Poisson Brackets and Lie Operators

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1 Symplecticity and Poisson Brackets

1.1 Symplecticity

Consider an n-dimensional (2n-dimensional phase space) linear system. Let the canonical coordinates of the system be

$$X = \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \\ \cdots \\ q_n \\ p_n \end{pmatrix} \tag{1}$$

When we index X_i here, $0 < i < 2n$, including both q and p. Let M be the $2n \times 2n$ matrix that describes the map that brings the coordinates of the particles from the initial position $s = 0$ to the time of observation s in this linear dynamical system: $X = MX_0$. Then M must satisfy the symplecticity condition

$$M^{\mathrm{T}} S M = S \tag{2}$$

where the matrix S is the block-diagonal symplectic form:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_n \tag{3}$$

Note that all symplectic matrices are necessarily even dimensional. Since $S^2 = -I$, S may be thought of as the matrix equivalent of $i = \sqrt{-1}$. This analogy extends to the point that exponentials of symplectic matrices are similar to rotations.

What is remarkable is that the symplecticity condition, Eq. (2), applies also to a nonlinear system if we identify M to be the Jacobian matrix of the map, whose elements are defined as

$$X = MX_0 \qquad M_{ij} = \frac{\partial X_i}{\partial (X_0)_j} \tag{4}$$

where $(X_0)_j$ is the j^{th} component of the initial coordinates of a particle at $s = 0$ (including both coordinates and momenta!), and X_i is the i^{th} component of the final state X of the particle at arbitrary time s. In a linear system, the Jacobian is just the transformation matrix, and is independent of the particle coordinates. In a nonlinear system the Jacobian matrix M depends on the coordinates of X_0 and the symplecticity condition Eq. (2) must be satisfied for all X_0.

The symplecticity condition resembles a unitarity condition since the left hand side of Eq. (2) is quadratic in M, while the right hand side is almost a unit matrix. This imposes very strong constraints on M. Some things immediately follow from the symplecticity condition:

1. S and I are both symplectic.
2. If M is symplectic, then $\det(M) = \pm 1$. (We restrict ourselves to $\det(M) = +1$.)
3. M is invertible, with $M^{-1} = S^{-1} M^T S = S^T M^T S$.
4. If M is symplectic, so are M^{T} and M^{-1}.
5. If both M and N are symplectic, then MN is symplectic.
6. If λ is an eigenvalue of a symplectic matrix M, then so is $1/\lambda$.

This is already starting to look algebraic.

One bit of magic is that *all Hamiltonian systems are symplectic*. This includes both linear and nonlinear Hamiltonian systems, or even when the Jacobian depends on the coordinates! You proved this in your homework last week. For linear systems, the map is independent of X and X_0, so the symplectic condition only has to hold for all time s. However, for nonlinear systems where the map depends on X_0, the symplectic condition must hold for all s and X_0. That's a strong constraint!

The symplectic condition imposes a total of $n(2n-1)$ constraints since $M^{\mathrm{T}} S M = S$ is antisymmetric, so the $2n \times 2n$ matrix M therefore has $n(2n+1)$ independent elements. In the $n = 1$ case, there is only one constraint (unit determinant), and 3 independent elements; for $n = 2$ there are 6 constraints and 10 independent elements. When we get to treating the group of symplectic matrices as a Lie algebra, the independent elements give rise to the generators of the group.

1.2 Poisson Brackets

The Poisson bracket between functions $f(X; s)$ and $g(X; s)$ of canonical coordinates might be familiar from Hamiltonian mechanics:

$$[f, g] \equiv \sum_{i=0}^{2n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{5}$$

$$= \sum_{i,j} \frac{\partial f}{\partial X_i} S_{ij} \frac{\partial g}{\partial X_j} \tag{6}$$

where again the i, j range over all coordinates and momenta. Just like symplectic matrices, the Poisson bracket has some handy properties:

1. Antisymmetry: $[f, g] = -[g, f]$
2. Real distributivity: $[af + bg, h] = a[f, h] + b[g, h]$ for $\forall a, b \in \Re$
3. Functional distributivity: $[f, gh] = [f, g]h + g[f, h]$
4. Jacobi identity: $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$
5. Fundamental Poisson brackets: $[X_i, X_j] = S_{ij}$
6. $[f, g] = 0$ if f or g are constant with respect to X.

