USPAS Accelerator Physics 2015
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Chapter 5 Review and
Chapter 6:

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Matrix Example: Strong Focusing

- Consider a doublet of thin quadrupoles separated by drift $L$

\[
M_{\text{doublet}} = \begin{pmatrix} \frac{1}{f_D} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{f_F} & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{f_D} - \frac{L}{f_F} f_D & L \\ \frac{1}{f_D} - \frac{1}{f_F} f_F & 1 + \frac{L}{f_D} \end{pmatrix}
\]

\[
\frac{1}{f_{\text{doublet}}} = \frac{1}{f_D} - \frac{1}{f_F} f_F - \frac{L}{f_F f_D}
\]

(C&M 5.1 with $f_F = -f_D$)

\[
f_D = f_F = f \quad \Rightarrow \quad \frac{1}{f_{\text{doublet}}} = -\frac{L}{f^2}
\]

There is net focusing given by this alternating gradient system

A fundamental point of optics, and of accelerator strong focusing
Strong Focusing: Another View

For this to be focusing, \( x' \) must have opposite sign of \( x \) where these are coordinates of transformation of incoming paraxial ray

\[
M_{\text{doublet}} = \begin{pmatrix}
\frac{1}{f_D} & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{f_F} & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix}
\frac{1}{f_D} - \frac{1}{f_F} & \frac{L}{f f_D} \\
\frac{1}{f_D} - \frac{1}{f_F} & 1 + \frac{L}{f_D}
\end{pmatrix}
\]

incoming paraxial ray \( \begin{pmatrix} x \\ x' \end{pmatrix} = M_{\text{doublet}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix}
\frac{1}{f_D} - \frac{1}{f_F} & \frac{L}{f f_D} \\
\frac{1}{f_D} - \frac{1}{f_F} & 1 + \frac{L}{f_D}
\end{pmatrix} x_0 \)

For this to be focusing, \( x' \) must have opposite sign of \( x \) where these are coordinates of transformation of incoming paraxial ray

\[
f_F = f_D \quad x' < 0 \quad \text{BUT} \quad x > 0 \text{ iff } f_F > L
\]

Equal strength doublet is net focusing under condition that each lens’ focal length is greater than distance between them
More Math: Hill’s Equation

- Let’s go back to our quadrupole equations of motion for $R \to \infty$

$$x'' + Kx = 0 \quad y'' - Ky = 0 \quad K \equiv \frac{1}{(B\rho)} \left( \frac{\partial B_y}{\partial x} \right)$$

What happens when we let the focusing $K$ vary with $s$?

Also assume $K$ is periodic in $s$ with some periodicity $C$

$$x'' + K(s)x = 0 \quad K(s) \equiv \frac{1}{(B\rho)} \left( \frac{\partial B_y}{\partial x} \right)(s) \quad K(s + C') = K(s)$$

This periodicity can be one revolution around the accelerator or as small as one repeated “cell” of the layout

(Such as a FODO cell in the previous slide)

The simple harmonic oscillator equation with a periodically varying spring constant $K(s)$ is known as Hill’s Equation
Hill’s Equation Solution Ansatz

\[ x'' + K(s)x = 0 \quad K \equiv \frac{1}{(B\rho)} \left( \frac{\partial B_y}{\partial x} \right)(s) \]

- Solution is a quasi-periodic harmonic oscillator

\[ x(s) = A w(s) \cos[\Psi(s) + \Psi_0] \]

where \( w(s) \) is periodic in \( C \) but the phase \( \Psi(s) \) is not!!

Substitute this educated guess ("ansatz") to find

\[ x' = A w' \cos[\Psi + \Psi_0] - A w \Psi' \sin[\Psi + \Psi_0] \]
\[ x'' = A (w'' - w \Psi'^2) \cos[\Psi + \Psi_0] - A (2w' \Psi' + w \Psi'') \sin[\Psi + \Psi_0] \]
\[ x'' + K(s)x = -A(2w' \Psi' + w \Psi'') \sin(\Psi + \Psi_0) + A (w'' - w \Psi'^2 + Kw) \cos(\Psi + \Psi_0) = 0 \]

For \( w(s) \) and \( \Psi(s) \) to be independent of \( \Psi_0 \), coefficients of the sin and cos terms must vanish identically.
Courant-Snyder Parameters

\[2w w' \Psi' + w^2 \Psi'' = (w^2 \Psi')' = 0 \quad \Rightarrow \quad \Psi' = \frac{k}{w(s)^2}\]

\[w'' - \left(\frac{k^2}{w^3}\right) + Kw = 0 \quad \Rightarrow \quad w^3(w'' + Kw) = k^2\]

