Outline

- Liouville's theorem
- Canonical coordinates
- Hamiltonian
- Symplecticity
- Symplectic integration
- Symplectification algorithm

Liouville's Theorem

In the local region of a particle, the particle density in phase space is constant provided that the particles move in a general field consisting of magnetic fields and of fields whose forces are independent of velocity.

• Density function = number of particles per unit volume of six-dimensional phase space

$$f(\vec{r},\,\vec{p},t)$$

• Particle current

$$\vec{J}_6 = (f\dot{\vec{r}},\, f\dot{\vec{p}}) = (f\vec{\upsilon},\, f\vec{F})$$

• Continuity equation = preservation of the number of particles

$$\partial f/\partial t + \nabla_6 \cdot \vec{J}_6 = 0$$

$$\nabla_6 \cdot \vec{J}_6 = \nabla \cdot (f\vec{\upsilon}) + \nabla_p (f\vec{F}) = (\nabla f)\cdot\vec{\upsilon} + f\underbrace{(\nabla\cdot\vec{\upsilon})}_{0} + (\nabla_p f)\cdot\vec{F} + f(\nabla_p\cdot\vec{F})$$

$$\nabla_p\cdot\vec{F} = \underbrace{\nabla_p\cdot[\vec{g}(\vec{r})}_{0} + q\vec{\upsilon}\times\vec{B}(\vec{r})] = q\nabla_p\cdot(\vec{\upsilon}\times\vec{B}) = q\vec{B}\cdot(\nabla_p\times\vec{\upsilon}) - q\vec{\upsilon}\cdot(\underbrace{\nabla_p\times\vec{B}}_{0})$$

$$\left[\nabla_p \times \left(\frac{\vec{p}}{\sqrt{p^2c^2+m^2c^4}}\right)\right]_z = \frac{\partial}{\partial p_x}\left(\frac{p_y}{\sqrt{p^2c^2+m^2c^4}}\right) - \frac{\partial}{\partial p_y}\left(\frac{p_x}{\sqrt{p^2c^2+m^2c^4}}\right) = 0 \Rightarrow$$

$$\nabla_p \times \vec{\upsilon} = 0$$

Liouville's Theorem

$$\nabla_6 \cdot \vec{J}_6 = (\nabla f) \cdot \vec{\upsilon} + (\nabla_p f) \cdot \vec{F}$$

- Continuity equation becomes

$$\partial f / \partial t + (\nabla f) \cdot \dot{\vec{r}} + (\nabla_p f) \cdot \dot{\vec{p}} = df / dt = 0$$

Theorem proved.

- Limitations and exceptions
  – Linear and non-linear mismatch ⇒ effective emittance growth
  – Velocity-dependent Coulomb interaction between individual particles: IBS, Touschek
  – Dissipative mechanisms
    - Radiation, radiation damping
    - Charge-exchange injection
    - Electron cooling
    - Stochastic cooling
- An immediate consequence of the theorem

$$\vec{Y} = \vec{T}(\vec{X}), \quad \Delta\vec{T} = \sum_{j=1}^{6} \frac{\partial \vec{T}}{\partial X_j}(\vec{0})\Delta X_j, \quad \Delta Y_i = \sum_{j=1}^{6} \frac{\partial T_i}{\partial X_j}(\vec{0})\Delta X_j = \sum_{j=1}^{6} M_{ij}\Delta X_j$$

Since the determinant of the Jacobian is a ratio of volumes

$$\det M = 1$$

Canonical Momentum and Vector Potential

- Conservative force

$$\nabla \times \vec{F} = 0$$

- Lorentz force

$$\vec{F} = d\vec{p}\,/\,dt = q(\vec{E} + \vec{\upsilon} \times \vec{B})$$

can be velocity dependent and is not always conservative.