Why is the Poisson bracket useful? Consider $f(X(s);s)$: it changes in time s either because of explicit s dependence, or because it depends on $X(s)$. The total time derivative f' can then be written as:

$$
\begin{aligned}
f' &= \frac{\partial f}{\partial s} + \sum_i \frac{\partial f}{\partial X_i} X_i' && (7)\\
&= \frac{\partial f}{\partial s} + \sum_{i,j} \frac{\partial f}{\partial X_i} S_{ij} \frac{\partial H}{\partial X_j} && (8)\\
&= \frac{\partial f}{\partial s} + [f, H] && (9)
\end{aligned}
$$

So a quantity $f(X)$ is a constant of the motion described by H if it is not explicitly s-dependent, and if it has

$$[f, H] = 0 \tag{10}$$

Poisson brackets are therefore intimately related to the time evolution of phase space quantities. It looks like the only relevant Poisson brackets involve the Hamiltonian here, but we'll find out that Poisson brackets of other quantities are also useful. As we move from linear to nonlinear dynamics, we will see that Lie algebras in accelerator physics are basically a formalism to simplify calculations within the algebra of Poisson brackets.

Poisson brackets in differential algebras used in accelerator tracking are often computed between two Taylor series of X. We can see that if f is an n^{th} order and g is an m^{th} order Taylor series, their Poisson bracket is another Taylor series of order $m + n - 2$.

1.3 Example: Coupled harmonic oscillators

Consider a pair of simple degenerate harmonic oscillators described by the Hamiltonian

$$H = \frac{1}{2}(\omega^2 x^2 + p_x^2 + \omega^2 y^2 + p_y^2) \tag{11}$$

It's almost obvious (though we can show) that $f_1 = \omega^2 x^2 + p_x^2$ and $f_2 = \omega^2 y^2 + p_y^2$ are constants of the motion. For example,

$$
\begin{aligned}
[f_1, H] &= [\omega^2 x^2 + p_x^2, \frac{1}{2}[\omega^2 x^2 + p_x^2]]\\
&= \frac{\omega^2}{2}\left([x^2, p_x^2] + [p_x^2, x^2]\right) = 0 && (12)
\end{aligned}
$$

However, another constant of the motion is $g = xp_y - yp_x$; this corresponds to the angular momentum:

$$
\begin{aligned}
[g, H] &= \frac{1}{2}[xp_y - yp_x, \omega^2 x^2 + p_x^2 + \omega^2 y^2 + p_y^2]\\
&= \frac{1}{2}\left([xp_y, p_x^2] + \omega^2 [xp_y, y^2] - \omega^2 [yp_x, x^2] - [yp_x, p_y^2]\right) = 0 && (13)
\end{aligned}
$$

You can see here why this has to be a degenerate oscillator for the angular momentum to be a conserved quantity. By forming $[f_1, g]$ or $[f_2, g]$, we can also find that $h = \omega^2 xy + p_x p_y$ is a constant of the motion, but it's a combination of the other invariants: $\omega^2 g^2 + h^2 = f_1 f_2$.

2 Lie Operators and Lie Algebras

2.1 Back To The Map

Recall that for the past week, we've been learning how to solve this in a different way, by writing down a linear Jacobian transport matrix that propagates the coordinates X_0 to a new coordinate $X(s)$ with a transport matrix:

$$X = MX_0 \qquad M_{ij} = \frac{\partial X_i}{\partial (X_0)_j} \tag{14}$$

The question is how to extend this to nonlinear systems, since accelerators are nonlinear. The Jacobian is the way that output conditions vary from small variations in input conditions. One natural way to express a nonlinear transport map is in terms of a truncated power series of the original coordinates:

$$X = F(X_0) \quad \text{(Truncated power series)} \tag{15}$$

where $F(X_0)$ is, say, a collection of N^{th} order power series in the components of X_0 where N is the order to which we are truncating in a perturbative expansion.

