Integer resonances

Let's start with a simplified formalism for the horizontal motion equation:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = f(\theta),$$

where

- $\theta = s/R$ is the azimuthal angle around the ring with
- R being the average radius of the ring,
  i. e. we approximate by a circular ring.
- $f(\theta)$ is some source of perturbations from errors.

Fourier transform the function f and let's look at the m^{th} harmonic term:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \varepsilon \cos(m\theta). \tag{1}$$

$$\frac{d^2x}{d\theta^2} + {Q_{\rm H}}^2 x = \varepsilon \cos(m\theta). \tag{1}$$

Solution to Eq. (1) is of the form

$$x = \tilde{x} + \bar{x},$$

with homogeneous part

$$\tilde{x} = A\cos(Q_{\rm H}\theta) + B\sin(Q_{\rm H}\theta),$$

an inhomogeneous part

$$\bar{x} = \frac{\varepsilon}{{Q_{\rm H}}^2 - m^2}\left[\cos(m\theta) - \cos(Q_{\rm H}\theta)\right]$$

$$\bar{x} = \frac{\varepsilon\theta}{Q_{\rm H}+m}\ \sin\left(\frac{Q_{\rm H}+m}{2}\,\theta\right)\ \frac{2}{(Q_{\rm H}-m)\theta}\ \sin\left(\frac{Q_{\rm H}-m}{2}\,\theta\right).$$

which reduces to $\bar{x} \simeq \dfrac{\varepsilon\theta}{2Q_{\rm H}}\sin(Q_{\rm H}\theta), \qquad \text{for } Q_{\rm H} = m.$

Did that go by too fast?

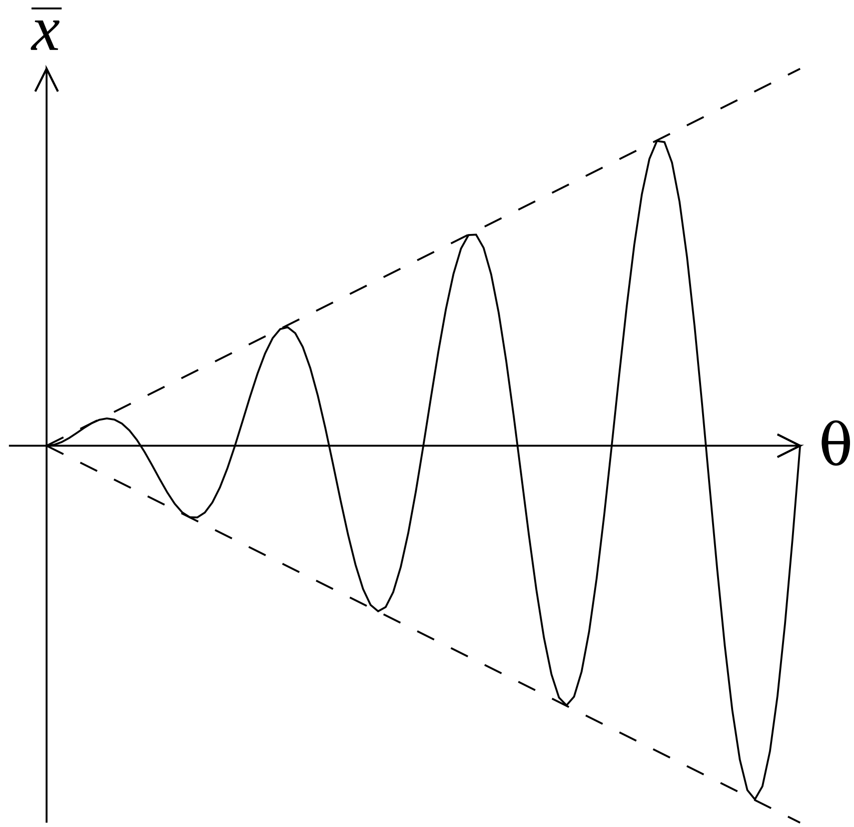
Trigonometric identity:

$$
\begin{aligned}
\sin(A+B)\sin(A-B) \\
&= (\sin A\cos B + \sin B\cos A)(\sin A\cos B - \sin B\cos A) \\
&= \sin^2 A\cos^2 B - \sin^2 B\cos^2 A \\
&= \frac{1}{2}(1+\cos 2A)\frac{1}{2}(1-\cos 2B) - \frac{1}{2}(1-\cos 2A)\frac{1}{2}(1+\cos 2B) \\
&= \frac{1}{2}[\cos 2A - \cos 2B],
\end{aligned}
$$

having used the double-angle formulae for $\cos^2\theta$ and $\sin^2\theta$:

$$
\begin{aligned}
\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta), \\
\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta).
\end{aligned}
$$

Linear growth from integer resonance



$$\bar{x}(\theta) = \frac{\varepsilon}{2Q_{\mathrm{H}}}\theta\,\sin(Q_{\mathrm{H}}\theta).$$

Simpler argument

- On an integer resonance, μ is a multiple of 2π, we expect $\mathbf{M} = \mathbf{I}$.

If there is a small path-length error δl in one drift section, then the 1-turn matrix becomes

$$\mathbf{M} = \begin{pmatrix} 1 & \delta l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \delta l \\ 0 & 1 \end{pmatrix}.$$

Any particle with $x_0' \neq 0$ will propagate as

$$\begin{pmatrix} x_n \\ x_n' \end{pmatrix} = \begin{pmatrix} 1 & \delta l \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x_0 + n\, x_0'\, \delta l \\ x_0' \end{pmatrix}.$$

This grows linearly with turn number n.

Linear coupling resonances

Now we consider a slight amount of H-V coupling with equations:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \varepsilon \cos(m\theta)\, y, \quad \text{and}$$
$$\frac{d^2y}{d\theta^2} + {Q_\mathrm{V}}^2 y = \varepsilon \cos(m\theta)\, x.$$

- Assume ε is very small, and substitute the solutions of the homogeneous equations for x and y into the corresponding inhomog. terms on the rhs:

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \frac{1}{2}\varepsilon_y \left[\cos(Q_\mathrm{V}+m)\theta + \cos(m-Q_\mathrm{V})\theta\right], \quad \text{and}$$
$$\frac{d^2y}{d\theta^2} + {Q_\mathrm{V}}^2 y = \frac{1}{2}\varepsilon_x \left[\cos(Q_\mathrm{H}+m)\theta + \cos(m-Q_\mathrm{H})\theta\right],$$

where ε_x and ε_y contain the respective amplitude information of the homogeneous solutions.

The same arguments as in the previous section lead to the resonance conditions

$$
\begin{aligned}
Q_{\rm H} + Q_{\rm V} &= m, \quad \text{and} \\
|Q_{\rm H} - Q_{\rm V}| &= m,
\end{aligned}
$$

which classify linear sum and difference resonances, respectively.

The sum and difference resonances behave differently as a little coupling is added to an ideal uncoupled lattice.
Consider the uncoupled 1-turn transfer matrix:

$$
\mathbf{T} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_2 \end{pmatrix} =
$$

$$
\begin{pmatrix}
\cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 & 0 & 0 \\
-\gamma_1 \sin\mu_1 & \cos\mu_1 - \alpha_1 \sin\mu_1 & 0 & 0 \\
0 & 0 & \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \\
0 & 0 & -\gamma_2 \sin\mu_2 & \cos\mu_2 - \alpha_2 \sin\mu_2
\end{pmatrix}
$$

- Diff. res. condition: $\sin\mu_1 = \sin\mu_2$,
- Sum res. condition: $\sin\mu_1 = -\sin\mu_2$.