- Notice that in both equations \(w^2 \propto k\) so we can scale this out and define a new set of functions, Courant-Snyder Parameters or Twiss Parameters

\[\beta(s) \equiv \frac{w^2(s)}{k}\]

\[\alpha(s) \equiv -\frac{1}{2} \beta'(s)\]

\[\gamma(s) \equiv 1 + \alpha(s)^2 \frac{1}{\beta(s)}\]

\[\Psi'(s) = \frac{1}{\beta(s)} \quad \Psi(s) = \int \frac{ds}{\beta(s)}\]

\[\Rightarrow \quad K \beta = \gamma + \alpha'\]

\(\beta(s), \alpha(s), \gamma(s)\) are all periodic in \(C\)

\(\Psi(s)\) is not periodic in \(C\)
Towards The Matrix Solution

- What is the matrix for this Hill’s Equation solution?

\[ x(s) = A \sqrt{\beta(s)} \cos \Psi(s) + B \sqrt{\beta(s)} \sin \Psi(s) \]

Take a derivative with respect to \( s \) to get \( x' \equiv \frac{dx}{ds} \)

\[ \Psi' = \frac{1}{\beta(s)} \quad x'(s) = \frac{1}{\sqrt{\beta(s)}} \left\{ [B - \alpha(s)A] \cos \Psi(s) - [A + \alpha(s)B] \sin \Psi(s) \right\} \]

Now we can solve for \( A \) and \( B \) in terms of initial conditions \((x(0), x'(0))\)

\[ x_0 \equiv x(0) = A \sqrt{\beta(0)} \quad x'_0 \equiv x'(0) = \frac{1}{\sqrt{\beta(0)}} [B - \alpha(0)A] \]

\[ A = \frac{x_0}{\sqrt{\beta(0)}} \quad B = \frac{1}{\sqrt{\beta(0)}} [\beta(0)x'_0 + \alpha(0)x_0] \]

And take advantage of the periodicity of \( \beta, \alpha \) to find \( x(C), x'(C) \)
Hill’s Equation Matrix Solution

\[ x(s) = A \sqrt{\beta(s)} \cos \Psi(s) + B \sqrt{\beta(s)} \sin \Psi(s) \]

\[ x'(s) = \frac{1}{\sqrt{\beta(s)}} \{ [B - \alpha(s)A] \cos \Psi(s) - [A + \alpha(s)B] \sin \Psi(s) \} \]

\[ A = \frac{x_0}{\sqrt{\beta(0)}} \quad B = \frac{1}{\sqrt{\beta(0)}} [\beta(0)x'_0 + \alpha(0)x_0] \]

\[ x(C) = [\cos \Psi(C) + \alpha(0) \sin \Psi(C)]x_0 + \beta(0) \sin \Psi(C)x'_0 \]

\[ x'(C) = -\gamma(0) \sin \Psi(C)x_0 + [\cos \Psi(C) - \alpha(0) \sin \Psi(C)]x'_0 \]

We can write this down in a matrix form where \( \mu = \Psi(C) - \Psi(0) \) is the betatron phase advance through one period \( C \)

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{s_0+C} =
\begin{pmatrix}
  \cos \mu + \alpha(0) \sin \mu & \beta(0) \sin \mu \\
  -\gamma(0) \sin \mu & \cos \mu - \alpha(0) \sin \mu
\end{pmatrix}
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{s_0}
\]

\[
\mu = \int_{s_0}^{s_0+C} ds \frac{1}{\beta(s)}
\]

phase advance per cell
Interesting Observations

- $\mu$ is independent of $s$: this is the **betatron phase advance** of this periodic system
- Determinant of matrix $M$ is still 1!
- Looks like a rotation and some scaling
- $M$ can be written down in a **beautiful** and **deep** way

\[
\begin{align*}
\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} &= \begin{pmatrix} \cos \mu + \alpha(0) \sin \mu & \beta(0) \sin \mu \\ -\gamma(0) \sin \mu & \cos \mu - \alpha(0) \sin \mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \\
\mu &= \int_{s_0}^{s_0+C} ds \frac{1}{\beta(s)} 
\end{align*}
\]

\[M = I \cos \mu + J \sin \mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) \equiv \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}\]

\[J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}\]

remember \[x(s) = A \sqrt{\beta(s)} \cos[\Psi(s) + \Psi_0]\]
Convenient Calculations

- If we know the transport matrix $M$, we can find the lattice parameters (periodic in $C$)

$$
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{s_0+C} =
\begin{pmatrix}
  \cos \mu + \alpha(0) \sin \mu & \beta(0) \sin \mu \\
  -\gamma(0) \sin \mu & \cos \mu - \alpha(0) \sin \mu
\end{pmatrix}
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{s_0}
$$

