, quadrupoles and cavities a vector potential of the form

$$\begin{aligned} qA_s &= q\left(1+\frac{x}{\rho}\right)(\vec{A}\cdot\hat{s}) \\ &= -\frac{p_{\rm sy}}{\rho}\,x - \frac{p_{\rm sy}K}{2}\,(x^2-y^2)+\ldots \\ &\quad + \frac{qV}{\omega_{\rm rf}}\sum_{j=-\infty}^{\infty}\delta(s-jL)\,\cos(\omega_{\rm rf}t+\phi_0) \end{aligned} \tag{7.64}$$

is sufficient. Here the circumference is L, and the magnetic guide field parameter is

$$K = \frac{1}{\rho^2} + \frac{q}{p_{\rm sy}}\left(\frac{\partial B_y}{\partial x}\right)_0, \tag{7.65}$$

and $p_{\rm sy}$ is the momentum of the synchronous design particle. The effective rf phase as the synchronous particle passes the cavities is ϕ_0, to give a net energy gain per turn of $[qV\cos(\phi_0)]$. For simplicity in Eq. (7.64) the effect of all rf cavities has been lumped at the location $s = 0$ in the ring.

The time coordinate may be broken up into the time for the synchronous particle to arrive at the location s plus a deviation Δt for the particular particle's arrival time:

$$t = t_{\rm sy}(s) + \Delta t(s) = \frac{2\pi h}{\omega_{\rm rf}L}\,s + \Delta t = \frac{s}{\beta c} + \Delta t. \tag{7.66}$$

If the beam is held at constant energy, then we may make a canonical transformation of the time coordinate Δt to rf phase φ given by

$$\varphi = \omega_{\rm rf}\Delta t. \tag{7.67}$$

If acceleration is assumed to be adiabatically slow, so that $\omega_{\rm rf}$ changes very slowly, and the magnetic guide fields track the momentum of the synchronous particle, keeping the synchronous particle on a fixed trajectory, we can allow for an adiabatic energy ramp according to

$$U_{\rm sy} = U_0 + \frac{qV\sin\phi_0}{L}\,s, \tag{7.68}$$

where the energy gain per turn $[qV\sin\phi_0]$ is much less than the total energy U_s. In this case it might not unreasonable to use φ as the longitudinal coordinate, so long as we are prepared to allow for adiabatic damping of the phase space areas. To convert the time coordinate into an rf phase angle relative to the phase of the synchronous particle, we can use the generating function

$$F_2(x, p_x, t, W; s) = xp_x + \left[\omega_{\text{rf}} W - \left(U_0 + \frac{qV \sin\phi_0}{L}\, s\right)\right] t - \frac{2\pi h}{L}\, W s + \frac{qV\pi h \sin\phi_0}{\omega_{\text{rf}} L^2}\, s^2, \tag{7.69}$$

to find a new canonical momentum W corresponding to the phase coordinate. This is what was used to arrive at Eq. (CM: 7.61)[1]

Before proceeding down this path it will behoove us to examine the effect of ramping the energy. The deviation in energy of another particle of energy U from the synchronous particle may be defined as

$$\Delta U = U - U_{\text{sy}}. \tag{7.70}$$

For the synchronous particle the phase of the rf cavity should be

$$\begin{aligned} \phi_{\text{sy}} &= \phi_0 + \int_0^{t_{\text{sy}}} \omega_{\text{rf}}\, dt \\ &= \phi_0 + \int_0^{t_{\text{sy}}} \frac{2\pi h \beta c}{L}\, dt_{\text{sy}} \end{aligned} \tag{7.71}$$

With changing energy and the velocity dependence of ω_{rf}, calculation of this integral becomes a problem and φ does not appear to be such an attractive candidate for a canonical coordinate. This is why Chris Iselin chose to take $\zeta = -c\Delta t$ as the longitudinal coordinate variable in the MAD program[14]. Of course there are other parameters which are not necessarily constants in real accelerators. It is quite common to vary the radial position of the closed orbit, as well as the synchronous phase of the rf – particularly during the phase jump at transition crossing. Pulsed quadrupoles are frequently used to cause a rapid change in the transition energy at transition during acceleration.