Recall that for periodic systems, we often wrote the transport matrix in terms of an exponential. This exponential depended on the details of our magnets and the lattice layout, but did not depend on the coordinates themselves. It turns out that another natural way to extend the Jacobian transport of Eq. (14) is with a Lie map, which is the exponential of a differential operator $:G(s):$ (which we'll define in a moment) that can depend on the coordinates:

$$X = \mathrm{e}^{:G(X):} X \Big|_{X=X_0} \qquad \text{(Lie map)} \tag{16}$$

Note some subtle differences between the truncated power series and the Lie map:

- The Lie map is symplectic by definition, while the truncated power series is not!
- $G(X)$ is a function of the coordinates, while the truncated power series is only a function of the initial coordinates X_0.
- There are many functions F, one for each coordinate, while there is only one $G(X)$, called the generator of the Lie transformation. It is a linear combination of the generators we saw earlier.
- To get the same equivalent order, we need to write the Lie map to the $(N+1)^{\text{st}}$ order. (We'll see we need an extra term from the Poisson bracket differentiation.)

Some group-theoretic babble before we continue: a Lie group is a mathematical group which is also a finite-dimensional real smooth manifold, and in which the group operations of multiplication (or concatenation) and inversion are smooth maps. Generally Lie groups are generated by infinitesimal generators. Some examples of Lie groups are:

- $\Re^n$ is an abelian Lie group under addition
- The orthogonal group $\mathrm{O_n}(\Re)$, the group of all rotations and reflections of an n-dimensional vector space. The subgroup of elements of determinant one is called the $\mathrm{SO_n}(\Re)$ special orthogonal group, or rotation group.
- The group $\mathrm{Sp_{2n}}(\Re)$, or group of all symplectic matrices.
- U(1)×SU(2)×SU(3), the composition group of the Standard Model.

2.2 Lie Operators

Since it's a pain to keep writing brackets all over the place, and because we're not confused enough yet, we'll rewrite the Poisson bracket in another notation that emphasizes its operator nature:

$$:f:g \equiv [f,g] = \sum_{i,j} \frac{\partial f}{\partial X_i} S_{ij} \frac{\partial g}{\partial X_j} \tag{17}$$

$f(X)$ is known as a Lie operator that operates on the function $g(X)$. Antisymmetry immediately follows: $:f:g = -:g:f$. One convenience of this notation is that powers of this operator are easier to write, so

$$(:f:)^2 g = [f,[f,g]] \qquad (:f:)^3 g = [f[f,[f,g]]] \qquad \ldots \tag{18}$$

$$(:f:)^k(gh) = \sum_{n=0}^{k} \frac{k!}{n!(k-n)!}[(:f:)^n g][(:f:)^{k-n} h] \tag{19}$$

The Jacobi identity also helps us find that the commutator of two Lie operators $:f:$ and $:g:$ is given by the Poisson brackets:

$$\{f,g\} \equiv :f::g: - :g::f: = :[f,g]: \tag{20}$$

This gives a cool variation of the Jacobi identity that can be used to simplify commutators of Lie operators:

$$\{f,g\}h = :h::g:f \tag{21}$$

Eq. (20) is the reason Poisson brackets play a prominent role in Lie algebra of operators. Commutators of operators occur often, and this equation states that they can be calculated in terms of Poisson brackets. So, for example, $:f:$ and $:g:$ commute if $[f,g]$ is constant.

Recall that we are expanding our map in higher order terms in a way similar to expanding a power series or Taylor series using the exponential of a differential operator:

$$e^{:f:} = \sum_{k=0}^{\infty} \frac{1}{k!}(:f:)^k \tag{22}$$

This exponential operator a Lie transformation, with $:f:$ as its generator. It is particularly useful when $:f:$ is nilpotent, i.e. $:f:^n = 0$ for some n.