Betatron phase advance per cell

$$
\cos \mu = \frac{1}{2} \text{Tr } M
$$

$$
\beta(0) = \beta(C') = \frac{m_{12}}{\sin \mu}
$$

$$
\alpha(0) = \alpha(C') = \frac{m_{11} - \cos \mu}{\sin \mu}
$$

$$
\gamma(0) \equiv \frac{1 + \alpha^2(0)}{\beta(0)}
$$
General Non-Periodic Transport Matrix

- We can parameterize a general non-periodic transport matrix from $s_1$ to $s_2$ using lattice parameters and $\Delta \Psi = \Psi(s_2) - \Psi(s_1)$

\[
M_{s_1 \rightarrow s_2} = \begin{pmatrix}
\sqrt{\frac{\beta(s_2)}{\beta(s_1)}} [\cos \Delta \Psi + \alpha(s_1) \sin \Delta \Psi] & \sqrt{\beta(s_1)\beta(s_2)} \sin \Delta \Psi \\
- \frac{[\alpha(s_2) - \alpha(s_1)] \cos \Delta \Psi + [1 + \alpha(s_1)\alpha(s_2)] \sin \Delta \Psi}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}} [\cos \Delta \Psi - \alpha(s_2) \sin \Delta \Psi]
\end{pmatrix}
\]

(C&M Eqn 5.52)

- This does not have a pretty form like the periodic matrix

However both can be expressed as

\[
M = \begin{pmatrix}
C & S \\
C' & S'
\end{pmatrix}
\]

where the C and S terms are cosine-like and sine-like; the second row is the s-derivative of the first row!

A common use of this matrix is the $m_{12}$ term:

\[
\Delta x(s_2) = \sqrt{\beta(s_1)\beta(s_2)} \sin(\Delta \Psi) x'(s_1)
\]

Effect of angle kick on downstream position
(Deriving the Non-Periodic Transport Matrix)

\[ x(s) = Aw(s) \cos \Psi(s) + Bw(s) \sin \Psi(s) \]

\[ x'(s) = A \left( w'(s) \cos \Psi(s) - \frac{\sin \Psi(s)}{w(s)} \right) + B \left( w'(s) \sin \Psi(s) + \frac{\cos \Psi(s)}{w(s)} \right) \]

Calculate \( A, B \) in terms of initial conditions \((x_0, x'_0)\) and \((w_0, \Psi_0)\)

\[ A = \left( w'_0 \sin \Psi_0 + \frac{\cos \Psi_0}{w_0} \right) x_0 - (w_0 \sin \Psi_0) x'_0 \]

\[ B = - \left( w'_0 \cos \Psi_0 - \frac{\sin \Psi_0}{w_0} \right) x_0 + (w_0 \cos \Psi_0) x'_0 \]

Substitute \((A,B)\) and put into matrix form:

\[
\begin{pmatrix}
  x(s) \\
  x'(s)
\end{pmatrix}
= \begin{pmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
\end{pmatrix}
\begin{pmatrix}
  x_0 \\
  x'_0
\end{pmatrix}
\]

\[ m_{11}(s) = \frac{w(s)}{w_0} \cos \Delta \Psi - w(s) w'_0 \sin \Delta \Psi \quad \Delta \Psi \equiv \Psi(s) - \Psi_0 \]

\[ m_{12}(s) = w(s) w_0 \sin \Delta \Psi \]

\[ m_{21}(s) = - \frac{1 + w(s) w_0 w'(s) w'_0}{w(s) w_0} \sin \Delta \Psi - \left[ \frac{w'_0}{w(s)} - \frac{w'(s)}{w_0} \right] \cos \Delta \Psi \]

\[ m_{22}(s) = \frac{w_0}{w(s)} \cos \Delta \Psi + w_0 w' \sin \Delta \Psi \]
Review

Hill’s equation \( x'' + K(s)x = 0 \)

quasi-periodic ansatz solution \( x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0] \)

\[
\begin{align*}
\beta(s) &= \beta(s + C') \\
\gamma(s) &\equiv \frac{1 + \alpha(s)^2}{\beta(s)} \\
\alpha(s) &\equiv -\frac{1}{2}\beta'(s) \\
\Psi(s) &= \int \frac{ds}{\beta(s)}
\end{align*}
\]

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos \mu + \alpha(0)\sin \mu & \beta(0)\sin \mu \\ -\gamma(0)\sin \mu & \cos \mu - \alpha(0)\sin \mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}
\]

betatron phase advance

\[
\mu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)} \quad \text{Tr } M = 2\cos \mu
\]

\[
M = I\cos \mu + J\sin \mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) = \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}
\]

\[
J^2 = -I \quad \Rightarrow \quad M = e^{J(s)\mu}
\]
Transport Matrix Stability Criteria

