Single Gaussian bunch

$$dQ(s,t) = \frac{q}{\sqrt{2\pi}\sigma_s}\, e^{\frac{-(s-vt)^2}{2\sigma_s}}\, ds,$$

The distribution of current passing the pickup is then

$$\begin{aligned} i(t) = \frac{dQ}{dt} = v\,\frac{dQ}{ds} &= \frac{qv}{\sqrt{2\pi}\sigma_s}\, e^{\frac{-(s-vt)^2}{2\sigma_s}} \\ &= \frac{q}{\sqrt{2\pi}\sigma_t}\, e^{\frac{-(t-t_0)^2}{2\sigma_t}}, \end{aligned}$$

where $t_0 = \frac{s}{v}$. Fourier transform to get harmonic content: $(j = \sqrt{-1}.)$

$$\begin{aligned} \hat{\imath}(\omega) &= \frac{q}{\sqrt{2\pi}\sigma_t} \int_{-\infty}^{\infty} e^{-j\omega t}\, e^{-\frac{t^2}{2\sigma_t^2}}\, dt = \frac{q}{\sqrt{2\pi}\sigma_t} e^{-\frac{\sigma_t^2\omega^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(t+j\sigma_t^2\omega)^2}{2\sigma_t^2}}\, dt, \\ &= q\, e^{-\frac{\sigma_t^2\omega^2}{2}}. \end{aligned}$$

Infinitessimal bunch-length approximation:

$$i(t) = \lim_{\sigma_t \to 0} \frac{q}{\sqrt{2\pi}\sigma_t} e^{-\frac{t^2}{2\sigma_t}} = q\,\delta(t).$$

(Remember that the δ-function has the units of the inverse of its argument.)

So the spectral content becomes flat:

$$\hat{\imath}(\omega) = q \int_{-\infty}^{\infty} e^{-j\omega t}\,\delta(t)\,dt = q.$$

Circulating bunches

For N_p equally spaced bunches:

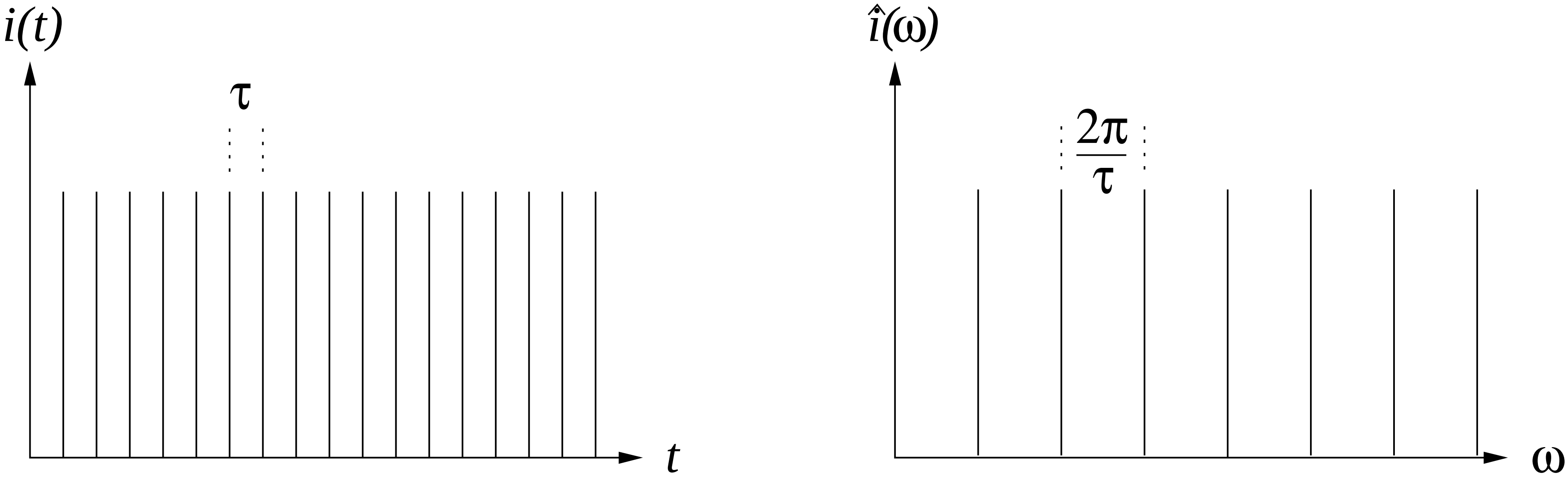
$$i(t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{N_p} \int_{-\infty}^{\infty} Q_m(t')\, \delta\left(t - t' - \frac{nmL}{N_p v}\right) dt',$$

