Horizontal-Vertical coupling

With the exception of the solenoid matrix, all of our transport matrices have had no coupling terms between x and y motion.

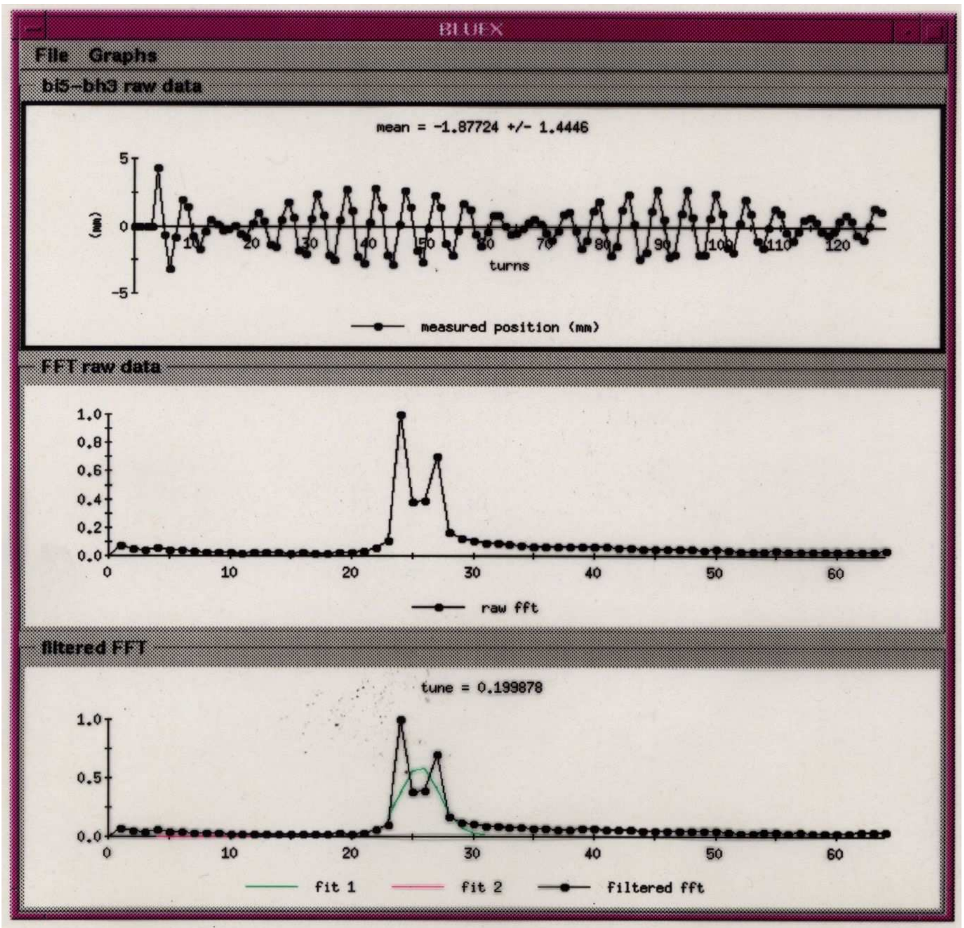
- In 2×2 block form: $\mathbf{T} = \begin{pmatrix} \mathbf{M}_{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathrm{V}} \end{pmatrix}.$
- A quadrupole rotated by 45° degrees is called a *skew quadrupole*:

$$\mathbf{Q}_{\mathrm{skew}} = \begin{pmatrix} \mathbf{I}\cos\frac{\pi}{4} & \mathbf{I}\sin\frac{\pi}{4} \\ -\mathbf{I}\sin\frac{\pi}{4} & \mathbf{I}\cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_{\mathrm{f}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{\mathrm{d}} \end{pmatrix} \begin{pmatrix} \mathbf{I}\cos\frac{\pi}{4} & -\mathbf{I}\sin\frac{\pi}{4} \\ \mathbf{I}\sin\frac{\pi}{4} & \mathbf{I}\cos\frac{\pi}{4} \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} \mathbf{Q}_{\mathrm{f}} + \mathbf{Q}_{\mathrm{d}} & \mathbf{Q}_{\mathrm{d}} - \mathbf{Q}_{\mathrm{f}} \\ \mathbf{Q}_{\mathrm{d}} - \mathbf{Q}_{\mathrm{f}} & \mathbf{Q}_{\mathrm{d}} - \mathbf{Q}_{\mathrm{f}} \end{pmatrix}.$$