A common source of transverse coupling is a slight roll of quad by angle θ:
Let's assume that the last element in $\mathbf{T}$ is the thin quadrupole:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 1 \end{pmatrix}.$$

Estimate effect of rolled thin quad:

$$\mathbf{T}' = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} = \mathbf{R}\mathbf{Q}\mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{T}$$

$$= \begin{pmatrix} \mathbf{I}\cos\theta & \mathbf{I}\sin\theta \\ -\mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{I}\cos\theta & -\mathbf{I}\sin\theta \\ \mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_2 \end{pmatrix}.$$

Fast forward skipping a bit of algebra:

$$\mathbf{T}' = \begin{pmatrix} (\mathbf{I}\cos^2\theta + \mathbf{D}^2\sin^2\theta)\mathbf{u}_1 & (\mathbf{I} - \mathbf{F}^2)\mathbf{u}_2\cos\theta\sin\theta \\ (\mathbf{D}^2 - \mathbf{I})\mathbf{u}_1\cos\theta\sin\theta & (\mathbf{I}\cos^2\theta + \mathbf{F}^2\sin^2\theta)\mathbf{u}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{f}\sin^2\theta & 1 \end{pmatrix}\mathbf{u}_1 & \begin{pmatrix} 0 & 0 \\ \frac{2}{f}\cos\theta\sin\theta & 0 \end{pmatrix}\mathbf{u}_2 \\ \begin{pmatrix} 0 & 0 \\ \frac{2}{f}\cos\theta\sin\theta & 0 \end{pmatrix}\mathbf{u}_1 & \begin{pmatrix} 1 & 0 \\ -\frac{2}{f}\sin^2\theta & 1 \end{pmatrix}\mathbf{u}_2 \end{pmatrix}.$$

More algebra . . .

$$\mathbf{M} = \mathbf{u}_1 \cos^2\theta + \frac{2\sin^2\theta}{f} \begin{pmatrix} 0 & 0 \\ \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 \end{pmatrix},$$

$$\mathbf{N} = \mathbf{u}_2 \cos^2\theta - \frac{2\sin^2\theta}{f} \begin{pmatrix} 0 & 0 \\ \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \end{pmatrix},$$

$$\mathbf{m} = \begin{pmatrix} 0 & 0 \\ \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 \end{pmatrix} \frac{\sin 2\theta}{f},$$

$$\mathbf{n} = \begin{pmatrix} 0 & 0 \\ \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \end{pmatrix} \frac{\sin 2\theta}{f}.$$

Recall from our previous discussion of coupling (in CM:§ 6.9):

$$\kappa = \lambda + \lambda^{-1} = \frac{\mathrm{tr}(\mathbf{M}+\mathbf{N})}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(\mathbf{M}-\mathbf{N})}{2}\right)^2 + |\mathbf{m}+\tilde{\mathbf{n}}|}.$$

For either resonance condition, $\cos\mu_1 = \cos\mu_2$, so $\mathrm{tr}(\mathbf{u}_1) = \mathrm{tr}(\mathbf{u}_2)$.