If we consider a ramp with a constant increase of energy per turn

$$\begin{aligned} U_{\text{sy}} &= U_0 + Rs \quad \text{with} \\ R &= \frac{qV}{L} \sin\phi_0 \end{aligned} \tag{7.72}$$

then the time evolution as a function of path length of the synchronous

[1] The formalism of Suzuki[35] was followed in writing CM: § 7.6.

particles is given by

$$
\begin{aligned}
t_{\rm sy}(s) &= \int_0^s \frac{ds}{\beta c} \\
&= \int_0^s \left[1 - \left(\frac{mc^2}{U_0 + Rs'}\right)^2\right]^{1/2} ds' \\
&= \frac{mc^2}{R} \int_{\frac{U_0}{mc^2}}^{\frac{U_0+Rs}{mc^2}} \sqrt{1-\xi^{-2}}\, d\xi, && s' = \frac{\xi mc^2 - U_0}{R} \\
&= -\frac{mc^2}{R} \int_{\frac{mc^2}{U_0}}^{\frac{mc^2}{U_0+Rs}} \sqrt{1-\eta^2}\, \frac{d\eta}{\eta^2}, && \xi = \frac{1}{\eta} \\
&= \frac{mc^2}{R} \int_{\cos^{-1}\left(\frac{mc^2}{U_0}\right)}^{\cos^{-1}\left(\frac{mc^2}{U_0+Rs}\right)} \tan^2\theta\, d\theta, && \eta = \cos\theta \\
&= \frac{mc^2}{R} \left[\tan\theta - \theta\right] \Big|_{\cos^{-1}\left(\frac{mc^2}{U_0}\right)}^{\cos^{-1}\left(\frac{mc^2}{U_0+Rs}\right)} \\
&= \frac{mc^2}{2R} \left[\frac{U_0+Rs}{mc^2}\sqrt{1-\left(\frac{mc^2}{U_0+Rs}\right)^2} - \frac{U_0}{mc^2}\sqrt{1-\left(\frac{mc^2}{U_0}\right)^2}\right. \\
&\qquad \left. + \cos^{-1}\left(\frac{mc^2}{U_0+Rs}\right) - \cos^{-1}\left(\frac{mc^2}{U_0}\right)\right] \\
&= \frac{mc^2}{2R} \left[\beta\gamma - \beta_0\gamma_0 + \cos^{-1}\left(\frac{1}{\gamma}\right) - \cos^{-1}\left(\frac{1}{\gamma_0}\right)\right] \qquad (7.73)
\end{aligned}
$$

Provided that the ramping is sufficiently slow, then acceleration may be treated adiabatically.

At least in the adiabatic case, then we can find the new canonical coordinate and Hamiltonian from Eq. (7.69):

$$
-U = \frac{\partial F_2}{\partial t} = \omega_{\rm rf} W - \left(U_0 + \frac{qV\sin\phi_0}{L}\, s\right), \qquad (7.74)
$$

$$
\varphi = \frac{\partial F_2}{\partial W} = \omega_{\rm rf} t - \frac{2\pi h}{L}\, s, \qquad (7.75)
$$

with the derivative of the generating function:

$$\begin{aligned}\frac{\partial F_2}{\partial s} &= -\frac{qV\sin\phi_0}{L}\,t - \frac{2\pi h}{L}\,s + \frac{2\pi h qV\sin\phi_0}{\omega_{\rm rf}L^2}\,s \\ &= -\frac{qV\sin\phi_0}{L}\left[\frac{2\pi h}{\omega_{\rm rf}L}\,s + \Delta t\right] - \frac{2\pi h}{L}\,s + \frac{2\pi h qV\sin\phi_0}{\omega_{\rm rf}L^2}\,s \\ &= -\frac{qV\sin\phi_0}{L}\,\frac{\varphi}{\omega_{\rm rf}} - \frac{2\pi h}{L}\,s \\ &= -\frac{qV\sin\phi_0}{L}\,\frac{\varphi}{\omega_{\rm rf}} - \frac{s}{\lambda\!\!\!^{-*}_{\rm rf}},\end{aligned} \tag{7.76}$$

where we have written $\lambda\!\!\!^{-*}_{\rm rf} = L/2\pi h$ as a constant effective rf wavelength corrected for the velocity.[2] Ignoring the vertical motion, the new Hamiltonian is