where $Q_m(t') = dq_m/dt'$ is the longitudinal profile of charge in the m^{th} bunch.
Frequency spectrum:

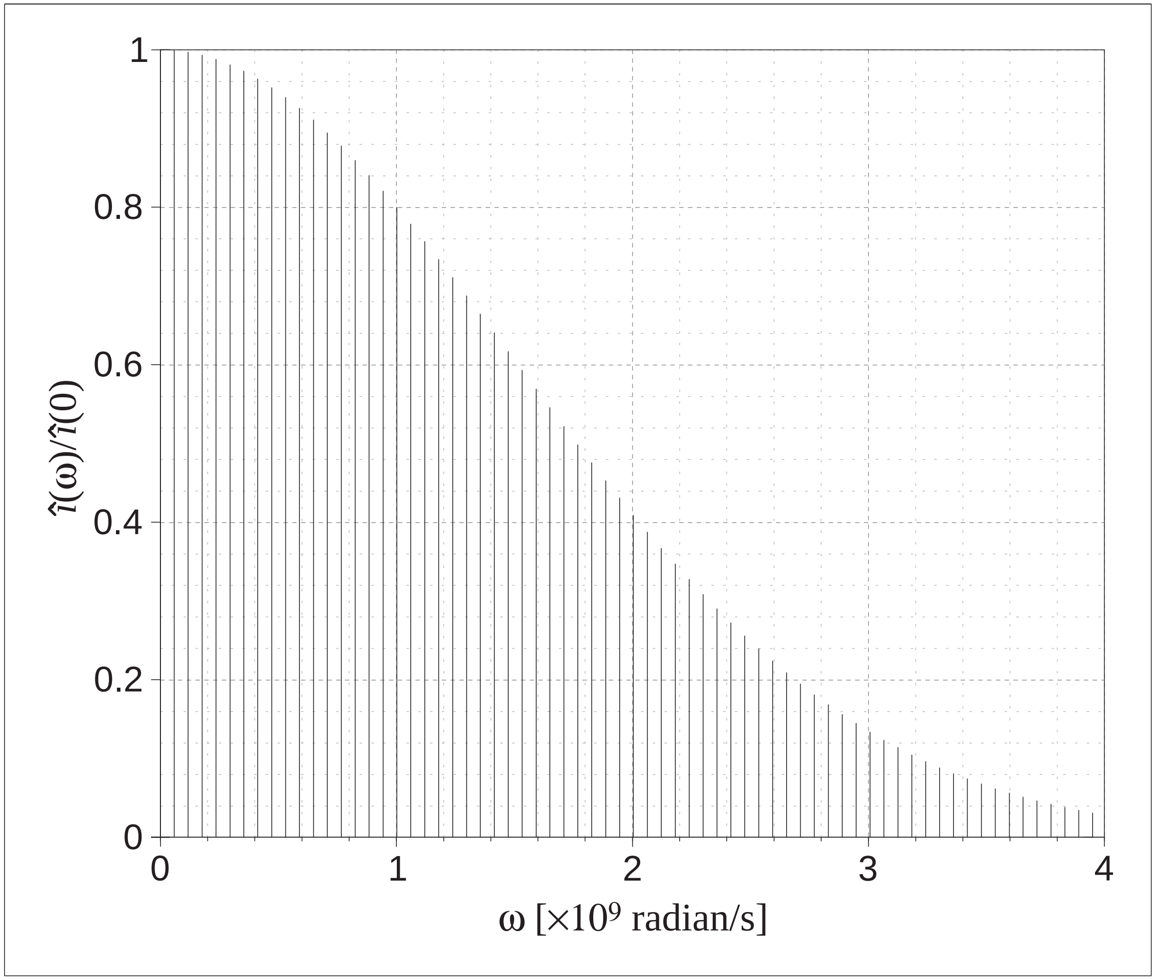
$$\begin{aligned}\hat{\imath}(\omega) &= \sum_{m=1}^{N_p} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\omega t}\, Q_m(t')\, \delta\left(t - t' - \frac{nmL}{N_p v}\right) dt'\, dt, \\ &= \sum_{m=1}^{N_p} \sum_{n=-\infty}^{\infty} \hat{Q}_m(\omega) \exp\left(-j\frac{nmL}{N_p v}\omega\right).\end{aligned}$$