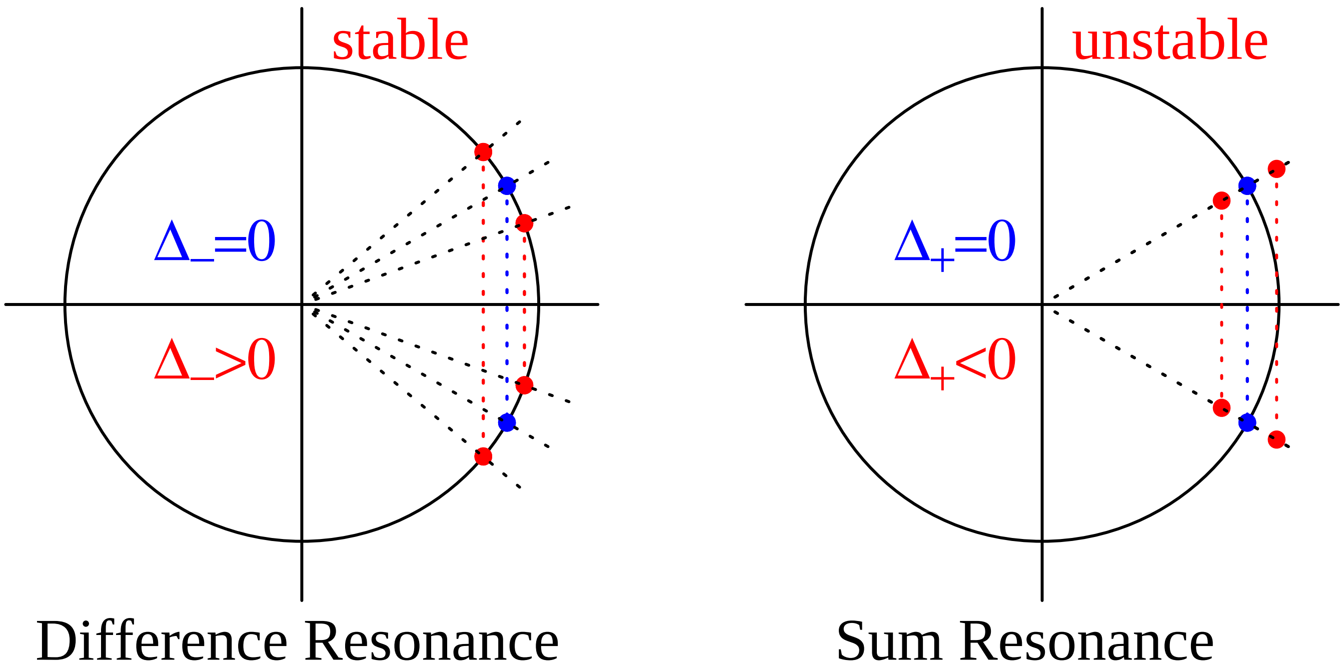
$$\frac{\mathrm{tr}(\mathbf{M}-\mathbf{N})}{2} = \frac{\beta_1 \sin\mu_1 - \beta_2 \sin\mu_2}{f} \sin^2\theta,$$

$$|\mathbf{m}+\tilde{\mathbf{n}}| = \frac{\beta_1\beta_2}{f^2} \sin^2(2\theta) \sin\mu_1 \sin\mu_2,$$

- Notice that $|\mathbf{m}+\tilde{\mathbf{n}}| \neq 0$ if there is a slight roll of the quadrupole.
- The sign of $|\mathbf{m}+\tilde{\mathbf{n}}|$ is determined solely by the product $\sin\mu_1 \sin\mu_2$. For the slightly coupled $\mathbf{T}'$, the argument of the radical is

$$\begin{aligned}
\Delta_\pm &= \left(\frac{\mathrm{tr}(\mathbf{M}-\mathbf{N})}{2}\right)^2 + |\mathbf{m}+\tilde{\mathbf{n}}| \\
&= \frac{\sin^4\theta}{f^2}(\beta_1 \sin\mu_1 - \beta_2 \sin\mu_2)^2 + \frac{\beta_1\beta_2}{f^2} \sin^2(2\theta) \sin\mu_1 \sin\mu_2 \\
&= \frac{\sin^4\theta}{f^2}(\beta_1 \pm \beta_2)^2 \sin^2\mu_1 \mp \frac{\beta_1\beta_2}{f^2} \sin^2(2\theta) \sin^2\mu_1 \\
&\simeq \mp \frac{4\beta_1\beta_2 \sin^2\mu_1}{f^2} \theta^2, \qquad \text{for small } \theta.
\end{aligned}$$

- As θ increases away from zero, the degenerate eigenvalues are pushed apart:
  1. In the case of a **difference resonance**, $\Delta_- > 0$, and the degenerate λ_j eigenvalue pairs split apart by moving along the unit circle in the complex plane. Since the eigenvalues stay on the circle, the motion remains **stable** with $\lambda_j^* = \lambda_j^{-1}$.
  2. For a **sum resonance**, $\Delta_+ < 0$, and the λ_j eigenvalues move away from the unit circle out into the complex plane resulting in **unstable** motion with $\lambda_j^* \neq \lambda_j^{-1}$.