$$\begin{aligned}H_1(x,p_x,\varphi,W;s) &= \mathrm{H} + \frac{\partial F_2}{\partial s} \\ &= \frac{p_{\rm sy}}{\rho}\,x + \frac{p_{\rm sy}K}{2}\,x^2 + \frac{qV}{\omega_{\rm rf}}\sum_{j=-\infty}^{\infty}\delta(s-jL)\,\cos\left(\phi_0+\varphi+\frac{s}{\lambda\!\!\!^{-*}_{\rm rf}}\right) \\ &\quad - \left(1+\frac{x}{\rho}\right)\left[\frac{(U_{\rm sy}-\omega_{\rm rf}W)^2 - m^2c^4}{c^2} - p_x^2\right]^{1/2} - \frac{qV\sin\phi_0}{L}\,\frac{\varphi}{\omega_{\rm rf}} - \frac{s}{\lambda\!\!\!^{-*}_{\rm rf}} \\ &= \frac{p_{\rm sy}}{\rho}\,x + \frac{p_{\rm sy}K}{2}\,x^2 + \frac{qV}{\omega_{\rm rf}}\sum_{j=-\infty}^{\infty}\delta(s-jL)\,\cos\left(\phi_0+\varphi+\frac{s}{\lambda\!\!\!^{-*}_{\rm rf}}\right) \\ &\quad - \left(1+\frac{x}{\rho}\right)\left[p_{\rm sy}^2 - \frac{2\omega_{\rm rf}U_{\rm sy}}{c^2}\,W + \frac{{\omega_{\rm rf}}^2}{c^2}\,W^2 - p_x^2\right]^{1/2} \\ &\quad - \frac{qV\sin\phi_0}{L}\,\frac{\varphi}{\omega_{\rm rf}} - \frac{s}{\lambda\!\!\!^{-*}_{\rm rf}}.\end{aligned} \tag{7.77}$$

[2]The true rf wavelength is actually $\lambda\!\!\!^{-}_{\rm rf} = L/2\pi h\beta$ and is a function of the momentum of the particle, whereas $\lambda\!\!\!^{-*}_{\rm rf}$ is a constant depending only on the design circumference and harmonic number.

With a few approximations and a bit more algebra this becomes

$$
\begin{aligned}
&H_1 \\
&\simeq \frac{p_{\rm sy}}{\rho}\, x + \frac{p_{\rm sy}K}{2}\, x^2 + \frac{qV}{\omega_{\rm rf}} \sum_{j=-\infty}^{\infty} \delta(s-jL)\, \cos\left(\phi_0 + \varphi + \frac{s}{\lambdabar^*_{\rm rf}}\right) \\
&\quad - p_{\rm sy}\left(1+\frac{x}{\rho}\right)\left[1 - \frac{\omega_{\rm rf}U_{\rm sy}}{p_{\rm sy}^2c^2}\, W + \left(\frac{{\omega_{\rm rf}}^2}{2p_{\rm sy}^2c^2} - \frac{1}{8}\,\frac{4U_{\rm sy}^2{\omega_{\rm rf}}^2}{p_{\rm sy}^4c^4}\right) W^2 - \frac{1}{2}\,\frac{p_x^2}{p_{\rm sy}^2}\right] \\
&\quad - \frac{qV\sin\phi_0}{L}\,\frac{\varphi}{\omega_{\rm rf}} - \frac{s}{\lambdabar^*_{\rm rf}} \\
&\simeq p_{\rm sy}\left\{ -1 + \frac{K}{2}\, x^2 + \frac{qV}{\omega_{\rm rf}p_{\rm sy}} \sum_{j=-\infty}^{\infty} \delta(s-jL)\, \cos\left(\phi_0 + \varphi + \frac{s}{\lambdabar^*_{\rm rf}}\right)\right. \\
&\quad \left. + \left(1+\frac{x}{\rho}\right)\left[\frac{\omega_{\rm rf}U_{\rm sy}}{p_{\rm sy}^2c^2}\, W + \frac{m^2{\omega_{\rm rf}}^2}{2p_{\rm sy}^4}\, W^2 + \frac{1}{2}\,\frac{p_x^2}{p_{\rm sy}^2}\right] - \frac{qV\sin\phi_0}{Lp_{\rm sy}}\,\frac{\varphi}{\omega_{\rm rf}} - \frac{s}{\lambdabar^*_{\rm rf}}\right\} \\
&\simeq p_{\rm sy}\left\{ -1 + \frac{K}{2}\, x^2 + \frac{qV}{\omega_{\rm rf}p_{\rm sy}} \sum_{j=-\infty}^{\infty} \delta(s-jL)\, \cos\left(\phi_0 + \varphi + \frac{s}{\lambdabar^*_{\rm rf}}\right)\right. \\
&\qquad + \left(1+\frac{x}{\rho}\right)\left[\frac{1}{\lambdabar^*_{\rm rf}}\,\frac{W}{p_{\rm sy}} + \frac{1}{\gamma^2{\lambdabar^*_{\rm rf}}^2}\left(\frac{W}{p_{\rm sy}}\right)^2 + \frac{1}{2}\left(\frac{p_x}{p_{\rm sy}}\right)^2\right] \\
&\qquad \left. - \frac{qV\sin\phi_0}{Lp_{\rm sy}}\,\frac{\varphi}{\omega_{\rm rf}} - \frac{s}{\lambdabar^*_{\rm rf}}\right\}
\end{aligned}
\tag{7.78}
$$