For identical bunches this becomes

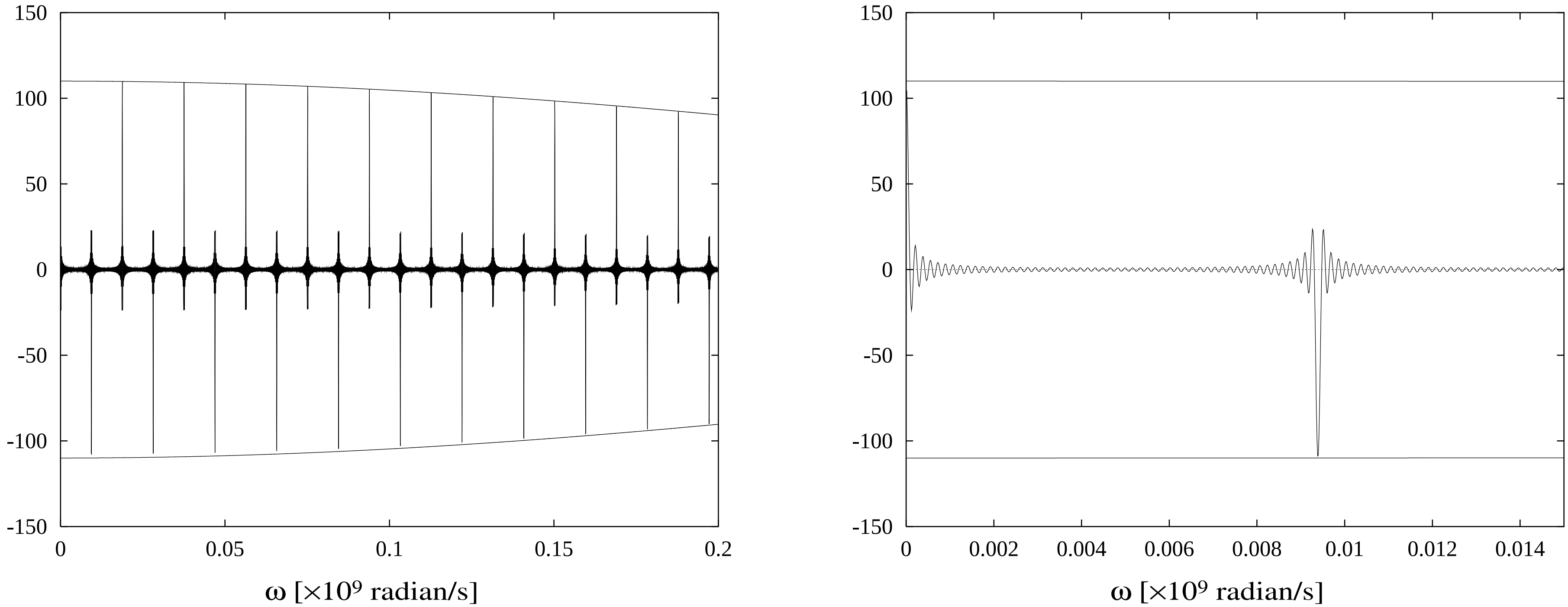
$$\hat{\imath}(\omega) = \hat{Q}(\omega) \sum_{n=-\infty}^{\infty} \exp\left(-j\frac{nL}{N_p v}\omega\right) = \frac{Q(\omega)}{N_p \omega_s} \sum_{n=-\infty}^{\infty} \delta\left(\omega - nN_p\omega_s\right).$$