In this case off-diagonal blocks will look something like:

$$\frac{\mathbf{Q}_{\mathrm{d}} - \mathbf{Q}_{\mathrm{f}}}{2} = \begin{pmatrix} \frac{1}{2}\left[\cosh(\sqrt{k}\,l) - \cos(\sqrt{k}\,l)\right] & \frac{1}{2\sqrt{k}}\left[\sinh(\sqrt{k}\,l) - \sin(\sqrt{k}\,l)\right] \\ \frac{\sqrt{k}}{2}\left[\sinh(\sqrt{k}\,l) + \sin(\sqrt{k}\,l)\right] & \frac{1}{2}\left[\cosh(\sqrt{k}\,l) - \cos(\sqrt{k}\,l)\right] \end{pmatrix}.$$

Tune measurement from turn-by-turn data



- Top: data from horiz. BPM.
  - Note beating of 2 freqs.
  - source: slight roll of quads.

- Middle: Raw FFT.
  - Tune from FFT (128 turns).
  - Two peaks.

$$q_u \simeq 0.187.$$

$$q_v \simeq 0.211.$$

- Bottom: Fit to single peak.
  - Double peak caused by HV coupling.
  - Fit peak at $q \sim 0.200$.

USPAS: Lectures on Coupling and Chromaticity
Waldo MacKay January, 2013

Is it possible to tranform to different canonical coordinates so that a transport matrix

$$\mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix}$$

is transformed into a new matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}?$$

- In most cases of stable motion, the answer is yes.
  Two methods are commonly used:

1. The Teng-Edwards symplectic rotation matrix formalism
   - Not totally robust.
   - Gives a nice interpretation for small couplings.
     Leaves interpretation almost with horiz-vertical meaning.
2. A method using eigenvectors to build the similarity tranformation.
   - More robust.
   - Can be generalized to higher dimensions.
   - Result may not correspond as easily to H-V interpretation.

Teng-Edwards rotation

Use the similarity transformation

$$\mathbf{T} = \mathbf{R}\mathbf{U}\mathbf{R}^{-1}.$$

The symplectic rotation

$$\mathbf{R} = \begin{pmatrix} \mathbf{I}\cos\phi & \mathbf{D}^{-1}\sin\phi \\ -\mathbf{D}\sin\phi & \mathbf{I}\cos\phi \end{pmatrix},$$

where $\mathbf{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1.$ (i. e. $\mathbf{D}$ is symplectic.)

The coordinates (x, P_x, y, P_y) change to *normal mode coordinates* (u, P_u, v, P_v):

$$\begin{pmatrix} u \\ P_u \\ v \\ P_v \end{pmatrix} = \mathbf{R}^{-1} \begin{pmatrix} x \\ P_x \\ y \\ P_y \end{pmatrix},$$

We have

$$\mathbf{U} = \mathbf{R}^{-1}\mathbf{T}\mathbf{R} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix},$$

where we may write

$$\mathbf{A} = \mathbf{I}\cos\mu_u + \mathbf{J}_u \sin\mu_u, \quad \text{and} \quad \mathbf{B} = \mathbf{I}\cos\mu_v + \mathbf{J}_v \sin\mu_v,$$

with

$$\mathbf{J}_u = \begin{pmatrix} \alpha_u & \beta_u \\ -\gamma_u & -\alpha_u \end{pmatrix}, \quad \text{and} \quad \mathbf{J}_v = \begin{pmatrix} \alpha_v & \beta_v \\ -\gamma_v & -\alpha_v \end{pmatrix}.$$

Teng and Edwards solution

$$\cos\mu_u - \cos\mu_v = \frac{1}{2}\mathrm{tr}(\mathbf{M}-\mathbf{N})\left[1 + \frac{2\det(\mathbf{m}) + \mathrm{tr}(\mathbf{nm})}{\left[\frac{1}{2}\mathrm{tr}(\mathbf{M}-\mathbf{N})\right]^2}\right]^{\frac{1}{2}}.$$

$$\cos(2\phi) = \frac{\frac{1}{2}\mathrm{tr}(\mathbf{M}-\mathbf{N})}{\cos\mu_u - \cos\mu_v}.$$

$$\mathbf{D} = -\frac{\mathbf{m}+\tilde{\mathbf{n}}}{(\cos\mu_u - \cos\mu_v)\sin(2\phi)}.$$

$$\mathbf{A} = \mathbf{M} - \mathbf{D}^{-1}\mathbf{m}\tan\phi.$$

$$\mathbf{B} = \mathbf{N} + \mathbf{D}\mathbf{n}\tan\phi.$$

Recall that $\tilde{\mathbf{n}} = \mathbf{S}\mathbf{n}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}$.