$$\begin{aligned}
\nabla \times \vec{F} &= \nabla \times d\vec{p}\,/\,dt = q[\nabla \times \vec{E} + \nabla \times (\vec{\upsilon} \times \vec{B})] \\
&= -q\partial\vec{B}\,/\,\partial t + q[(\vec{B}\cdot\underbrace{\nabla)\vec{\upsilon}}_{0} - (\vec{\upsilon}\cdot\nabla)\vec{B} + \underbrace{(\nabla\cdot\vec{B})}_{0}\vec{\upsilon} - \underbrace{(\nabla\cdot\vec{\upsilon})}_{0}\vec{B}] \\
&= -q\partial\vec{B}\,/\,\partial t - q(\vec{\upsilon}\cdot\nabla)\vec{B} = -qd\vec{B}\,/\,dt = -d(\nabla \times q\vec{A})\,/\,dt \Rightarrow
\end{aligned}$$

$$\nabla \times \frac{d\vec{p}}{dt} + \frac{d}{dt}(\nabla \times q\vec{A}) = \nabla \times \left[\frac{d}{dt}(\vec{p} + q\vec{A})\right] = 0$$

- *Canonical* momentum and *conservative* canonical force

$$\vec{P} = \vec{p} + q\vec{A}, \quad \vec{F}_{can} = d\vec{P}\,/\,dt$$

Hamiltonian

- Relativistic Hamiltonian of a free particle

$$H = \sqrt{p^2c^2 + m^2c^4}$$

- Hamiltonian in an electromagnetic field

$$\vec{B} = \nabla \times \vec{A}, \quad \vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$$

$$H = \sqrt{(\vec{P} - q\vec{A})^2c^2 + m^2c^4} + q\phi$$

- Hamilton's equations

$$d\vec{P}/dt = -\nabla H, \quad d\vec{x}/dt = \nabla_P H$$

- Frenet-Serret curvilinear coordinate system

$$d\vec{l} = dr\,\hat{r} + rd\theta\,\hat{\theta} + dy\,\hat{y} = dx\,\hat{x} + \frac{r}{\rho}ds\,\hat{s} + dy\,\hat{y} = dx\,\hat{x} + dy\,\hat{y} + (1 + \frac{x}{\rho})ds\,\hat{s}$$

- Hamiltonian becomes

$$H = \sqrt{m^2c^4 + c^2\left[p_x^2 + p_y^2 + \left(\frac{p_s}{1 + x/\rho}\right)^2\right]}$$

$$H = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + \left(\frac{P_s - qA_s}{1 + x/\rho}\right)^2} + q\phi$$

s-Hamiltonian

- Poincaré-Cartan invariant under a canonical variable transformation

$$\vec{P}\cdot d\vec{r} - Hdt = inv$$

$$P_x dx + P_y dy + P_s ds - Hdt = P_x dx + P_y dy + (-H)dt - (-P_s)ds$$

$$= P_x dx + P_y dy + (-U)dt - H_s ds$$

- Hamiltonian with the longitudinal coordinate as the independent variable

$$H_s = -P_s(x, P_x, y, P_y, t, -U; s)$$

$$= -qA_s - \left(1+\frac{x}{\rho}\right)\sqrt{\left(\frac{U-q\phi}{c}\right)^2 - m^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2}$$

- Curl in Frenet-Serret coordinates

$$\nabla\times\vec{A} = \frac{1}{1+x/\rho}\left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{x} + \frac{1}{1+x/\rho}\left(\frac{\partial A_x}{\partial s} - \frac{\partial A_s}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{s}$$

- For static transverse magnetic fields

$$\phi = 0,\quad A_x = A_y = 0$$

$$B_x = \frac{1}{1+x/\rho}\frac{\partial A_s}{\partial y},\quad B_y = -\frac{1}{1+x/\rho}\frac{\partial A_s}{\partial x}$$