- The left plot shows the current distribution for equal δ-function bunches of charge $Q_m(t') = q\delta(t')$ and spacing $\tau = L/N_p v$.
- The right plot shows the corresponding Fourier transform.



- If the bunches are Gaussian rather than δ-function shaped, then the comb has a Gaussian envelope.



- Leaving a gap of $N_p - N_b = 120 - 110$ bunches produces the comb with modulation by an *enhancement function*:

$$\mathcal{E}_n(\omega) = \frac{\sin\left(\frac{nN_b\tau_s}{2N_p}\omega\right)}{\sin\left(\frac{n\tau_s}{2N_p}\omega\right)}.$$

- The enlargement on the right shows the ripple caused by the gap of 10 missing bunches at a frequency of 78 kHz.

Long. Schottky spectrum for unbunched beam

- Particles of different momenta will have different revolution frequencies f_m.
- Current of m^{th} particle:

$$i_m(t) = qf_m \sum_{n=-\infty}^{\infty} e^{jn[\omega_m t+\psi_m]} = qf_m \left[1 + 2\sum_{n=1}^{\infty} \cos[n(\omega_m t + \psi_m)]\right]$$

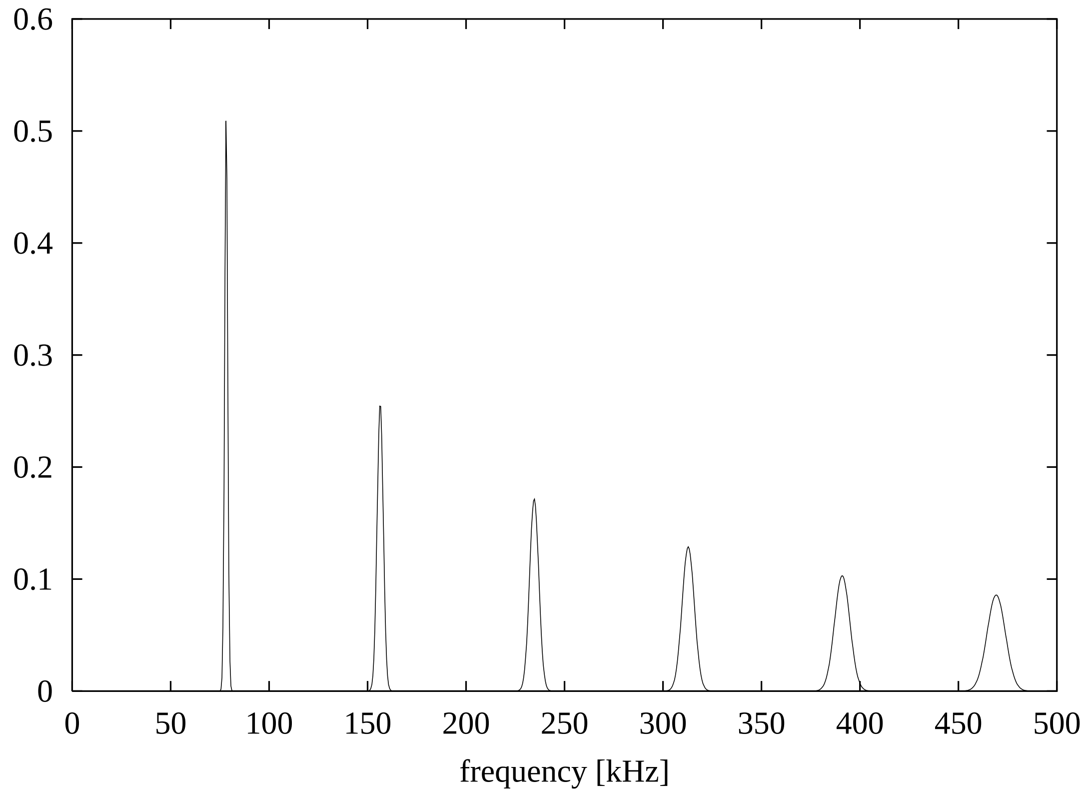
- n^{th} revolution harmonic has $\quad$ bandwidth $= n\sigma_f = nf_s\,|\eta_{\text{tr}}|\frac{\sigma_p}{p}$.