Note that the Lie algebra is not the same as your typical algebra, particularly because operators do not necessarily commute. For example, $\exp(\ln(x)) = x$, but $\exp(:\ln(x):) \neq :x:$. In particular, the map $\exp(:\ln(x):)$ is symplectic: $\exp(:\ln(x):)x = x$, $\exp(:\ln(x):)p = p + 1/x$, so its Jacobian has unit determinant:

$$M = \frac{\partial X_i}{\partial (X_0)_j} = \begin{pmatrix} 1 & 0 \\ -1/x^2 & 1 \end{pmatrix} \tag{23}$$

while the map $:x:$ is nonsymplectic ($:x:x = 0$ and $:x:p = 1$). Some simple Lie operators in $2n$ dimensions are:

$$\begin{aligned}
:q_i: &= \frac{\partial}{\partial p_i} \qquad :p_i: = -\frac{\partial}{\partial q_i} \\
:q_i p_i: &= p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i} \\
:q_i::p_i: &= :p_i::q_i: = -\frac{\partial^2}{\partial q_i \partial p_i} \qquad (\ :q_i:\ \text{and}\ :p_i:\ \text{commute}) \\
:q_i^2: &= 2q_i \frac{\partial}{\partial p_i} \qquad :p_i^2: = -2p_i \frac{\partial}{\partial q_i}
\end{aligned} \tag{24}$$

2.3 The One-Turn Map and the Hamilton-Cayley Theorem

How do we describe the general linear one-turn map of our "standard" Hamiltonian in this formalism? Consider the quadratic form

$$f_2 = -\frac{1}{2} X^T F X \tag{25}$$

where F is symmetric and positive-definite and F is linear, so it does not depend on X. Our linear Hamiltonians can be written this way, for example, since they are quadratic forms in x and p_x with symmetric terms. We can show that $:f_2\colon X = SFX$ —

$$\begin{aligned} :f_2\colon X_i &= \frac{\partial f_2}{X_j} S_{jk} \partial X_i \partial X_k \\ &= -\frac{\partial (F_{lm} X_l X_m)}{\partial X_j} S_{ji} \\ &= -(S_{ji} F_{kj} X_k) = -(SFX)_i \end{aligned}$$

This also gives

$$e^{:f_2:} X = e^{SF} X \qquad or \qquad e^{:f_2:} = e^{SF} \tag{26}$$

Recall that the matrix form of the one-turn map is $T = I\cos\mu + J\sin\mu = e^{\mu J}$ in the linear uncoupled case. From (26), $SF = \mu J$, and we can find F for the one-turn map in terms of the Courant-Snyder parameters:

$$F = \mu S^{-1} J = \mu S^T J = \mu \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \tag{27}$$

Note that det(F)=$\mu^2 \geq 0$ so this is positive-definite for a stable lattice. We can then write the one-turn Lie operator f_2 for the one-turn map from (25):

$$f_2 = -\frac{\mu}{2} X^T \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} X = -\frac{\mu}{2}\left(\gamma^2 x^2 + 2\alpha x p_x + \beta p_x^2\right) \tag{28}$$

More generally, we can find the Lie operator from the matrix and vice-versa for any map. (An example of doing this is above, and in your homework.) The above example had the advantage that the matrix form was already an exponential! For other simple cases, like the quadrupole, this is not the case. Then you will have to use the Cayley-Hamilton theorem: every square matrix satisfies its own characteristic equation. Another way of saying this is that if λ_i are the eigenvalues of an $N \times N$ matrix F, then any function

$$f(F) = \sum_{k=0}^{N-1} a_k F^k \tag{29}$$

where the a_k satisfy the $N-1$ equations

$$f(\lambda_i) = \sum_{k=0}^{N-1} a_k \lambda_i^k \tag{30}$$

We can use this to show that in the two-dimensional case (see homework again):

$$F = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \mu \equiv \sqrt{\det F} \qquad \Rightarrow e^{SF} = \cos\mu\, I + \frac{\sin\mu}{\mu} \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} \tag{31}$$