- For long systems (rings) we want $M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ stable as $n \to \infty$

- If 2x2 $M$ has eigenvectors $(V_1, V_2)$ and eigenvalues $(\lambda_1, \lambda_2)$:
  \[ M^n \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = A\lambda_1^n V_1 + B\lambda_2^n V_2 \]

- $M$ is also unimodular (det $M=1$) so $\lambda_{1,2} = e^{\pm i\mu}$ with complex $\mu$

- For $\lambda_{1,2}^n$ to remain bounded, $\mu$ must be real

- We can always transform $M$ into diagonal form with the eigenvalues on the diagonal (since det $M=1$); this does not change the trace of the matrix

  \[ e^{i\mu} + e^{-i\mu} = 2 \cos \mu = \text{Tr} M \]

- The **stability requirement** for these types of matrices is then

  $\mu$ real \quad \Rightarrow \quad -1 \leq \frac{1}{2} \text{Tr} M \leq 1$
Most accelerator lattices are designed in modular ways
  - Design and operational clarity, separation of functions

One of the most common modules is a FODO module
  - Alternating focusing and defocusing “strong” quadrupoles
  - Spaces between are combinations of drifts and dipoles
  - Strong quadrupoles dominate the focusing
  - Periodicity is one FODO “cell” so we’ll investigate that motion

- Horizontal beam size largest at centers of focusing quads
- Vertical beam size largest at centers of defocusing quads
Periodic Example: FODO Cell Phase Advance

- Select periodicity between centers of focusing quads
  - A natural periodicity if we want to calculate maximum $\beta(s)$

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & \frac{L^2}{4f} + L \\ \frac{L^2}{16f^3} - \frac{L}{4f^2} & 1 - \frac{L^2}{8f^2} \end{pmatrix}$$

$\text{Tr } M = 2 \cos \mu = 2 - \frac{L^2}{4f^2}$

$$1 - \frac{L^2}{8f^2} = \cos \mu = 1 - 2 \sin^2 \frac{\mu}{2} \quad \Rightarrow \quad \sin \frac{\mu}{2} = \pm \frac{L}{4f}$$

- $\mu$ only has real solutions (stability) if $\frac{L}{4} < f$
Periodic Example: FODO Cell Beta Max/Min

- What is the maximum beta function, \( \hat{\beta} \)?
  - A natural periodicity if we want to calculate maximum \( \beta(s) \)

\[
M = \left( \frac{1 - \frac{L^2}{8f^2}}{\frac{L^2}{16f^3} - \frac{L}{4f^2}} \right) \frac{L^2}{4f} + L
\]

\[
\hat{\beta} \sin \mu = \frac{L^2}{4f} + L = L \left( 1 + \sin \frac{\mu}{2} \right)
\]

\[
\hat{\beta} = \frac{L}{\sin \mu} \left( 1 + \sin \frac{\mu}{2} \right)
\]

- Follow a similar strategy reversing F/D quadrupoles to find the minimum \( \beta(s) \) within a FODO cell (center of D quad)

\[
\hat{\beta} = \frac{L}{\sin \mu} \left( 1 - \sin \frac{\mu}{2} \right)
\]
FODO Betatron Functions vs Phase Advance

Generally want $\beta/L$ small

- Strong focusing
- Smaller magnets
- Less expensive accelerator
This is a picture of a FODO lattice, showing contours of $\pm \sqrt{\beta(s)}$ since the particle motion goes like $x(s) = A\sqrt{\beta(s)} \cos[\Psi(s) + \Psi_0]$

- This also shows a particle oscillating through the lattice
- Note that $\sqrt{\beta(s)}$ provides an “envelope” for particle oscillations
  - $\sqrt{\beta(s)}$ is sometimes called the envelope function for the lattice
- Min beta is at defocusing quads, max beta is at focusing quads
- 6.5 periodic FODO cells per betatron oscillation

$$\Rightarrow \mu = 360^\circ / 6.5 \approx 55^\circ$$
Example: RHIC FODO Lattice

- 1/6 of one of two RHIC synchrotron rings, injection lattice
  - FODO cell length is about $L = 30$ m
  - Phase advance per FODO cell is about $\mu = 77^\circ = 1.344$ rad

\[
\hat{\beta} = \frac{L}{\sin \mu} \left( 1 + \sin \frac{\mu}{2} \right) \approx 53 \text{ m}
\]

\[
\ddot{\beta} = \frac{L}{\sin \mu} \left( 1 - \sin \frac{\mu}{2} \right) \approx 8.7 \text{ m}
\]
Propagating Lattice Parameters