$$\langle i\rangle = \sum_{i=1}^{N} i_m(t) = Nq\langle f\rangle = Nqf_s,$$

$$\sigma_i^2 = \langle(i - \langle i\rangle)^2\rangle = \left\langle \left[\sum_{m=1}^{N} qf_m \left(1 + 2\sum_{n=1}^{\infty}\cos(n\omega_m t + \psi_m)\right) - Nqf_s\right]^2\right\rangle$$
$$= 2q^2 f_s^2 N.$$

So the rms current component of the n^{th} harmonic is independent of harmonic number:

$$\sigma_i = q f_s \sqrt{2N}.$$



- Plot of 1st six revolution harmonics. Area under each peak is identical.

Synchrotron oscillation of single particle

- Arrival time of particle at detector becomes:

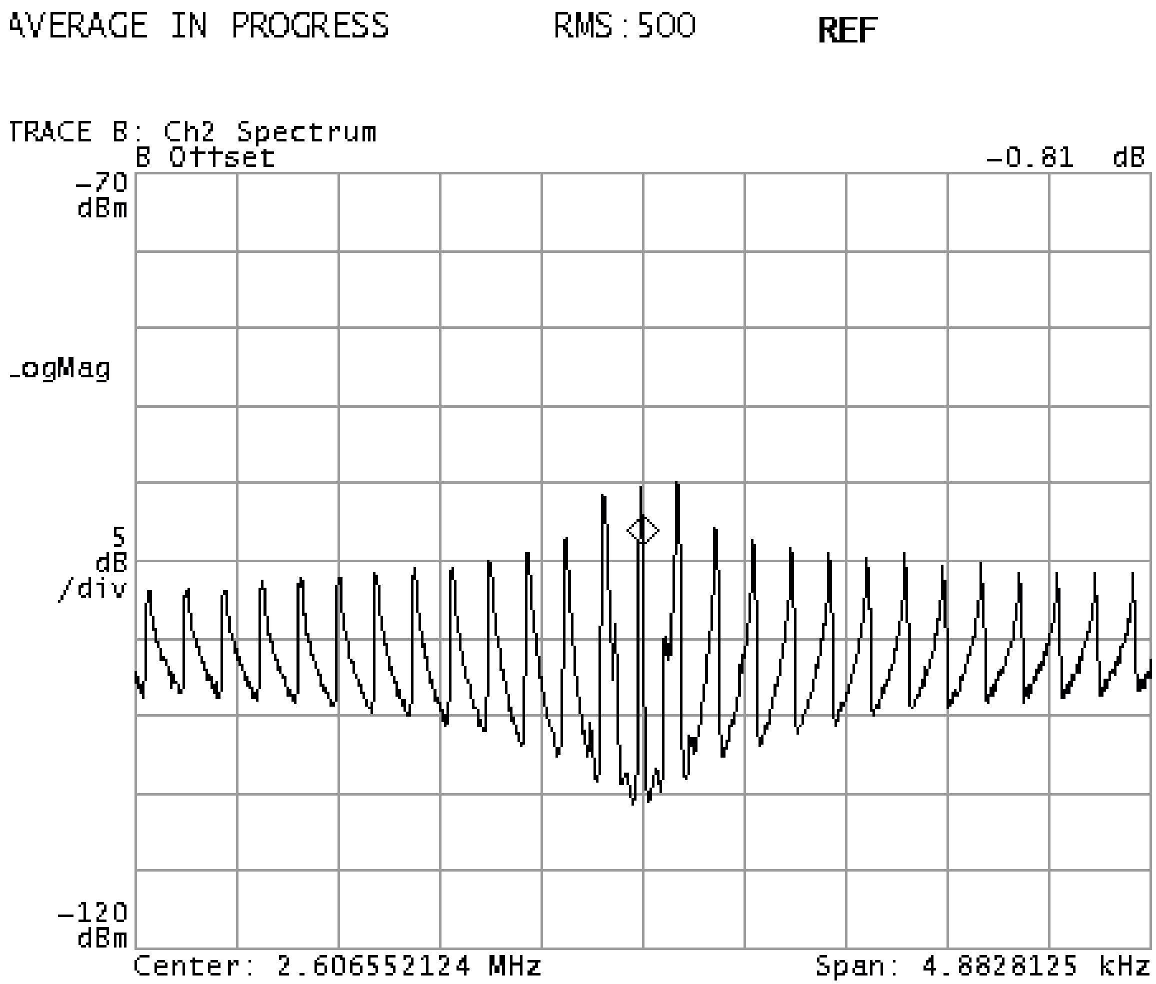
$$t \to t + a\sin(\Omega_s t + \psi)$$

- Current seen by detector on $n^{\rm th}$ turn:

$$\begin{aligned} i_n &= \sum_{n=-\infty}^{\infty} q\,\delta\left(t - n\left[\tau_s + a\sin(\Omega_s n\tau_s + \psi)\right]\right) \\ &\simeq \frac{q}{\tau_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s[\tau_s + a\sin(n\Omega_s\tau_s+\psi)]}, \end{aligned}$$

- Identity: $\quad e^{jz\sin\theta} = \sum_{m=-\infty}^{\infty} J_m(z)\,e^{j\theta}.$

$$i_n = q \sum_{n=-\infty}^{\infty} e^{-jn\omega_s\tau_s} \sum_{m=-\infty}^{\infty} J_m(n\omega_s a)\, e^{-jm(n\omega_s\tau_s+\psi)}.$$



Transverse oscillation

- Transverse signal of single particle is proportional to dipole oscillation:

$$d_m = a_m(t) i_m(t),$$

- $i_m(t)$ has components of synchrotron oscillation.
- $a_m(t) = a_m \cos(\mathrm{q}_m \omega_m t + \psi_m)$ has components of betatron oscillation.
  - q_m is fractional betatron tune.
  - ω_m is particle's revolution frequency.
  - ψ_m is particle's initial betatron phase.
- Sum over N particles

$$\langle d \rangle = 0.$$

$$\sigma_d = \sqrt{\langle (d_m - \langle d_m \rangle)^2 \rangle} = q f_s \sigma_a \sqrt{\frac{N}{2}}.$$

- Momentum spread and betatron tune spread contribute to give a nonzero width to the betatron line.

