University Physics 226N/231N
Old Dominion University

Ch 10: Angular Work, Power, Momentum

Dr. Todd Satogata (ODU/Jefferson Lab)

satogata@jlab.org

http://www.toddsatogata.net/2016-ODU

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Reminder: The Third Midterm will be Mon Nov 21 2016

Happy Birthday to John Keats, Dan Alderson, Nick Saban,
Neal Stephenson, Rob Schneider, and Peter Jackson!
Happy Halloween!

Please set your cell phones to “vibrate” or “silent” mode. Thanks!

Review: Constant Angular Acceleration

- Problems with constant angular acceleration are exactly analogous to similar problems involving linear motion in one dimension.
  - The **exact** same equations apply, with $x \to \theta, \quad v \to \omega, \quad a \to \alpha$

Table 10.1 Angular and Linear Position, Velocity, and Acceleration

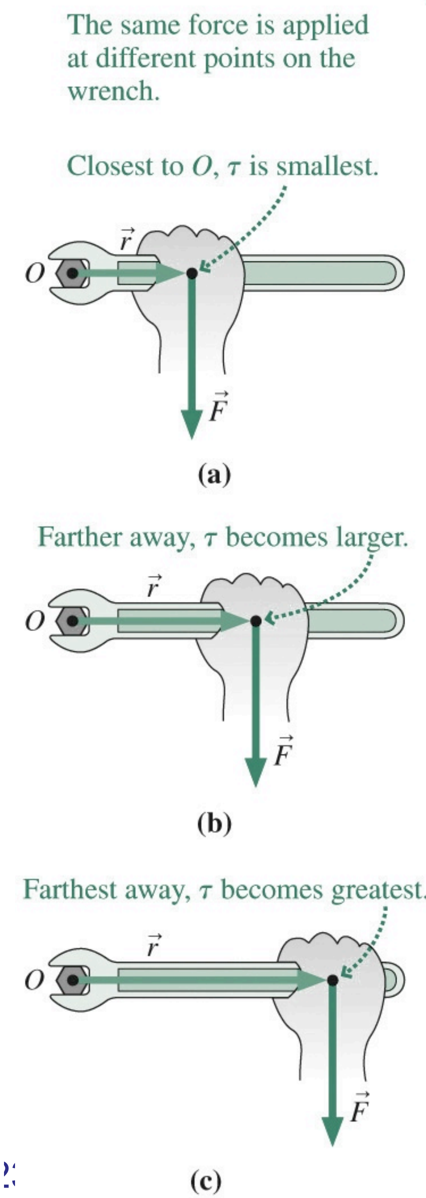
Linear Quantity	**Angular Quantity**
Position x	Angular position θ
Velocity $v = \frac{dx}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

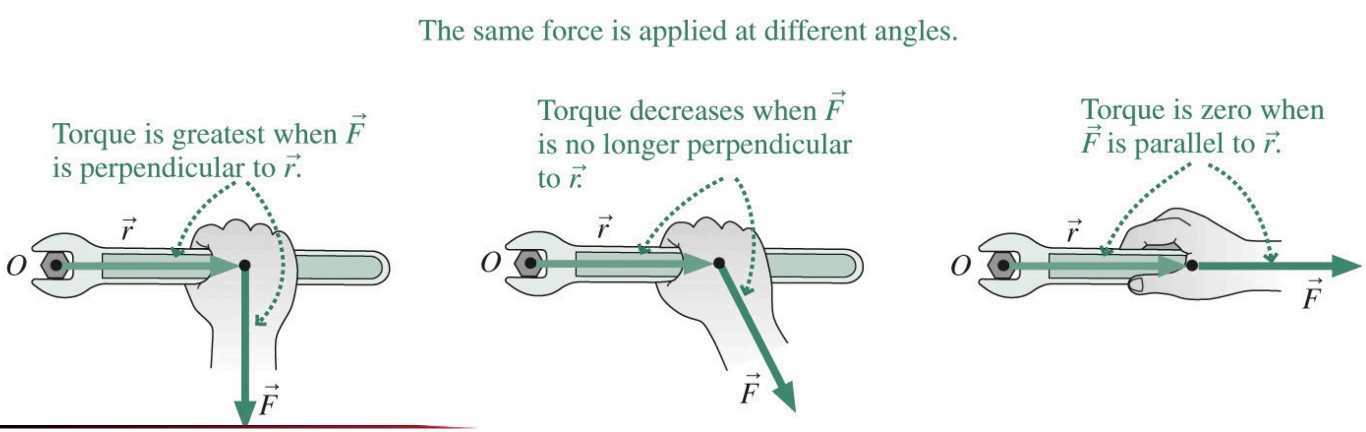
Equations for Constant Linear Acceleration		**Equations for Constant Angular Acceleration**	
$\bar{v} = \frac{1}{2}(v_0 + v)$	(2.8)	$\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$	(10.6)
$v = v_0 + at$	(2.7)	$\omega = \omega_0 + \alpha t$	(10.7)
$x = x_0 + v_0 t + \frac{1}{2}at^2$	(2.10)	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	(10.8)
$v^2 = v_0^2 + 2a(x - x_0)$	(2.11)	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10.9)