Hamiltonian in Standard Canonical Coordinates

- For static transverse magnetic fields

$$\phi = 0, \quad A_x = A_y = 0$$

$$\frac{H_s}{p_0} = -\frac{qA_s}{p_0} - \left(1+\frac{x}{\rho}\right)\sqrt{\left(\frac{U}{p_0 c}\right)^2 - \left(\frac{mc}{p_0}\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2}$$

$$= -\frac{qA_s}{p_0} - \left(1+\frac{x}{\rho}\right)\sqrt{\left(\frac{U}{p_0 c}\right)^2 - \left(\frac{mc}{p_0}\right)^2 - x'^2 - y'^2}$$

$$= H(x, x', y, y', t, -U/p_0; s)$$

$$p_x / p_0 \simeq dx/ds = x', \quad p_y / p_0 \simeq dy/ds = y'$$

- Canonical change of longitudinal coordinates

$$(t, -U/p_0) \to (z = s - \upsilon_0 t, \delta = \Delta p / p_0)$$

$$\tilde{H}(x, x', y, y', z, \delta; s) = H(x, x', y, y', t, -U/p_0; s) + \partial F_2(t, \delta; s)/\partial s$$

$$z = \partial F_2 / \partial \delta, \quad -U/p_0 = \partial F_2 / \partial t$$

Hamiltonian in Standard Canonical Coordinates

$$F_2 = -\frac{U}{p_0} t + F(\delta, s)$$

$$\frac{U}{p_0} = \frac{U_0}{p_0}\left(1 + \frac{\Delta U}{U}\right) = \frac{c}{\beta_0}(1 + \beta_0^2 \delta)$$

$$F_2 = -\frac{c}{\beta_0}(1 + \beta_0^2 \delta) t + \delta s + F(s)$$

$$F_2 = \frac{c}{\beta_0}(1 + \beta_0^2 \delta)\left(\frac{s}{\upsilon_0} - t\right) - \frac{s}{\beta_0^2} + s$$

- Hamiltonian in the new canonical longitudinal coordinates

$$\tilde{H} = -\frac{qA_s}{p_0} - \left(1 + \frac{x}{\rho}\right)\sqrt{\left(\frac{p}{p_0}\right)^2 - x'^2 - y'^2} + \frac{\partial F_2}{\partial s}$$

$$= -\frac{qA_s}{p_0} - \left(1 + \frac{x}{\rho}\right)\sqrt{1 + 2\delta + \delta^2 - x'^2 - y'^2} + 1 + \delta$$

Hamiltonian in Standard Canonical Coordinates

• Expansion of the Hamiltonian keeping the lower-order terms

$$\tilde{H} = -\frac{qA_s}{p_0} - \left(1+\frac{x}{\rho}\right)\left(1+\frac{1}{2}[2\delta+\delta^2-(x'^2+y'^2)+\ldots]-\frac{1}{8}[4\delta^2+\ldots]+\ldots\right)+1+\delta$$

$$= -\frac{qA_s}{p_0} - \left(1+\frac{x}{\rho}\right)\left(1+\delta-\frac{1}{2}(x'^2+y'^2)+\ldots\right)+1+\delta$$

$$= -\frac{qA_s}{p_0} + \frac{1}{2}(x'^2+y'^2) - \frac{x}{\rho} - \frac{x\delta}{\rho}$$

Symplectic Transformations and Matrices

- Hamilton's equations

$$H = H(x, P_x, y, P_y, t, -U; s), \quad \vec{X} = (x, P_x, y, P_y, t, -U)$$

$$\frac{dx_i}{ds} = \frac{\partial H}{\partial P_i}, \quad \frac{dP_i}{ds} = -\frac{\partial H}{\partial x_i}$$

$$\frac{dX_i}{ds} = \sum_{j=1}^{6} S_{ij} \frac{\partial H}{\partial X_j}$$

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Symplectic Transformations and Matrices

