USPAS Graduate Accelerator Physics Homework 3

Due date: Thursday January 22, 2015

1 C-M 5.4: Floquet Transformation

(a) (5 points) Show that the coordinate transformation
\[
\begin{pmatrix}
\xi \\
\zeta
\end{pmatrix} = \begin{pmatrix}
\beta - \frac{1}{2} \\
\alpha \beta - \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
z \\
z'
\end{pmatrix}
\]
transforms the transfer matrix \( M = e^{J\mu} \) into the matrix
\[
N = \begin{pmatrix}
\cos \mu & \sin \mu \\
-\sin \mu & \cos \mu
\end{pmatrix}.
\]
These new coordinates \((\xi, \zeta)\) are sometimes referred to as Floquet or Courant-Snyder coordinates. Note that the ellipse of the Courant-Snyder invariant has been transformed into a circle. Show that the invariant \( W \) remains unchanged under this transformation. (The fact that it will preserves phase space area helps with your third homework problem of this set!)

(b) (15 points) Consider a Gaussian distribution of particles in the new coordinates,
\[
f(\xi, \zeta) = \frac{N}{2\pi \epsilon} \exp \left( -\frac{\xi^2 + \zeta^2}{2\epsilon} \right).
\]
Find the distribution in the old coordinates \((z, z')\). Evaluate the variances \( \sigma_z^2 = \langle (z - \langle z \rangle)^2 \rangle \), and \( \sigma_{z'}^2 = \langle (z' - \langle z' \rangle)^2 \rangle \), and the covariance \( \sigma_{zz'}^2 = \langle (z - \langle z \rangle)(z' - \langle z' \rangle) \rangle \).

2 C-M 5.5: Twiss Parameter Propagation

(10 points) Using the Courant-Snyder invariant
\[
W = \gamma z^2 + 2\alpha zz' + \beta z'^2,
\]
show that the Twiss parameters transform from \( s_1 \) to \( s_2 \) by the matrix transformation
\[
\begin{pmatrix}
\beta_2 \\
\alpha_2 \\
\gamma_2
\end{pmatrix} = \begin{pmatrix}
M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\
-M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\
M_{21}^2 & -2M_{21}M_{22} & M_{22}^2
\end{pmatrix} \begin{pmatrix}
\beta_1 \\
\alpha_1 \\
\gamma_1
\end{pmatrix},
\]
if the one-dimensional transport matrix is given by
\[
\begin{pmatrix}
z_2 \\
z'_2
\end{pmatrix} = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix} \begin{pmatrix}
z_1 \\
z'_1
\end{pmatrix}.
\]

3 C-M 5.10: Conversion of Emittances

(10 points) Show that the conversion from rms to 90% and 95% emittances are approximately
\[
\epsilon_{90\%} = 4.605 \epsilon_{\text{rms}} \quad \text{and} \quad \epsilon_{95\%} = 5.991 \epsilon_{\text{rms}}
\]
for a Gaussian distribution. Hint: It is by far easiest to do this problem in the normalized coordinates \((\xi, \zeta)\) of the first problem.