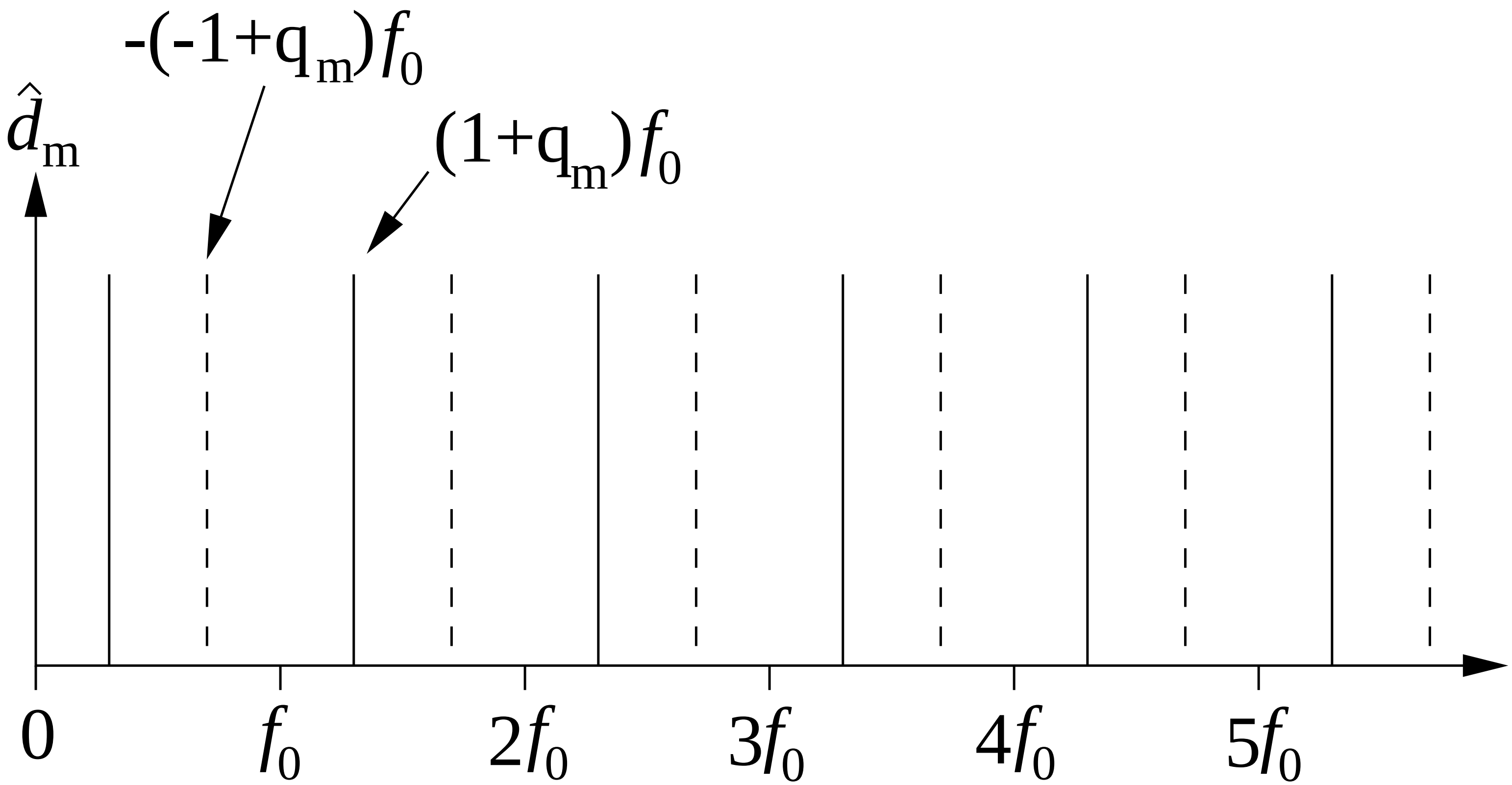
- If the betatron tune spread is only due to chromaticity, then

$$\sigma_{\mathrm{q}} = |(n+\mathrm{q})\eta_{\mathrm{tr}} + Q\xi|\,\frac{\sigma_p}{p}, \tag{1}$$

  where Q is the total betatron tune (including integer part) and ξ is the chromaticity

$$\xi = \frac{p}{Q}\,\frac{dQ}{dp}.$$

- Other contributions to the betatron tune spread which are not chromatic should be added in quadrature with Eq. (1), since they would be independent of the momentum oscillation.