- Canonical coordinate transformation in passing through a beam line

$$\vec{X}_1 = \vec{X}_1(\vec{X}_0)$$

$$\frac{dX_{1i}}{ds} = \sum_j \frac{\partial X_{1i}}{\partial X_{0j}} \frac{dX_{0j}}{ds} = \sum_j M_{ij} \frac{dX_{0j}}{ds}$$

$$\frac{dX_{1i}}{ds} = \sum_j S_{ij} \frac{\partial H}{\partial X_{1j}} = \sum_{j,k} S_{ij} \frac{\partial H}{\partial X_{0k}} \underbrace{\frac{\partial X_{0k}}{\partial X_{1j}}}_{\left(\left(M^{-1}\right)^T\right)_{jk}} = \sum_{j,k} S_{ij} \left(\left(M^{-1}\right)^T\right)_{jk} \underbrace{\frac{\partial H}{\partial X_{0k}}}_{-\sum_l S_{kl} \frac{dX_{0l}}{ds}}$$

$$\frac{dX_{1i}}{ds} = -\sum_{j,k} S_{ij} \left(\left(M^{-1}\right)^T\right)_{jk} S_{kl} \frac{dX_{0l}}{ds} = \sum_j M_{ij} \frac{dX_{0j}}{ds} \Rightarrow$$

$$M = -S(M^T)^{-1} S, \quad S = M^T S M$$

Definition of a symplectic matrix.

Symplectic Integration

- Expanding the equation of motion in the neighborhood of the reference trajectory $\hat{X}(s)$

$$\frac{d}{ds}(\hat{X}_i + \Delta X_i) = \sum_{j=1}^{6} S_{ij} \frac{\partial H}{\partial X_j}(\hat{X} + \Delta X) = \sum_{j=1}^{6} S_{ij} \left[\frac{\partial H}{\partial X_j}(\hat{X}) + \sum_{k=1}^{6} \frac{\partial^2 H}{\partial X_j \partial X_k}(\hat{X}) \Delta X_k + \ldots \right]$$

$$\frac{d\Delta X_i}{ds} = \sum_{j=1}^{6} \sum_{k=1}^{6} S_{ij} \underbrace{\frac{\partial^2 H}{\partial X_j \partial X_k}(\hat{X})}_{C_{jk}} \Delta X_k + \ldots = \sum_{j=1}^{6} \sum_{k=1}^{6} S_{ij} C_{jk} \Delta X_k + \ldots$$

- In matrix notation

$$\frac{d\Delta \vec{X}}{ds} = SC\Delta \vec{X}$$

- If *C* is constant

$$\Delta \vec{X}(s) = e^{SCs} \Delta \vec{X}(0) \quad \Rightarrow \quad M(s) = e^{SCs}$$

- If *C* is a function of *s* we approximate it in a piecewise constant fasion as

$$M(s) = e^{SC(s-ds)ds} \cdots e^{SC(2ds)ds} e^{SC(ds)ds} e^{SC(0)ds}$$

Symplectic Matrices

- Since any $2n\times 2n$ symplectic matrix can be expressed in terms of a symmetric matrix it has the same number of degrees of freedom

$$\frac{(2n)^2-2n}{2}+2n=(2n+1)n$$

E.g. a 4×4 symplectic matrix has 10 independent parameters.

- Choosing a basis for 4×4 symmetric matrices

$$c_j:\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

Symplectic Matrices

- Any 4×4 symmetric matrix can be written in the form

$$C = \sum_{j=1}^{10} \alpha_j c_j$$

- Any 4×4 symplectic matrix can be written as

$$\exp(SC) = \exp\left(\sum_{j=1}^{10} \underbrace{Sc_j}_{G_j} \alpha_j \right) = \exp\left(\sum_{j=1}^{10} G_j \alpha_j \right)$$

Manipulations with Exponentials

- The Baker-Campbell-Hausdorff (BCH) formula

$$e^X e^Y = e^Z$$

$$Z = \log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,X,Y]+[Y,Y,X]) + \frac{1}{24}[X,Y,Y,X] + \ldots$$

$$[X,X,Y] \equiv [X,[X,Y]],\quad [X,Y,Y,X] \equiv [X,[Y,[Y,X]]]$$

- The Zassenhaus formula

$$e^{(A+B)h} = e^{Ah} e^{Bh} e^{-[A,B]h^2/2} e^{(2[B,A,B]+[A,A,B])h^3/6} e^{O(h^4)}$$