- If I have $(\beta, \alpha, \gamma)(s_1)$ and I have the transport matrix $M(s_1, s_2)$ that transports particles from $s_1 \rightarrow s_2$, how do I find the new lattice parameters $(\beta, \alpha, \gamma)(s_2)$?

$$M(s_1, s_1 + C) = I \cos \mu + J \sin \mu = \begin{pmatrix} \cos \mu + \alpha(s_1) \sin \mu & \beta(s_1) \sin \mu \\ -\gamma(s_1) \sin \mu & \cos \mu - \alpha(s_1) \sin \mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

**Homework 😊**

$$\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}$$
Propagating Lattice Parameters

- If I have \((\beta, \alpha, \gamma)(s_1)\) and I have the transport matrix \(M(s_1, s_2)\) that transports particles from \(s_1 \rightarrow s_2\), how do I find the new lattice parameters \((\beta, \alpha, \gamma)(s_2)\)?

\[
M(s_1, s_1 + C) = I \cos \mu + J \sin \mu = \begin{pmatrix}
\cos \mu + \alpha(s_1) \sin \mu & \beta(s_1) \sin \mu \\
-\gamma(s_1) \sin \mu & \cos \mu - \alpha(s_1) \sin \mu
\end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\
m_{21} & m_{22} \end{pmatrix}
\]

The J(s) matrices at \(s_1, s_2\) are related by

\[
J(s_2) = M(s_1, s_2)J(s_1)M^{-1}(s_1, s_2)
\]

Then expand, using \(\det M = 1\)

\[
J(s_2) = \begin{pmatrix}
\alpha(s_2) & \beta(s_2) \\
-\gamma(s_2) & -\alpha(s_2)
\end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\
m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \alpha(s_1) & \beta(s_1) \\
-\gamma(s_1) & -\alpha(s_1) \end{pmatrix} \begin{pmatrix} m_{22} & -m_{12} \\
-m_{21} & m_{11} \end{pmatrix}
\]

Quadratic: Lattice elements repeat themselves for \(M = \pm I\)
**What’s the Ellipse?**

- Area of an ellipse that envelops a given percentage of the beam particles in phase space is related to the **emittance**.

We can express this in terms of our lattice functions!
Invariants and Ellipses

\[ x(s) = A \sqrt{\beta(s)} \cos[\phi(s) + \phi_0]. \]

- We assumed \( A \) was constant, an invariant of the motion.
  
  A can be expressed in terms of initial coordinates to find
  
  \[ \mathcal{W} \equiv A^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 x'_0 + \beta_0 x'_0^2 \]

  This is known as the **Courant-Snyder invariant**: for all \( s \),
  \[ \mathcal{W} = \gamma(s) x(s)^2 + 2\alpha(s) x(s) x'(s) + \beta(s) x'(s)^2 \]

  Similar to total energy of a simple harmonic oscillator

  \( \mathcal{W} \) looks like an elliptical area in \((x, x')\) phase space

  Our matrices look like scaled rotations (ellipses) in phase space
Emittance

- The area of the ellipse inscribed by any given particle in phase space as it travels through our accelerator is called the **emittance** $\epsilon$: it is “constant” and given by

\[
\epsilon = \pi \mathcal{W} = \pi \left[ \gamma(s) x(s)^2 + 2 \alpha(s) x(s) x'(s) + \beta(s) x'(s)^2 \right]
\]

Emittance is often quoted as the area of the ellipse that would contain a certain fraction of all (Gaussian) beam particles

- e.g. RMS emittance contains 39% of 2D beam particles

Related to RMS beam size $\sigma_{RMS}$

\[
\sigma_{RMS} = \sqrt{\epsilon \beta(s)}
\]

- RMS beam size depends on $s$!

RMS emittance convention is fairly standard for electron rings, with units of mm-mrad
Adiabatic Damping and Normalized Emittance

- But we introduce electric fields when we accelerate
  - When we accelerate, invariant emittance is not invariant!
  - We are defining areas in \((x, x')\) phase space
  - The definition of \(x\) doesn’t change as we accelerate
  - But \(x' \equiv dx/ds = p_x/p_0\) does since \(p_0\) changes!
  - \(p_0\) scales with relativistic beta, gamma: \(p_0 \propto \beta \gamma\)
  - This has the effect of compressing \(x'\) phase space by \(\beta \gamma\)

- **Normalized emittance** is the invariant in this case
  - unnormalized emittance goes down as we accelerate
  - This is called adiabatic damping, important in, e.g., linacs

\[
\begin{align*}
\epsilon_N & \equiv \beta \gamma \epsilon
\end{align*}
\]
Phase Space Ellipse Geography