2.4 Lie operators for other accelerator elements

The transport maps for accelerator elements can be represented as Lie transformations. For example, consider the one-dimensional drift. We know that it's usual map is

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{32}$$

The Lie transformation corresponding to this is $\exp(:-\frac{1}{2}Lp^2:)$. We can see this by writing out a few terms:

$$\begin{aligned} :p^2:x &= -2p \qquad (:p^2:)^n x = 0 \quad \forall n > 1 \\ :p^2:p &= 0 \qquad\;\;\, (:p^2:)^n p = 0 \quad \forall n > 1 \end{aligned} \tag{33}$$

From this it's apparent that $\exp(:-\frac{1}{2}Lp^2:)x = x + Lp$, and $\exp(:-\frac{1}{2}Lp^2:)p = p$.

We can similarly establish Lie operators for other elements, including nonlinear terms such as thin-lens multipoles. We couldn't do this with the simple linear matrix formalism before, but now we can apply the full power of Lie operators and Lie algebras to concatenate these maps, simulate accelerator maps more efficiently, and solve nonlinear dynamics problems. Some examples of these elements are listed here in Table 1.

Table 1: Lie Operators for Common Accelerator Elements

Element	Map	Lie Operator
Drift space	$x = x_0 + Lp_0$	$\exp(:-\frac{1}{2}Lp^2:)$
	$p = p_0$	
Thin-lens quadrupole	$x = x_0$	$\exp(:-\frac{1}{2f}x^2:)$
	$p = p_0 - \frac{1}{f}x_0$	
Thin-lens kick	$x = x_0$	$\exp(:\lambda x^n:)$
	$p = p_0 + \lambda n x_0^{n-1}$	
Thick focusing quad	$x = x_0 \cos\sqrt{k}L + \frac{p_0}{\sqrt{k}}\sin\sqrt{k}L$	$\exp(:-\frac{1}{2}L(kx^2 + p^2):)$
	$p = -kx_0 \sin\sqrt{k}L + p_0\cos\sqrt{k}L$	
Thick defocusing quad	$x = x_0 \cosh\sqrt{k}L + \frac{p_0}{\sqrt{k}}\sinh\sqrt{k}L$	$\exp(:-\frac{1}{2}L(kx^2 - p^2):)$
	$p = -kx_0 \sinh\sqrt{k}L + p_0\cosh\sqrt{k}L$	
Coordinate shift	$x = x_0 - b$	$\exp(:ax + bp:)$
	$p = p_0 + a$	
Coordinate rotation	$x = x_0\cos\mu + p_0\sin\mu$	$\exp(:-\frac{\mu}{2}(x^2 + p^2):)$
(Phase advance μ)	$p = -x_0\sin\mu + p_0\cos\mu$	
Full-turn Hamiltonian	(lots of things)	$\exp(C:H_{\text{eff}}:)$ or
		$\exp(:-\frac{\mu}{2}(\gamma x^2 + 2\alpha xp + \beta p^2):)$

Note that Lie representations are really useful for generalizations to nonlinear systems, and for power series analysis when performed by computers. However, Lie operators like those listed in this table really aren't useful for simple linear accelerator problems. For example, consider the thin-lens FODO lattice: its Lie representation is given by the concatenation

$$\exp\left(:-\frac{1}{2f}x^2:\right)\exp\left(:-\frac{1}{2}Lp^2:\right)\exp\left(:\frac{1}{2f}x^2:\right)\exp\left(:-\frac{1}{2}Lp^2:\right) \tag{34}$$

Note the reverse ordering; these are operators, after all! Considering that these are infinite series before losing terms when they are applied to (x, p), expanding this is a complete headache compared to the simple 2×2 or 4×4 matrix approach.