Example where Teng-Edwards method fails

$$\mathbf{T} = \begin{pmatrix} \cos\mu & \sin\mu + \frac{a^2}{\sin\mu} & a & 0 \\ -\sin\mu & \cos\mu & 0 & -a \\ a & 0 & \cos\mu & \sin\mu + \frac{a^2}{\sin\mu} \\ 0 & -a & -\sin\mu & \cos\mu \end{pmatrix}.$$

- I leave it to you to verify that $\mathbf{TST}^{\mathrm{T}} = \mathbf{S}$.
- Characteristic equation (if you work it out):

$$\begin{aligned} 0 &= \lambda^4 - \mathrm{tr}(\mathbf{T})(\lambda^3 + \lambda) + [2 + \mathrm{tr}(\mathbf{M})\mathrm{tr}(\mathbf{N}) - |\mathbf{m} + \tilde{\mathbf{n}}]\lambda^2 \\ &= (\lambda - e^{i\mu})^2(\lambda - e^{-i\mu})^2. \end{aligned}$$

  - The tunes are equal: $Q_u = Q_v = \frac{\mu}{2\pi}$.

◦ Note that $\mathbf{m} + \tilde{\mathbf{n}} = \mathbf{0}$, and therefore $\mathbf{D} = \mathbf{0}$, which does not satisfy $|\mathbf{D}| = 1$.

Eigenvector method

Assuming the motion is stable, this consists of finding the eigenvectors for $\mathbf{T}$:

$$\mathbf{T}\mathbf{v}_j = \lambda_j \mathbf{v}_j,$$

and constructing a similarity transformation: $\quad \mathbf{T} \to \mathbf{W}\mathbf{T}\mathbf{W}^{-1} = \mathbf{U}$
from the real and imaginary parts of the eigenvectors $\quad \mathbf{v}_j = \mathbf{a}_j \pm i\mathbf{b}_j,$
where $\mathbf{a}_j$ and $\mathbf{b}_j$ are the respective real and imaginary parts of $\mathbf{v}_j$.

It can be shown that the "rotation" matrix

$$\mathbf{W} = \begin{pmatrix} a_{1,1} & b_{1,1} & a_{2,1} & b_{2,1} & \cdots & b_{n,1} \\ a_{1,2} & b_{1,2} & a_{2,2} & b_{2,2} & \cdots & b_{n,2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1,2n} & b_{1,2n} & a_{2,2n} & b_{2,2n} & \cdots & b_{n,2n} \end{pmatrix}$$

block-diagonalizes the matrix $\mathbf{T}$, and is indeed symplectic if the eigenvectors are appropriately scaled.

For more details, see Iselin's unfinished (symphony):

F. Christoph Iselin, "The MAD Program (Methodical Accelerator Design) Version 8.13 **Physical Methods Manual**, CERN/SL/92-?? (AP) (1994).

- Warning: There are some typeohs (what's new) in that unfinished report.

Eigenvalues for 4x4 symplectic matrix

$$0 = |\mathbf{T} - \lambda \mathbf{I}| = \lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda^1 + A_0 = \prod_{j=1}^{4}(\lambda - \lambda_j).$$

The product of all four eigenvalues must give the determinant, so

$$A_0 = \lambda_1\lambda_2\lambda_3\lambda_4 = 1.$$

- Also remember that if λ is an eigenvalue, then so is λ^* since $\mathbf{T}$ is a real matrix.
- In the 2×2 case, the reciprocal of an eigenvalue must equal its complex conjugate.
- With four eigenvalues, the reciprocal and complex conjugate may be different eigenvalues.
- For stable motion, all four eigenvalues must lie on the unit circle in the complex plane.