- Now we can figure out some things from a phase space ellipse at a given s coordinate:

\[
\begin{align*}
  x_1 &= \sqrt{W/\gamma(s)} \\
  x_2 &= \sqrt{W/\beta(s)} \\
  y_1 &= \sqrt{W/\beta(s)} \\
  y_2 &= \sqrt{W/\gamma(s)}
\end{align*}
\]
Rings and Tunes

- A synchrotron is by definition a periodic focusing system
  - It is very likely made up of many smaller periodic regions too
  - We can write down a periodic one-turn matrix as before

\[
M = I \cos \mu + J \sin \mu \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J(s_0) = \begin{pmatrix} \alpha(0) & \beta(0) \\ -\gamma(0) & -\alpha(0) \end{pmatrix}
\]

- We define **tune** as the total betatron phase advance in one revolution around a ring divided by the total angle \(2\pi\)

\[
Q_{x,y} = \frac{\Delta \mu_{x,y}}{\Delta \theta} = \frac{1}{2\pi} \int ds \frac{1}{\beta_{x,y}(s)}
\]

Horizontal Betatron Oscillation with tune: \(Q_h = 6.3\), i.e., 6.3 oscillations per turn.

Vertical Betatron Oscillation with tune: \(Q_v = 7.5\), i.e., 7.5 oscillations per turn.
Tunes

- There are horizontal and vertical tunes
  - turn by turn oscillation frequency

- Tunes are a direct indication of the amount of focusing in an accelerator
  - Higher tune implies tighter focusing, lower $\langle \beta_{x,y}(s) \rangle$

- Tunes are a critical parameter for accelerator performance
  - Linear stability depends greatly on phase advance
  - Resonant instabilities can occur when $nQ_x + mQ_y = k$
  - Often adjusted by changing groups of quadrupoles

$$M_{\text{one turn}} = I \cos(2\pi Q) + J \sin(2\pi Q)$$
6.2: Stability Diagrams

- Designers often want or need to change the focusing of the two transverse planes in a FODO structure.
  - What happens if the focusing/defocusing strengths differ?

- Recalculate the M matrix and use dimensionless quantities:
  
  \[
  F \equiv \frac{L}{2f_F}, \quad D \equiv \frac{L}{2f_D}
  \]

  then take the trace for stability conditions to find:

  \[ \cos \mu = 1 + D - F - \frac{FD}{2} \]
  \[ \sin^2 \frac{\mu}{2} = \frac{FD}{4} + \frac{F - D}{2} \]
Stability Diagrams II

\[
\cos \mu = 1 + D - F - \frac{F D}{2} \quad \sin^2 \frac{\mu}{2} = \frac{F D}{4} + \frac{F - D}{2}
\]

- For stability, we must have \(-1 < \cos \mu < 1\)
- Using \(\cos \mu = 1 - 2 \sin^2 \frac{\mu}{2}\), stability limits are where
  \[
  \sin^2 \frac{\mu}{2} = 0 \quad \sin^2 \frac{\mu}{2} = 1
  \]

- These translate to an a “necktie” stability diagram for FODO

![Stability Diagram](image)

Figure 6.1 Stability or “necktie” diagram for an alternate focusing lattice. The shaded area is the region of stability.
6.3: Dispersion

- There is one more important lattice parameter to discuss
- **Dispersion** $\eta(s)$ is defined as the change in particle position with fractional momentum offset $\delta \equiv \Delta p/p_0$

$$x(s) = \text{betatron} + \eta_x(s)\delta \quad \eta_x(s) \equiv \frac{dx}{d\delta}$$

Dispersion originates from momentum dependence of dipole bends
Equivalent to separation of optical wavelengths in prism
This is known in accelerator lattice design language as a "double bend achromat".
Dispersion

- Add explicit momentum dependence to equation of motion again

\[ x'' + K(s)x = \frac{\delta}{\rho(s)} \]

Assume our ansatz solution and use initial conditions to find

\[
\begin{align*}
x(s) &= C(s)x_0 + S(s)x'_0 + D(s)\delta_0 \\
x'(s) &= C'(s)x_0 + S'(s)x'_0 + D'(s)\delta_0
\end{align*}
\]

\[ D(s) = S(s) \int_0^s \frac{C(\tau)}{\rho(\tau)} d\tau - C(s) \int_0^s \frac{S(\tau)}{\rho(\tau)} d\tau \]

Particular solution of inhomogeneous differential equation with periodic \( \rho(s) \)

\[
\begin{pmatrix}
x(s) \\
x'(s) \\
\delta(s)
\end{pmatrix} =
\begin{pmatrix}
C(s) & S(s) & D(s) \\
C'(s) & S'(s) & D'(s) \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x'_0 \\
\delta_0
\end{pmatrix}
\]