All of the elements above are exponentials of Poisson bracket operators, so they are Lie operators. Lie operators, like exponentials, have plenty of useful properties. Many are intuitive if $:f:$ and $:g:$ commute, such as $\exp(:f:)\exp(:g:) = \exp(:f+g:)$.

2.5 Example: Sextupole Taylor Map from Lie Operator Hamiltonian

As mentioned before, the one-turn Lie map is simply $\exp(:-CH:)$ where H is the Hamiltonian and C is the circumference of the accelerator. This can be extended to $\exp(:-LH:)$ where L is any length of integration, including multiple turns. Let's take the general sextupole Hamiltonian as an example, where

$$H = \frac{1}{3}S(x^3 - 3xy^2) + \frac{1}{2}(p_x^2 + p_y^2) \tag{35}$$

We can then calculate orders of the Hamiltonian:

$$\begin{aligned}
:H:x &= -\frac{\partial H}{\partial p_x} = -p_x \\
:H:^2 x &= -:H:p_x = -\frac{\partial H}{\partial x} = -S(x^2 - y^2) \\
:H:^3 x &= -S\left(\frac{\partial H}{\partial p_x}(2x) + \frac{\partial H}{\partial p_y}(2y)\right) = 2S(x\,p_x - y\,p_y) \\
:H:^4 x &= 2S\left(-\frac{\partial H}{\partial p_x}p_x + x\frac{\partial H}{\partial x} + \frac{\partial H}{\partial p_y}p_y - \frac{\partial H}{\partial y}y\right) \\
&= 2S\left[-p_x^2 + p_y^2 + Sx\left(x^2 + y^2\right)\right] \\
:H:^5 x &= O(S^2)
\end{aligned} \tag{36}$$

We can then obtain the Taylor map up to a modest order:

$$\begin{aligned}
\exp(-L:H:)x &= x + p_x L - \frac{1}{2}SL^2(x^2 - y^2) - \frac{1}{3}SL^3(xp_x - yp_y) \\
&\quad + \frac{1}{12}SL^4[-p_x^2 + p_y^2 + Sx(x^2 + y^2)] + O(S^2L^5)
\end{aligned} \tag{37}$$

where $O(S^2L^5)$ means terms same-or-higher order than S^2 in S and same-or-higher order than L^5 in L.

We can work through all the math (Mathematica *really* is your friend) to find the mappings of other coordinates as well:

$$\begin{aligned}
\exp(-L:H:)p_x &= p_x + SL(x^2 - y^2) - SL^2(xp_x - yp_y) - \frac{1}{3}SL^3[p_x^2 - p_y^2 - Sx(x^2 + y^2)] \\
&+ \frac{1}{12}S^2L^4(5x^2p_x - y^2p_x + 6xyp_y) + O(S^2L^5) \\
\exp(-L:H:)y &= y + Lp_y + SL^2xy + \frac{1}{3}SL^3(xp_y + yp_x) \\
&+ \frac{1}{12}SL^4[2p_xp_y + Sy(x^2 + y^2)] + O(S^2L^5) \\
\exp(-L:H:)p_y &= p_y + 2SLxy + SL^2(xp_y + yp_x) + \frac{1}{3}SL^3[2p_xp_y + Sy(x^2 + y^2)] \\
&+ \frac{1}{12}SL^4(x^2p_y - 6xyp_x - 5y^2p_y) + O(S^2L^5)
\end{aligned} \tag{38}$$