Up to 2nd order in *h*

$$e^{(A+B)h} = e^{Ah/2} e^{Bh} e^{Ah/2} + O(h^3)$$

$$M(h) = e^{SCh} = \exp\left(\sum_{j=1}^{10} \alpha_j G_j h\right) = \exp\left(\alpha_1 G_1 h + \sum_{j=2}^{10} \alpha_j G_j h\right)$$

$$= \underbrace{\exp(\alpha_1 G_1 h/2)}_{\text{half-step drift}} \underbrace{\exp\left(\sum_{j=2}^{10} \alpha_j G_j h\right)}_{\text{thin element kick}} \underbrace{\exp(\alpha_1 G_1 h/2)}_{\text{half-step drift}}$$

Symplectic Integrator

- The matrix representing a canonical coordinate transformation over a small step during which the fields can be considered constant can be represented as a thin kick sandwiched between two drifts with all coefficients determined by the Hamiltonian

$$M(h) = \underbrace{\exp\left(\alpha_1 G_1 h/2\right)}_{\text{half-step drift}} \underbrace{\exp\left(\sum_{j=2}^{10} \alpha_j G_j h\right)}_{\text{thin element kick}} \underbrace{\exp\left(\alpha_1 G_1 h/2\right)}_{\text{half-step drift}}$$

$$G_1 = Sc_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\exp(\alpha_1 G_1 h/2) = \begin{pmatrix} 1 & \alpha_1 h/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha_1 h/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} + O(h^3)$$

- This is the basis of symplectic drift-kick codes, kicks can be non-linear, e.g. for sextupoles, octupoles, etc.

Time-Symmetric Integrators

- Apply the BCH formula to the time symmetric product

$$e^W = e^{Xh}e^{Yh}e^{Xh} = \exp(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,X,Y] + [Y,Y,X]) + \frac{1}{24}[X,Y,Y,X])e^{Xh}$$

$$= \exp\{(X+Y)h + \frac{1}{2}[X,Y]h^2 + \frac{1}{12}([X,X,Y] + [Y,Y,X])h^3 + \frac{1}{24}[X,Y,Y,X]h^4 + Xh$$

$$+ \frac{1}{2}[X+Y,X]h^2 + \frac{1}{4}[[X,Y],X]h^3 + \frac{1}{24}([[X,X,Y],X] + [[Y,Y,X],X])h^4$$

$$+ \frac{1}{12}([X+Y,X+Y,X] + [X,X,X+Y])h^3 + \frac{1}{24}[X,X,X,Y]h^4 + O(h^5)\}$$

$$= \exp\{(2X+Y)h + \frac{1}{6}([Y,Y,X] - [X,X,Y])h^3 + O(h^5)\}$$

Time-Symmetric Integrators

- If an integrator

$$I(h) = e^{g_1 h + g_2 h^2 + g_3 h^3 + \dots}$$

  is time reversible then

$$I(h)I(-h) = 1$$

- Applying the BCH formula gives

$$I(h)I(-h) = e^{g_1 h + g_2 h^2 + g_3 h^3 + \dots} e^{-g_1 h + g_2 h^2 - g_3 h^3 + \dots} = e^{2 g_2 h^2 + O(h^3)} = 1 \quad \Rightarrow$$

$$g_2 = 0$$

- Repeating we find by induction that all even powers vanish

$$g_{2j} = 0$$

  i.e. $\log(I(h))$ must be an odd function of *h*.