The trajectory has two parts:

\[ x(s) = \text{betatron} + \eta_x(s)\delta \quad \eta_x(s) \equiv \frac{dx}{d\delta} \]
Dispersion Continued

- Substituting and noting dispersion is periodic, \( \eta_x(s + C) = \eta_x(s) \)

\[
\begin{pmatrix}
\eta_x(s) \\
\eta'_x(s) \\
\delta(s)
\end{pmatrix} =
\begin{pmatrix}
C(s) & S(s) & D(s) \\
C'(s) & S'(s) & D'(s) \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\eta_x(s) \\
\eta'_x(s) \\
\delta_0
\end{pmatrix}
\]

achromat : \( D = D' = 0 \)

- If we take \( \delta_0 = 1 \) we can solve this in a clever way

\[
\begin{pmatrix}
\eta_x(s) \\
\eta'_x(s)
\end{pmatrix} =
\begin{pmatrix}
C(s) & S(s) \\
C'(s) & S'(s)
\end{pmatrix}
\begin{pmatrix}
\eta_x(s) \\
\eta'_x(s)
\end{pmatrix} +
\begin{pmatrix}
D(s) \\
D'(s)
\end{pmatrix}
\]

\[
(I - M)
\begin{pmatrix}
\eta_x(s) \\
\eta'_x(s)
\end{pmatrix} =
\begin{pmatrix}
D(s) \\
D'(s)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\eta_x(s) \\
\eta'_x(s)
\end{pmatrix} = (I - M)^{-1}
\begin{pmatrix}
D(s) \\
D'(s)
\end{pmatrix}
\]

- Solving gives

\[
\eta(s) = \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos \mu)}
\]

\[
\eta'(s) = \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos \mu)}
\]
FODO Cell Dispersion

- A periodic lattice without dipoles has no intrinsic dispersion.
- Consider FODO with long dipoles and thin quadrupoles.
  - Each dipole has total length $\rho\theta_C/2$ so each cell is of length $L = \rho\theta_C$.
  - Assume a large accelerator with many FODO cells so $\theta_C \ll 1$.

\[
M_{-2f} = \begin{pmatrix}
1 & 0 & 0 \\
-\frac{1}{2f} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
M_{\text{dipole}} = \begin{pmatrix}
1 & \frac{L}{2} & \frac{L\theta_C}{\theta_C^2} \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad
M_f = \begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{f} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
M_{\text{FODO}} = M_{-2f} M_{\text{dipole}} M_f M_{\text{dipole}} M_{-2f}
\]

\[
M_{\text{FODO}} = \begin{pmatrix}
1 - \frac{L^2}{8f^2} & L \left(1 + \frac{L}{4f}\right) & \frac{L}{2} \left(1 + \frac{L}{8f}\right) \theta_C \\
-\frac{L}{4f^2} \left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right) \theta_C \\
0 & 0 & 1
\end{pmatrix}
\]
FODO Cell Dispersion

- Like $\hat{\beta}$ before, this choice of periodicity gives us $\hat{\eta}_x$
  \[
  \hat{\eta}_x = \frac{L\theta_C}{4} \left[ \frac{1 + \frac{1}{2} \sin \frac{\mu}{2}}{\sin^2 \frac{\mu}{2}} \right] \quad \eta_x' = 0 \text{ at max}
  \]

- Changing periodicity to defocusing quad centers gives $\tilde{\eta}_x$
  \[
  \tilde{\eta}_x = \frac{L\theta_C}{4} \left[ \frac{1 - \frac{1}{2} \sin \frac{\mu}{2}}{\sin^2 \frac{\mu}{2}} \right] \quad \eta_x' = 0 \text{ at min}
  \]

\[
\theta_c = \frac{2\pi}{25} \quad (25 \text{ FODO cells})
\]
RHIC FODO Cell

Horizontal dispersion

Half quadrupole

Dipole

Quadrupole

Half quadrupole

Vertical
6.6: Dispersion Suppressor

- The FODO dispersion solution is non-zero everywhere
  - But in straight sections we often want $\eta_x = \eta'_x = 0$
    - e.g. to keep beam small in wigglers/undulators in a light source
  - We can “match” between these two conditions with with a dispersion suppressor, a non-periodic set of magnets that transforms FODO $(\eta_x, \eta'_x)$ to zero.