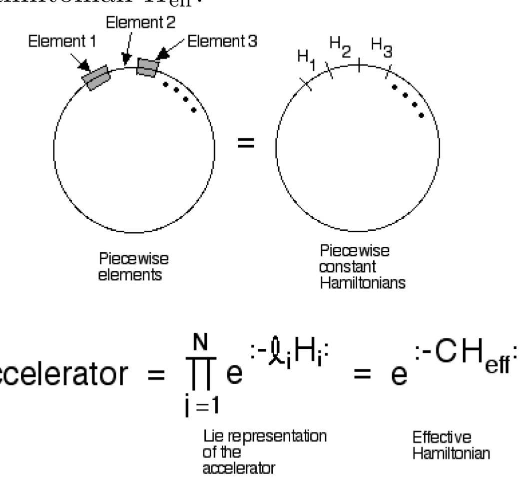
This is not too exciting, as we still would have to expand using the Floquet transformation to find resonance driving terms and strengths, but the real advantage here is that this expression can be explicitly calculated to any order in S. (Mathematica is your friend!) When you do this, you find that higher orders of sextupole powers drive higher order resonances, similar to octupoles, and even higher orders after that. The reason that accelerators still work despite an infinite number of resonance driving terms from nonlinearities is that these driving forces are perturbatively small — so small that the resonances are all tiny and isolated, and tend not to overlap according to the Chirikov resonance overlap criterion that Vladimir mentioned in the nonlinear dynamics lecture.

2.6 The Ring and the Baker-Campbell-Hausdorff Formula

How do we apply Lie techniques to piecewise continuous Hamiltonians in rings? We have individual elements (dipoles, quadrupoles, sextupoles, etc) that we chain together, so we end up with a Lie map that is their product:

$$\prod_{i=1} N\mathrm{e}^{:-L_i H_i:} = \mathrm{e}^{:-CH_{\mathrm{eff}}:} \tag{39}$$

This can be seen in the following figure, which in some sense expresses our desire to have a full-ring effective Hamiltonian that carries all the nonlinearity of the system. The goal is then to find the effective Hamiltonian H_{eff}.



If $: f :$ and $: g :$ do not commute, how do we relate $\exp(: f :) \exp(: g :)$ to a single Lie operator $\exp(: h :)$ — that is, how do we concatenate Lie maps? The basic formula that allows concatenation of Lie operators is called the Baker-Campbell-Hausdorff formula. It comes in many useful forms, but we'll just state two here for convenience — it's already long enough! Given $\exp(:f:)\exp(:g:) = \exp(:h:)$, h is related to f and g by:

$$\begin{aligned} h \;=\; & f + g + \frac{1}{2} :f: g + \frac{1}{12} :f:^2 g + \frac{1}{12} :g:^2 f + \frac{1}{24} :f::g:^2 f \\ & -\frac{1}{720} :g:^4 f - \frac{1}{720} :f:^4 g + \frac{1}{360} :g::f:^3 g + \frac{1}{360} :f::g:^3 f \\ & +\frac{1}{120} :f:^2:g:^2 f + \frac{1}{120} :g:^2:f:^2 g + O((f,g)^6) \end{aligned} \tag{40}$$

We can rewrite this in terms of commutators, so we can see more of the functional nesting, but it really doesn't help much. The coefficients of this expansion of the original BCH formula don't seem to have a convenient pattern.

If one of the terms in the BCH expansion is perturbatively small, we can sum the infinite power series in the first form over the function f or g (whichever is NOT perturbative) to find

$$
\begin{aligned}
e^{:f:}e^{:g:} &= \exp\left[:g + \left(\frac{:g:}{\exp(:g:) - 1}\right) f + O(f^2):\right] \qquad (41) \\
e^{:f:}e^{:g:} &= \exp\left[:f + \left(\frac{:f:}{1 - \exp(:f:)}\right) g + O(g^2):\right] \qquad (42)
\end{aligned}
$$

This and other forms of the BCH theorem, along with the Taylor expansion seen in the last section, allow us to calculate nonlinear accelerator maps to high order. These can be used for computer simulation, and for analysis. To determine dynamic aperture, or long-term beam stability, we often "track" for millions or tens of millions of turns around the accelerator. However, the maps that we've generated are not symplectic if they are just arbitrarily truncated! This leads to the field of "symplectification", where additional higher-order terms are added that make the map symplectic again, yet are high enough order that they do not dominate the dynamics.