Higher-Order Integrators

- Higher-order integrators may be constructed from the 2nd-order integration function

$$I_2(h)=e^{Ah/2}e^{Bh}e^{Ah/2}=e^{g_1h+g_3h^3+\dots}$$

- 4th-order integrator may be constructed using time-symmetric product of 2nd-order integrators

$$I_4(h)=I_2(ah)I_2(bh)I_2(ah)$$

$$I_4(h)=e^{(2a+b)g_1h+(2a^3+b^3)g_3h^3+\dots}=e^{(A+B)h+O(h^5)}$$

- To 4th-order this requires

$$\begin{cases}2a+b=1\\2a^3+b^3=0\end{cases}\Rightarrow\quad a=\frac{1}{2-2^{1/3}},\quad b=-\frac{2^{1/3}}{2-2^{1/3}}$$

Symplectification Algorithm

- A symplectic matrix *M* may be written in the form

$$M = (I + SW)(I - SW)^{-1}$$

  if and only if *W* is a symmetric matrix.

- If *W* is symmetric

$$M^T SM = (I - W^T S^T)^{-1}(I + W^T S^T)S(I + SW)(I - SW)^{-1}$$

$$= (I + WS)^{-1}(I - WS)S(I + SW)(I - SW)^{-1} = (I + WS)^{-1}(I - WS)(I + WS)S(I - SW)^{-1}$$

$$= (I + WS)^{-1}(I + WS)(I - WS)S(I - SW)^{-1} = (I + WS)^{-1}(I + WS)S(I - SW)(I - SW)^{-1}$$

$$= S$$

- If *M* is symplectic and $W = P + Q,\ W = P^T - Q^T$

$$M^T SM = S = (I - W^T S^T)^{-1}(I + W^T S^T)S(I + SW)(I - SW)^{-1}$$

$$= (I - W^T S^T)^{-1}(S + P - Q - P - Q + W^T SW)(I - SW)^{-1}$$

$$= (I - W^T S^T)^{-1}[(S - W^T)(I - SW) - 4Q](I - SW)^{-1}$$

$$= S - 4(I - W^T S^T)^{-1}Q(I - SW)^{-1} \quad \Rightarrow \quad Q = 0$$

Symplectification Algorithm

- Symmetric matrix from a symplectic matrix

$$W = S(I+M)^{-1}(I-M)$$

- For an almost symplectic matrix calculate the almost symmetric matrix

$$W_0 = S(I+M_0)^{-1}(I-M_0)$$

- Symmetrize the almost symmetric matrix

$$W_1 = (W_0 + W_0^T)/2$$

- Obtain a symplectic approximation of the original matrix

$$M_1 = (I+SW_1)(I-SW_1)^{-1}$$

Example: Sector Dipole Magnet

- Sector dipole magnet:

$$\vec{B} = \begin{cases} B_0 \hat{y}, \text{ for } 0 < s < L \\ 0, \text{ elsewhere} \end{cases}$$

$$B_y = -\frac{1}{1 + x/\rho} \frac{\partial A_s}{\partial x} \quad \Rightarrow \quad A_s = -B_0 \left(x + \frac{x^2}{2\rho} \right)$$

$$\tilde{H} = \left(\frac{x}{\rho} + \frac{x^2}{2\rho^2} \right) + \frac{1}{2}(x'^2 + y'^2) - \frac{x}{\rho} - \frac{x\delta}{\rho} = \frac{x^2}{2\rho^2} + \frac{1}{2}(x'^2 + y'^2) - \frac{x\delta}{\rho}$$

- Hessian matrix of the Hamiltonian

$$C = \frac{\partial H}{\partial X_i \partial X_j} \simeq \begin{pmatrix} 1/\rho^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = c_1 + \frac{1}{2\rho^2}(c_3 + c_4), \quad SC = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1/\rho^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M(s) = e^{SCs} = \sum_{n=0}^{\infty} \frac{(SCs)^n}{n!} = \begin{pmatrix} \cos(s/\rho) & \rho \sin(s/\rho) & 0 & 0 \\ -\sin(s/\rho)/\rho & \cos(s/\rho) & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$