Eigenvector equation:

$$\mathbf{T}\mathbf{v}_j = \lambda_j \mathbf{v}_j.$$

Also

$$\mathbf{T}^{-1}\mathbf{v}_j = \lambda_j^{-1}\mathbf{v}_j.$$

- $\mathbf{T}$ and $\mathbf{T}^{-1}$ have the same eigenvectors. Define $\mathbf{K} = \mathbf{T} + \mathbf{T}^{-1}$, then

$$\mathbf{K}v_j = (\mathbf{T} + \mathbf{T}^{-1})\mathbf{v}_j = (\lambda_j + \lambda_j^{-1})\mathbf{v}_j = \kappa \mathbf{v}_j.$$

- The characteristic equation of $\mathbf{K}$ is

$$0 = |\mathbf{K} - \mathbf{I}\kappa| = \sum_{j=0}^{2n} C_j \kappa^j = \left(\prod_{j=1}^{n} (\kappa - \kappa_j)\right)^2, \quad \text{with} \quad n = 2.$$

- In fact this holds for any $2n \times 2n$ symplectic $\mathbf{T}$. (See supplementary notes.)
- This reduces the problem from a $2n$-degree equation to only n-degrees.

The answer without the tedious algebra

For the 4×4 case:

$$\kappa = \lambda + \lambda^{-1} = \frac{\mathrm{tr}(\mathbf{M}+\mathbf{N})}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(\mathbf{M}-\mathbf{N})}{2}\right)^2 + |\mathbf{m}+\tilde{\mathbf{n}}|}.$$

$$\kappa_j = \lambda + \lambda^{-1} \quad \Rightarrow \quad 0 = \lambda^2 - \kappa_j\lambda + 1,$$

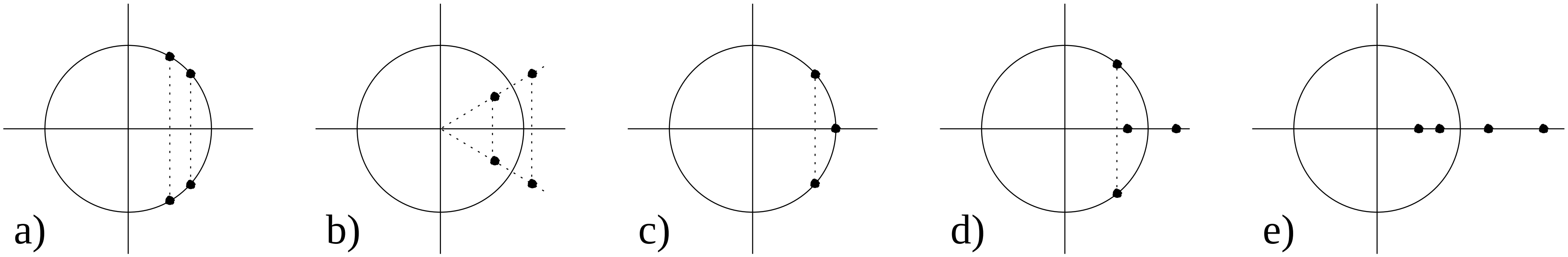
$$\lambda_{j\pm} = \frac{\kappa_j}{2} \pm \sqrt{\left(\frac{\kappa_j}{2}\right)^2 - 1}.$$

- For stability the λ's must be on the unit circle, so $\lambda = e^{\pm i\mu}$ for real μ.

$$\kappa = e^{i\mu} + e^{-i\mu} = 2\cos\mu.$$

- Require $-2 < \kappa < 2$ for guaranteed stability;
- values ± 2 on borderline.

Eigenvalues and Stability for 4x4 case



Possible configurations for eigenvalues.

- a: both planes stable.
- b, e: both planes unstable.
- c: one plane stable, other may or not be stable.
- d: one stable, other unstable.

Momentum dependence of focusing

Quadrupole focusing comes from

$$k = \frac{q}{p}\frac{\partial B_y(s)}{\partial x}, \qquad \text{where} \qquad p = p_0 \pm \Delta p.$$

So we can write:

$$k \simeq k_0\left(1 - \frac{\Delta p}{p}\right), \qquad \text{with} \qquad k_0 = \frac{e}{p_0}\frac{\partial B_y(s)}{\partial x}.$$

For an infinitesimal step through the quad, we find

$$d\mathbf{M}(0) = \begin{pmatrix} 1 & 0 \\ -k_0\,ds & 1 \end{pmatrix}, \qquad \text{and} \qquad d\mathbf{M}(\Delta p) = \begin{pmatrix} 1 & 0 \\ -k\,ds & 1 \end{pmatrix},$$

respectively for an on-momentum and off-momentum particle.

