USPAS Graduate Accelerator Physics Homework 3

Due date: Thursday January 22, 2015

1 C-M 5.4: Floquet Transformation

(a) (5 points) Show that the coordinate transformation

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta^{-\frac{1}{2}} & 0 \\ \alpha\beta^{-\frac{1}{2}} & \beta^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix}$$

transforms the transfer matrix $\mathbf{M} = \mathrm{e}^{\mathbf{J}\mu}$ into the matrix

$$\mathbf{N} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} .$$

These new coordinates (ξ, ζ) are sometimes referred to as Floquet or Courant-Snyder coordinates. Note that the ellipse of the Courant-Snyder invariant has been transformed into a circle. Show that the invariant $\mathcal{W}$ remains unchanged under this transformation. (The fact that it will preserves phase space area helps with your third homework problem of this set!)

(b) (15 points) Consider a Gaussian distribution of particles in the new coordinates,

$$f(\xi, \zeta) = \frac{N}{2\pi\epsilon} \exp\left(-\frac{\xi^2 + \zeta^2}{2\epsilon} \right) .$$

Find the distribution in the old coordinates (z, z'). Evaluate the variances $\sigma_z^2 = \langle (z - \langle z \rangle)^2 \rangle$, and $\sigma_{z'}^2 = \langle (z' - \langle z' \rangle)^2 \rangle$, and the covariance $\sigma_{zz'}^2 = \langle (z - \langle z \rangle)(z' - \langle z' \rangle) \rangle$.

2 C-M 5.5: Twiss Parameter Propagation

(10 points) Using the Courant-Snyder invariant

$$\mathcal{W} = \gamma z^2 + 2\alpha z z' + \beta z'^2 ,$$

show that the Twiss parameters transform from s_1 to s_2 by the matrix transformation

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} ,$$

if the one-dimensional transport matrix is given by

$$\begin{pmatrix} z_2 \\ z_2' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_1' \end{pmatrix} .$$

3 C-M 5.10: Conversion of Emittances

(10 points) Show that the conversion from rms to 90% and 95% emittances are approximately $\epsilon_{90\%} = 4.605\,\epsilon_{\mathrm{rms}}$ and $\epsilon_{95\%} = 5.991\,\epsilon_{\mathrm{rms}}$ for a Gaussian distribution. Hint: It is by far easiest to do this problem in the normalized coordinates (ξ, ζ) of the first problem.