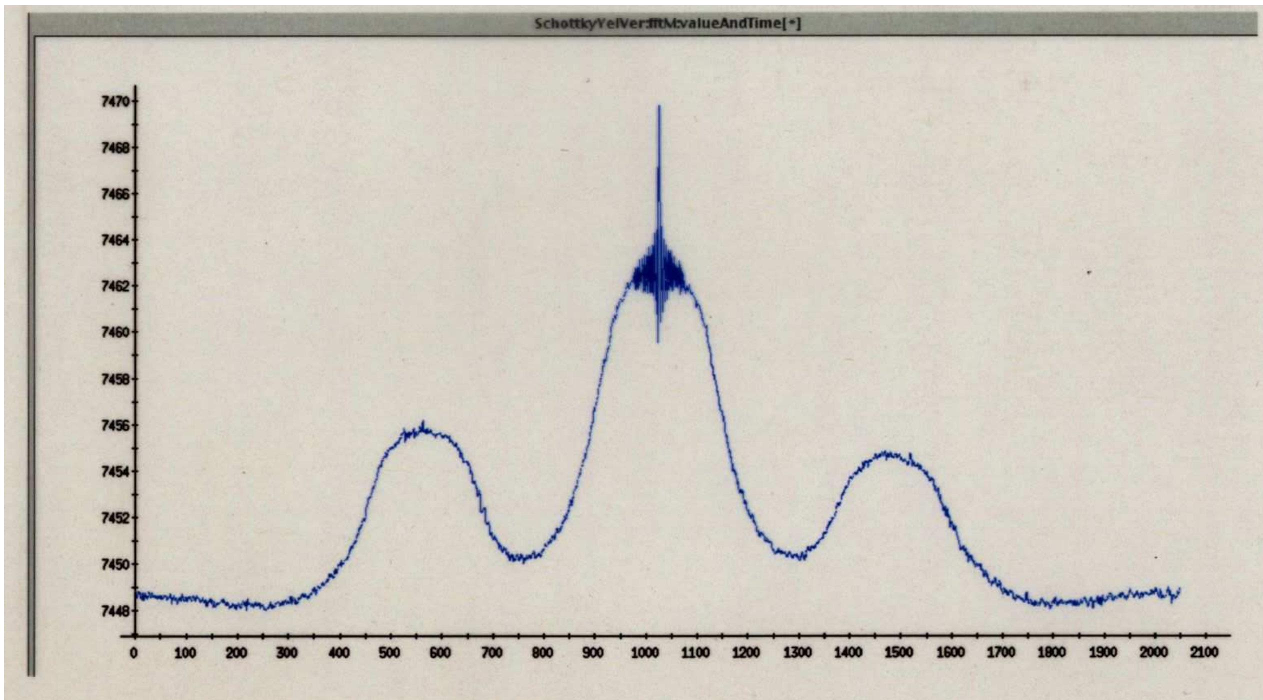


- Spectrum of betatron lines. The dashed lines are folded over from negative frequencies by plotting the absolute value $|(n + \mathrm{q})f_s|$. Here we have plotted lines for $\mathrm{q} < 0.5$.

For large values of n, the fractional part of the tune becomes negligible and the width of the band is

$$\sigma_{\mathrm{q}} \simeq |n\eta_{\mathrm{tr}} + Q\xi| \frac{\sigma_p}{p}.$$

We now see that the widths of the upper and lower sidebands are different since n can be either positive or negative. Whether the upper is narrower or wider depends on the signs of η_{tr}, n, and ξ.



- Measurement showing betatron and synchrotron sidebands around a revolution line.