Higher order (nonlinear) resonances

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \varepsilon \frac{\partial B_x}{\partial y} x \cos(m\theta), \quad \text{and}$$
$$\frac{d^2y}{d\theta^2} + {Q_\mathrm{V}}^2 y = \varepsilon \frac{\partial B_x}{\partial y} y \cos(m\theta)$$

$$\frac{\partial B_x}{\partial y} = \mathrm{Re}\left(i \sum_{n=0}^{\infty} n(a_n - ib_n)(x+iy)^{n-1}\right) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} nb_n(-1)^j \binom{n-1}{2j} x^{n-1-2j} y^{2j}.$$

Start with solutions,

$$x = A_1 \cos(Q_\mathrm{H}\theta), \quad \text{and} \quad y = A_2 \cos(Q_\mathrm{V}\theta),$$

$$\frac{d^2x}{d\theta^2} + {Q_\mathrm{H}}^2 x = \varepsilon n b_n \cos(m\theta) \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1}{2j} A_1^{n-2j} A_2^{2j} \cos^{n-2j}(Q_\mathrm{H}\theta) \cos^{2j}(Q_\mathrm{V}\theta).$$

Lots of algebra

$$\cos(m\theta)\cos^p(Q_\mathrm{H}\theta)\cos^q(Q_\mathrm{V}\theta)$$
$$= 2^{-(p+q)}\sum_{k=0}^{p}\sum_{l=0}^{q}\binom{p}{k}\binom{q}{l}\cos\{[(p-2k)Q_\mathrm{H}+(q-2l)Q_\mathrm{V}-m]\theta\}.$$

$$\vdots$$

For the x-equation, $p = n - 2j$, and $q = 2j$, giving resonances when

$$[n \pm 1 - 2(j+k)]Q_\mathrm{H} + 2(j-l)Q_\mathrm{V} = \pm m.$$

For the y-equation, $p = n-1-2j$, and $q = 2j+1$, giving the additional conditions

$$[n - 1 - 2(j+k)]Q_\mathrm{H} + [1 \pm 1 + 2(j-l)]Q_\mathrm{V} = \pm m.$$

$$\vdots$$

Results

- A normal quadrupole error excites the half-integer resonances:

$$2Q_{\mathrm{H}} = \pm m, \quad \text{and} \quad 2Q_{\mathrm{V}} = \pm m.$$

- Normal octopole:

$$\begin{aligned} \pm 4Q_{\mathrm{H}} &= m, \\ \pm 4Q_{\mathrm{V}} &= m, \\ \pm 2Q_{\mathrm{H}} &= m, \\ \pm 2Q_{\mathrm{V}} &= m, \quad \text{and} \\ \pm 2Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m. \end{aligned}$$

  Notice that the resonances driven by the normal quadrupole are also driven by the octopole.