This is essentially the same as Eq. (CM: 7.61) but with the longitudinal variables written as

$$\varphi = \omega_{\rm rf}\,\Delta t, \quad \text{and} \quad W = -\frac{\Delta U}{\omega_{\rm rf}}, \tag{7.79}$$

and where L is the circumference, and for magnets with transverse fields and no horizontal-vertical coupling

$$K = \frac{1}{\rho^2} + \frac{q}{p}\left(\frac{\partial B_y}{\partial x}\right)_0. \tag{7.80}$$

If we want to calculate matrices for the basic magnetic elements, i. e., normal quads and dipoles, then the summation drops out, since $\delta(s-jL) = 0$ and $V = 0$ away from the rf cavities. Then keeping only terms to second order in the canonical variables we have

$$\begin{aligned} H_1 &\simeq -p_s + \frac{p_s K}{2}x^2 + \frac{p_x^2}{2p_s} \\ &\quad + \left(\frac{U_{\rm sy}\omega_{\rm rf}}{p_{\rm sy}c^2} - \frac{2\pi h}{L}\right)W + \frac{m^2{\omega_{\rm rf}}^2}{2p_{\rm sy}^3}W^2 + \frac{U_{\rm sy}\omega_{\rm rf}}{\rho p_{\rm sy}c^2}Wx \\ &\simeq -p_{\rm sy} + \frac{p_{\rm sy}K}{2}x^2 + \frac{p_x^2}{2p_{\rm sy}} + \frac{m^2{\omega_{\rm rf}}^2}{2p_{\rm sy}^3}W^2 + \frac{U_{\rm sy}\omega_{\rm rf}}{\rho p_{\rm sy}c^2}Wx, \end{aligned} \tag{7.81}$$

since the two terms in the coefficient of W cancel. We may rescale the Hamiltonian by $1/p_{\rm sy}$ getting

$$H_{1.5} \simeq -1 + \frac{K}{2}x^2 + \frac{1}{2}{w_x}^2 + \frac{1}{\gamma^2 {ƛ^*_{\rm rf}}^2}w_\phi^2 + \frac{1}{\rho ƛ^*_{\rm rf}}w_\phi x, \tag{7.82}$$

with the new canonical momenta

$$w_x = \frac{p_x}{p_{\rm sy}}, \quad \text{and} \tag{7.83}$$

$$\begin{aligned} w_\phi &= \frac{W}{p_{\rm sy}} = -\frac{\Delta U}{\omega_{\rm rf}p_{\rm sy}} = -\frac{\frac{\beta\gamma mc^3}{\gamma mc^2}}{\frac{2\pi h\beta c}{L}}\frac{\Delta p}{p_{\rm sy}} \\ &= -ƛ^*_{\rm rf}\frac{\Delta p}{p_{\rm sy}}. \end{aligned} \tag{7.84}$$

In this case with φ and w_ϕ as canonically conjugate the longitudinal emittance would have units of length (meters), just like the horizontal and vertical planes. (Of course this should be obvious since then all three emittances come from the common Hamiltonian $H_{1.5}$.) In the paraxial approximation, we obviously have $w_x \simeq x'$.

7.4.1 *Variations of the longitudinal canonical variables*

There are a several different combinations for the longintudinal canonical variables, for example:

$$\left(z, \frac{\Delta p}{p_0}\right), \tag{7.85}$$

$$\left(-c\Delta t,\ \frac{\Delta u}{p_0 c}\right), \tag{7.86}$$

$$\left(-\frac{(t-t_0)v_0\gamma_0}{\gamma_0+1}, \frac{K-K_0}{K_0}\right), \tag{7.87}$$

$$(\phi,\ W), \tag{7.88}$$

$$(\phi,\ w_\phi), \tag{7.89}$$

The two pairs in Eqs. (7.88 and 7.84) were explained in the previous chapter Eqs. (7.79 and 7.84). Differentiating the equation

$$U^2 = p^2c^2 + m^2c^4, \tag{7.90}$$

leads to the relation

$$du = \beta c\, dp, \tag{7.91}$$

or on converting to fractional deviations

$$\frac{dp}{p_0} = \frac{1}{\beta}\frac{du}{p_0 c} = \frac{1}{\beta^2}\frac{du}{U_0}. \tag{7.92}$$