- Consider two FODO cells with different total bend angles $\theta_1, \theta_2$
  - Same quadrupole focusing to not disturb $\beta_x, \mu_x$ much
  - We want this to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0, 0)$
  - $\alpha_x = 0$ at ends to simplify periodic matrix
FODO Dispersion Suppressor

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
\cos 2\mu_x & \beta_x \sin 2\mu_x & D(s) \\
-\frac{\sin 2\mu_x}{\beta_x} & \cos 2\mu_x & D'(s) \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\hat{\eta}_x \\
0 \\
1
\end{pmatrix}
\]

multiply matrices \[\Rightarrow\]

\[D(s) = \frac{L}{2} \left( 1 + \frac{L}{8f} \right) \left[ \left( 3 - \frac{L^2}{4f^2} \right) \theta_1 + \theta_2 \right]\]

\[D'(s) = \left( 1 - \frac{L}{8f} - \frac{L^2}{32f^2} \right) \left[ \left( 1 - \frac{L^2}{4f^2} \right) \theta_1 + \theta_2 \right]\]

\[\hat{\eta}_x = \frac{4f^2}{L} \left( 1 + \frac{L}{8f} \right) (\theta_1 + \theta_2)\]

\[\theta_1 = \left( 1 - \frac{1}{4\sin^2 \frac{\mu}{2}} \right) \theta \quad \theta_2 = \left( \frac{1}{4\sin^2 \frac{\mu}{2}} \right) \theta\]

\[\theta = \theta_1 + \theta_2\]

two cells, one FODO bend angle \[\Rightarrow\] reduced bending
FODO Cell Dispersion and Suppressor

Dispersion-free Insertion
Mismatched Dispersion

- Someone in class asked what mismatched dispersion looks like
  - For example, this is what happens when the second dispersion suppressor is eliminated and the dipole-free FODO cells run right up against the FODO cells with dipoles
6.5: $\pi/2$ Insertion

- Insertions and matching: modular accelerator design
- FODO sections have very regular spacings of quads
  - Periodicity of quadrupoles => periodicity of focusing
- But we need some long quadrupole-free sections
  - RF, injections, extraction, experiments, long instruments

- Can we design a “module” that fits in a FODO lattice with a long straight section, and matches to FODO optics?
  - Yes: a minimal option is called the $\pi/2$ insertion
  - Matching lattice functions $(\beta, \alpha)_{x,y}$ at locations A,B
\[ \pi/2 \text{ Insertion} \]

\[
\mathbf{M} = \begin{pmatrix}
1 & l_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix}
\begin{pmatrix}
1 & l_2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{1}{f} & 1
\end{pmatrix}
\begin{pmatrix}
1 & l_1 \\
0 & 1
\end{pmatrix}
\]

\[
\mathbf{M} = \begin{pmatrix}
1 - \frac{l_1 l_2}{f^2} & 2l_1 + l_2 - \frac{l_1^2 l_2}{f^2} \\
-\frac{l_2}{f^2} & 1 - \frac{l_1 l_2}{f^2} - \frac{l_2}{f}
\end{pmatrix} = \begin{pmatrix}
\cos \mu + \alpha \sin \mu & \beta \sin \mu \\
-\gamma \sin \mu & \cos \mu - \alpha \sin \mu
\end{pmatrix}
\]

\[
\cos \mu = 1 - \frac{l_1 l_2}{f^2} \quad \beta \sin \mu = \left(2 - \frac{l_1 l_2}{f^2}\right) l_1 + l_2 \quad \gamma \sin \mu = \frac{l_2}{f^2}
\]

\[
m_{11} - m_{22} \text{ comparison: } l_2 = \alpha f \sin \mu
\]

Maximum \( l_2 \) when \( \sin \mu = 1, \mu = \frac{\pi}{2}, \cos \mu = 0 \)
\( \pi/2 \) Insertion

Design constraints:
\[
\begin{align*}
  f &= \frac{\alpha}{\gamma} \\
  l_2 &= \frac{\alpha^2}{\gamma} \\
  l_1 &= \beta - l_2
\end{align*}
\]

\( M_{\pi/2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = J \quad (\text{recall } J^2 = -I) \)

5m FODO drifts

\( l_1 = 7.95\text{m}, l_2 = 8.75\text{m} \)
Multiple $\pi/2$ Insertions

\[ M^2 = -I \]

\[ M^4 = I \]
Multiple $\pi/2$ Insertions

\[ M^2 = -I \]

Lattice elements repeat: see p. 22

\[ M^4 = I \]
Example: RHIC FODO Lattice Revisited

- Note modular design, including low-beta insertions
  - Used for experimental collisions
  - Minimum beam size $\sigma$ (with zero dispersion)
    - maximize luminosity
  - Large $s$, beam size in “low beta quadrupoles”
  - Other facilities also have longitudinal bunch compressors
    - Minimize longitudinal beam size (bunch length) for, e.g, FELs