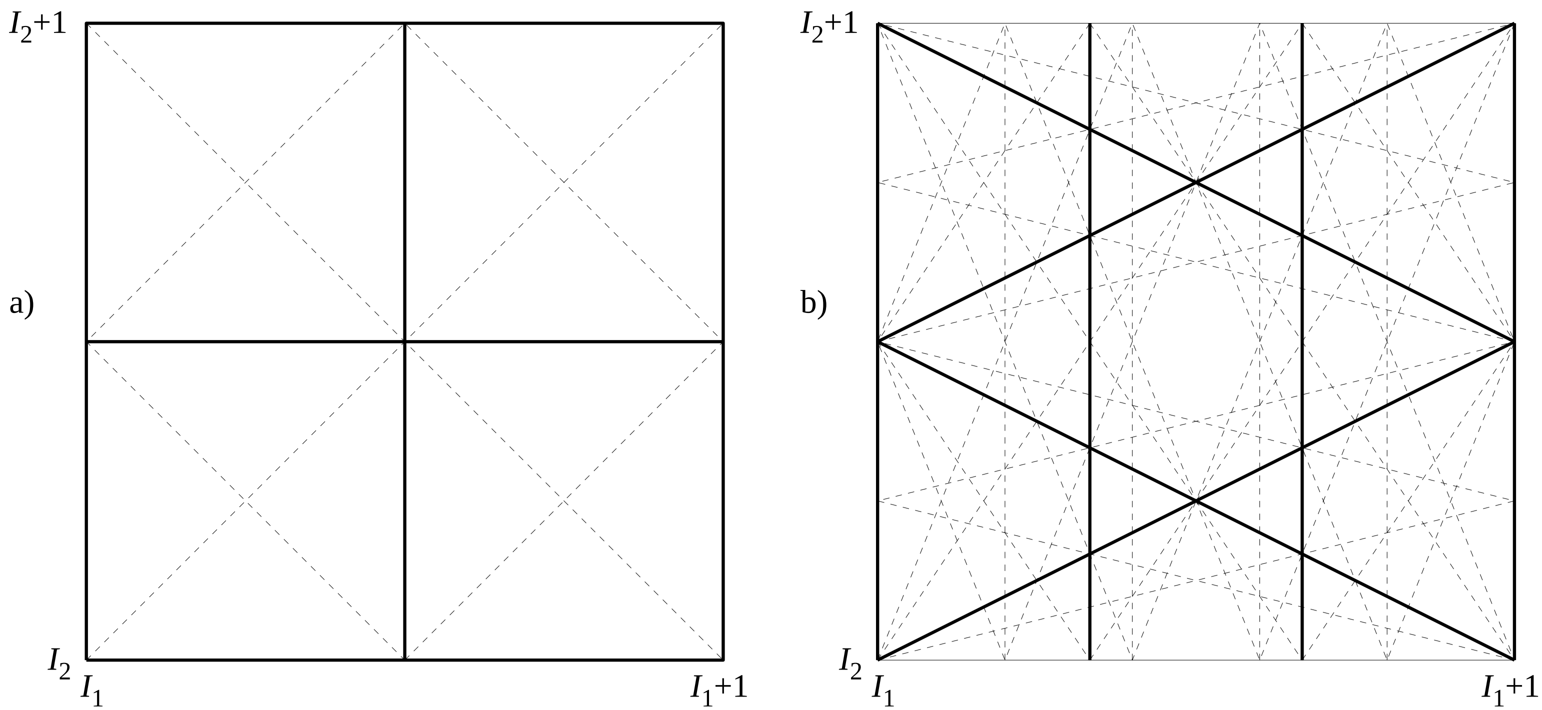
- Normal sextupole:

$$\begin{aligned} \pm 3Q_{\mathrm{H}} &= m, \\ \pm Q_{\mathrm{H}} &= m, \quad \text{and} \\ \pm Q_{\mathrm{H}} \pm 2Q_{\mathrm{V}} &= m. \end{aligned}$$

- Normal decapoles:

$$\begin{aligned}
\pm 5Q_{\rm H} \pm 2Q_{\rm V} &= m, \\
\pm 5Q_{\rm H} &= m, \\
\pm 3Q_{\rm H} \pm 2Q_{\rm V} &= m, \\
\pm 3Q_{\rm H} &= m, \\
\pm Q_{\rm H} \pm 4Q_{\rm V} &= m, \\
\pm Q_{\rm H} \pm 2Q_{\rm V} &= m, \quad \text{and} \\
\pm Q_{\rm H} &= m.
\end{aligned}$$