Thus, at location s, we find a change to the full turn transport matrix:

$$\mathbf{M} = [d\mathbf{M}(\Delta p)]\,[d\mathbf{M}(0)]^{-1} \begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix},$$

where μ_0 is the unperturbed phase advance of the whole machine with $\Delta p = 0$. Then

$$\mathbf{M}(s) = \begin{pmatrix} 1 & 0 \\ -k\,ds & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k_0\,ds & 1 \end{pmatrix} \begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}.$$

Multiplying the first two matrices yields:

$$\begin{pmatrix} 1 & 0 \\ -(k-k_0)ds & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k_0\delta\,ds & 1 \end{pmatrix}, \qquad \text{with} \qquad \delta = \frac{\Delta p}{p_0},$$

so the full product becomes:

$$\begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 + k_0\delta(\cos\mu_0 + \alpha\sin\mu_0)ds & \cos\mu_0 - \alpha\sin\mu_0 + k_0\delta\beta\sin\mu_0\,ds \end{pmatrix}$$

Taking half of the trace, we get

$$\begin{aligned}\cos\mu &= \frac{1}{2}\mathrm{tr}(\mathbf{M}) = \cos\mu_0 + \frac{k_0\delta}{2}\beta\sin\mu_0\,ds \\ &= \cos(\mu_0 + d\mu) \simeq \cos\mu_0 - \sin\mu_0\,d\mu.\end{aligned}$$

$$\frac{k_0\delta}{2}\,\beta\,\sin\mu_0\,ds = -\sin\mu_0\,d\mu.$$

$$d\mu = 2\pi\,dQ = -\frac{1}{2}\beta(s)k_0(s)\,ds\;\delta.$$

Integrating around the whole ring produces the tune shift:

$$\Delta Q = -\frac{1}{4\pi}\oint\beta(s)k_0(s)\,ds\;\frac{\Delta p}{p_0},$$

Natural chromaticity

The horizontal *natural chromaticity* is defined as

$$\xi_{x\mathrm{N}} = \left(\frac{\Delta Q_{\mathrm{H}}}{Q_{\mathrm{H}}}\right) \Big/ \left(\frac{\Delta p}{p_0}\right)$$
$$= -\frac{1}{4\pi Q_{\mathrm{H}}} \oint \beta_x(s) k_0(s)\, ds.$$

Similarly for vertical we have

$$\xi_{y\mathrm{N}} = \frac{1}{4\pi Q_{\mathrm{V}}} \oint \beta_y(s) k_0(s)\, ds,$$

since a horizontally focusing quad becomes defocusing in the vertical plane.

Residual chromaticity

If some magnetic field imperfections give rise to a perturbation of the kind

$$B_y = \sum_{n=2}^{\infty} b_n x^n,$$

$$\frac{\partial B_y}{\partial x} = 2b_2 x + \cdots, \quad \text{with} \quad x = \eta_x \delta + x_\beta,$$

then $k - k_0$ in $\begin{pmatrix} 1 & 0 \\ -(k-k_0)ds & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k_0\delta\, ds & 1 \end{pmatrix}$ must be replaced by

$$\left(-\frac{qG}{p_0} + 2\frac{q}{p_0} b_2 \eta_x\right)\frac{\Delta p}{p} + \ldots, \qquad \text{with} \qquad G = \left(\frac{\partial B_y}{\partial x}\right)_0 .$$

This yields the additional *residual chromaticity* in the horizontal plane:

$$\xi_{x\mathrm{R}} = \frac{1}{4\pi Q_{\mathrm{H}}}\frac{q}{p_0}\oint \beta_x(s)\, 2b_2(s)\eta_x(s)\, ds + \ldots .$$

Combining $\xi_{x\mathrm{N}}$ and $\xi_{x\mathrm{R}}$ gives

$$\xi_{x,\mathrm{total}} = -\frac{1}{4\pi Q_{\mathrm{H}}} \frac{q}{p} \oint \beta_x(s) \left[G(s) - 2b_2(s)\eta_x(s) \right] ds,$$

which may vanish, at least in principle, if

$$b_2(s) = \frac{G(s)}{2\eta_x(s)}.$$

- Sextupole lenses are generally used to compensate the natural chromaticity.

In the vertical plane we find the residual chromaticity

$$\xi_{y\mathrm{R}} = -\frac{1}{2\pi Q_{\mathrm{V}}} \frac{q}{p_0} \oint \beta_y(s) b_2(s) \eta_x(s)\, ds.$$

since the field components of a normal sextupole are

$$B_y = b_2(x^2 - y^2), \quad \text{and} \quad B_x = 2b_2 xy.$$