Conversion from the pair in Eq. (7.85) to the pair in Eq. (7.86) used in the MAD[3,4] program may be accomplished by the simple transformation

$$\begin{pmatrix} z \\ \frac{\Delta p}{p_0} \end{pmatrix} = \begin{pmatrix} -\beta_0 c(t-t_0) \\ \frac{\Delta p}{p_0} \end{pmatrix} = \begin{pmatrix} -\beta_0 c\Delta t) \\ \beta_0^{-1}\frac{\Delta u}{p_0 c} \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_0^{-1} \end{pmatrix}\begin{pmatrix} -c\Delta t \\ \frac{\Delta u}{p_0 c} \end{pmatrix} \tag{7.93}$$

The usual definition of dispersion defined as the particular solution to the inhomogeneous horizontal Hill's equation in Eq. (CM: 5.77) must be modified by a factor of β_0 to agree with the value calculated by MAD.

The pair in Eq. (7.87) are used in the program COSY Infinity[5] which can be obtained as follows. If we start from the good canonical pair:

$$(\Delta t, -\Delta U) \tag{7.94}$$

and rescale as usual by dividing by p_0, we may write

$$\Delta t\left(-\frac{\Delta U}{p_0}\right) = \frac{c\Delta t\,\Delta K}{p_0 c}, \tag{7.95}$$

where $\Delta K = \Delta U$ is the change in kinetic energy $K = (\gamma - 1)mc^2$. Writing p_0c in terms of the kinetic energy, we have

$$\begin{aligned} p_0c = mc^2\gamma_0\beta_0 &= mc^2(\gamma_0-1)\sqrt{\frac{\gamma_0+1}{\gamma_0-1}} = K_0\sqrt{\frac{\gamma_0+1}{\gamma_0-1}} \\ &= \frac{K_0(\gamma_0+1)}{\sqrt{\gamma_0^2-1}} = \frac{K_0(\gamma_0+1)}{\gamma_0\beta_0}. \end{aligned} \tag{7.96}$$

Substituting this into Eq. (7.95) we find

$$\Delta t\left(-\frac{\Delta U}{p_0}\right) = -\frac{\Delta t\,\gamma_0 v_0}{\gamma_0+1}\frac{\Delta K}{K_0}, \tag{7.97}$$

and we see that Eq. (7.87) must also be a good canonical pair that preserves the area elements of longitudinal phase space.

7.4.2 *Transport matrices for a few elements*

Using the Lie algebraic method demonstrated in CM § 3.7, matrices for a drift, quadrupole, and sector bend may obtained:

$$\mathbf{M}_{\text{drift}} = \begin{pmatrix} 1 & l & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{l}{\gamma^2 \lambda_{\text{rf}}^{*\,2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.98}$$

$$\mathbf{M}_{\text{quad}} = \begin{pmatrix} \cos(\sqrt{k}l) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}l) & 0 & 0 \\ -\sqrt{k}\sin(\sqrt{k}l) & \cos(\sqrt{k}l) & 0 & 0 \\ 0 & 0 & 1 & \frac{l}{\gamma^2 \lambda_{\text{rf}}^{*\,2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.99}$$

$$\mathbf{M}_{\text{sbend}} = \begin{pmatrix} \cos\left[\sqrt{1-n}\theta\right] & \frac{\rho\sin\left[\sqrt{1-n}\theta\right]}{\sqrt{1-n}} & 0 & \frac{\rho\left(1-\cos\left[\sqrt{1-n}\theta\right]\right)}{(1-n)\lambda_{\text{rf}}^*} \\ \frac{\sqrt{1-n}\,\sin\left[\sqrt{1-n}\theta\right]}{\rho} & \cos\left[\sqrt{1-n}\theta\right] & 0 & \frac{\sin\left[\sqrt{1-n}\theta\right]}{\sqrt{1-n}\lambda_{\text{rf}}^*} \\ -\frac{\sin\left[\sqrt{1-n}\theta\right]}{\sqrt{1-n}\lambda_{\text{rf}}^*} & -\frac{\rho\left(1-\cos\left[\sqrt{1-n}\theta\right]\right)}{(1-n)\lambda_{\text{rf}}^*} & 1 & \frac{\rho}{\lambda_{\text{rf}}^{*\,2}}\left\{\frac{\theta}{\gamma^2} - \frac{\sqrt{1-n}\theta - \sin\left[\sqrt{1-n}\theta\right]}{(1-n)^{-3/2}}\right\} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.100}$$

where we have used $H_{1.5}$ with the canonical pairs (x, w_x) and (ϕ, w_ϕ) for the radial and longitudinal canonical variables.