Review: Torque

- Torque τ is the rotational analog of force, and results from the application of one or more forces.
- Torque is relative to a chosen rotation axis.
- Torque depends on:
  - the distance from the rotation axis to the force application point.
  - the magnitude of the force $\vec{F}$
  - the orientation of the force relative to the displacement $\vec{r}$ from axis to force application point:

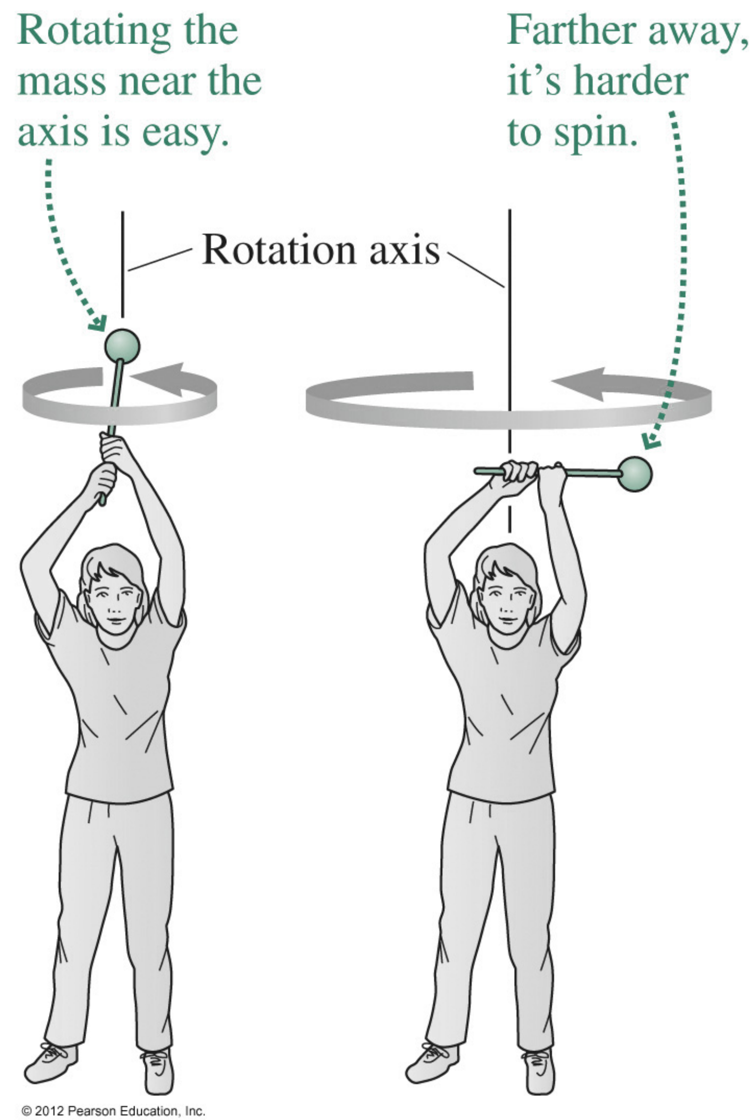
$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \tau = rF\sin\theta$$





Review: Rotational Analog of Newton's Law

- **Rotational inertia** I (or moment of inertia) is the rotational analog of mass.
  - Rotational inertia depends on the distribution of mass and its distance from the rotation axis, similar to center of mass.