Lines from normal multipoles

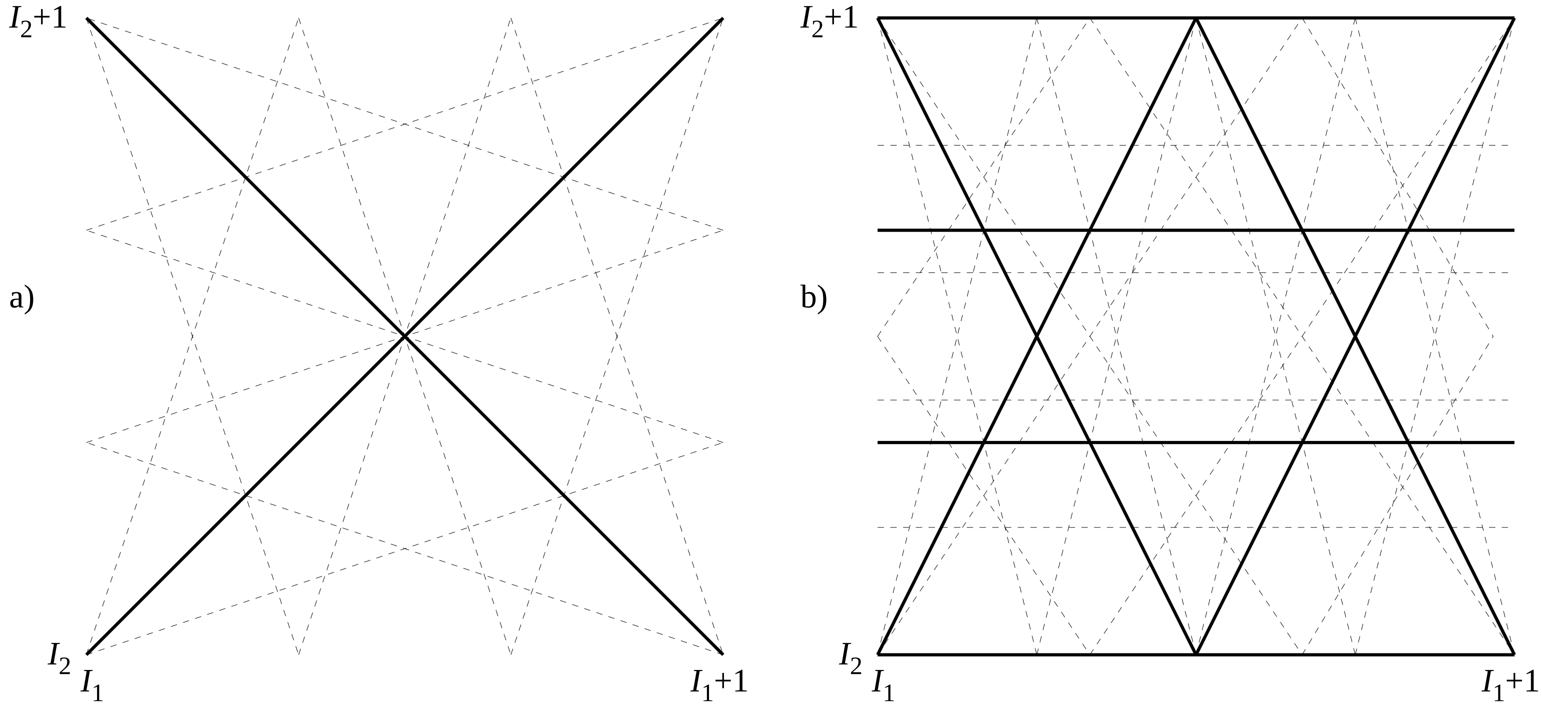


a) A tune plot showing the resonance lines driven by a normal quadrupole perturbation (heavy lines), and a normal octopole perturbation (all lines). I_1 and I_2 are arbitrary integers.

b) A tune plot showing the resonance lines driven by a normal sextupole (heavy lines), and a normal decapole (heavy and dashed lines).

- Typically: Positive slopes (diff res) OK; Negative slopes (sum res) bad.

Lines from skew multipoles



a) Skew quad lines (solid) and skew octopole lines (bold and dashed).

b) Skew sextupole (bold) and skew decapole (bold and dashed) lines.

- Again: Positive slopes (diff res) OK; Negative slopes (sum res) bad.

Periodicity