- Rotational acceleration, torque, and rotational inertia combine to give the rotational analog of Newton's second law $F = ma$

$$\tau = I\alpha$$

(or, more properly with vectors)

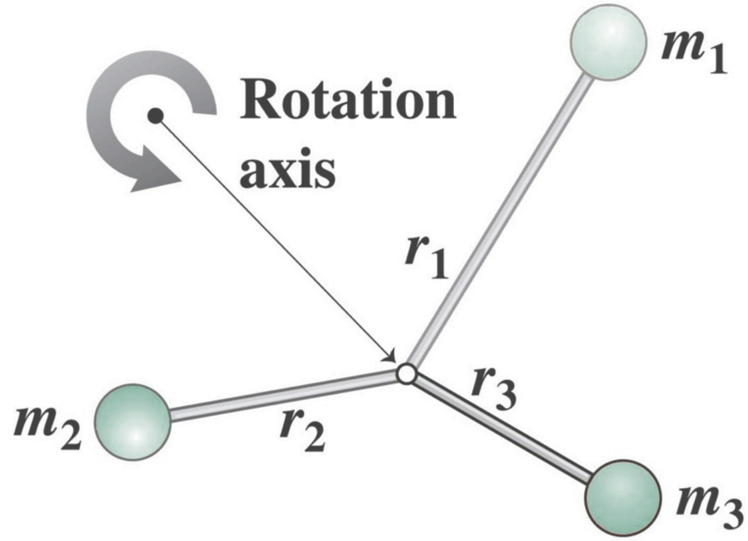
$$\vec{\tau} = I\vec{\alpha}$$

like $\vec{F} = m\vec{a}$

Review: Calculating Rotational Inertia

- For a single point mass *m*, rotational inertia is the product of mass with the square of the distance *r* from the rotation axis:

$$I = mr^2$$

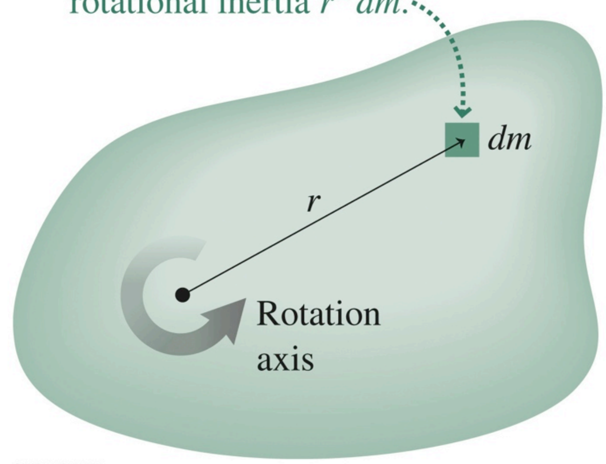


- For a system of discrete masses, the rotational inertia is the sum of the rotational inertias of the individual masses:

$$I = \sum m_i r_i^2$$

- For continuous matter, the rotational inertia is given by an integral over the distribution of matter:

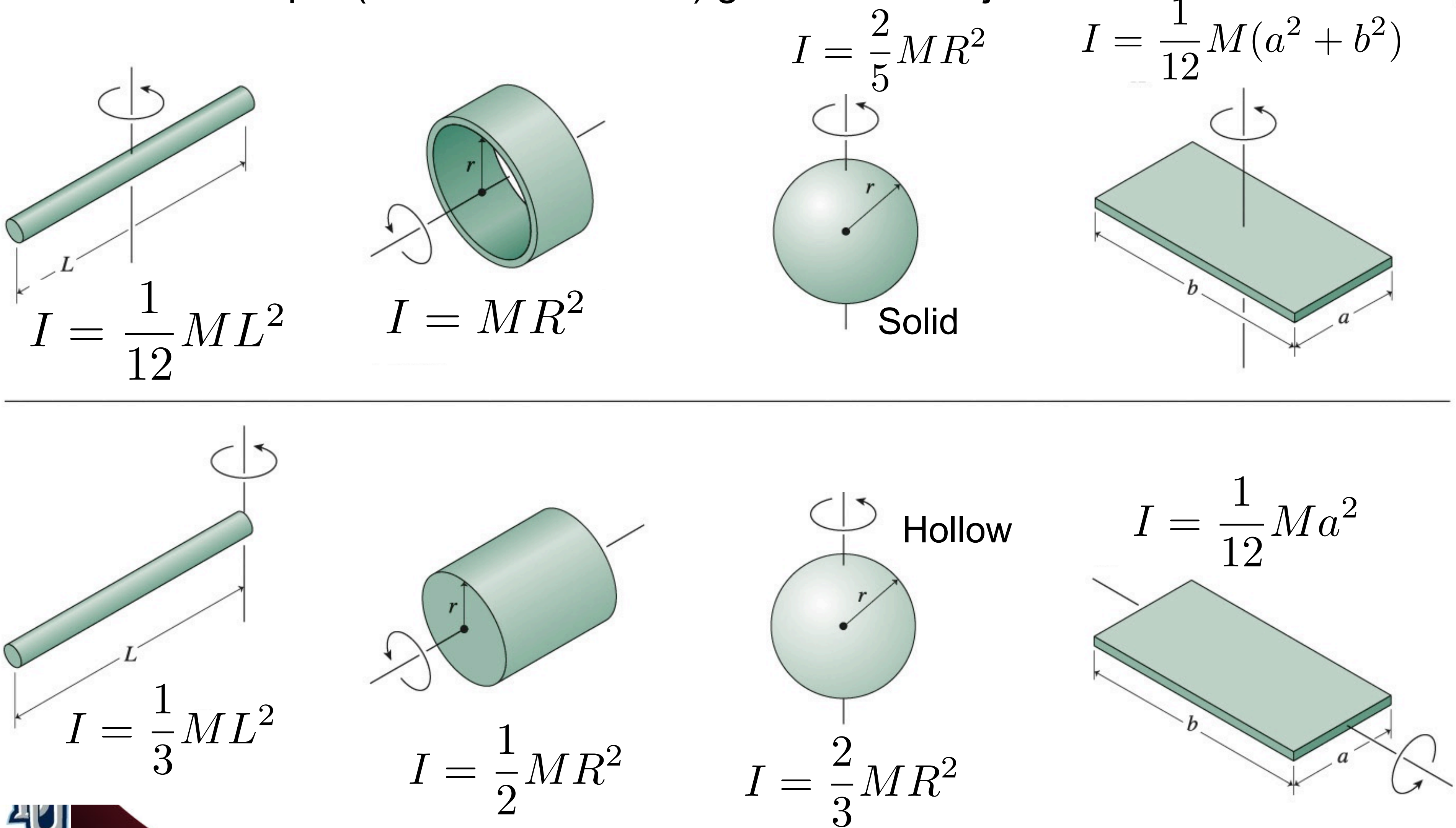
$$I = \int r^2 \, dm$$



Similar to center of mass: $\vec{r}_{\mathrm{cm}} = \dfrac{\int \vec{r}\, dm}{M}$

Some Rotational Inertias of Simple Objects

- We really do need to use calculus to figure out rotational inertias of most simple (three-dimensional) geometrical objects



$$I = \frac{1}{12}ML^2$$

$$I = MR^2$$

$$I = \frac{2}{5}MR^2$$

$$I = \frac{1}{12}M(a^2 + b^2)$$

$$I = \frac{1}{3}ML^2$$

$$I = \frac{1}{2}MR^2$$

$$I = \frac{2}{3}MR^2$$

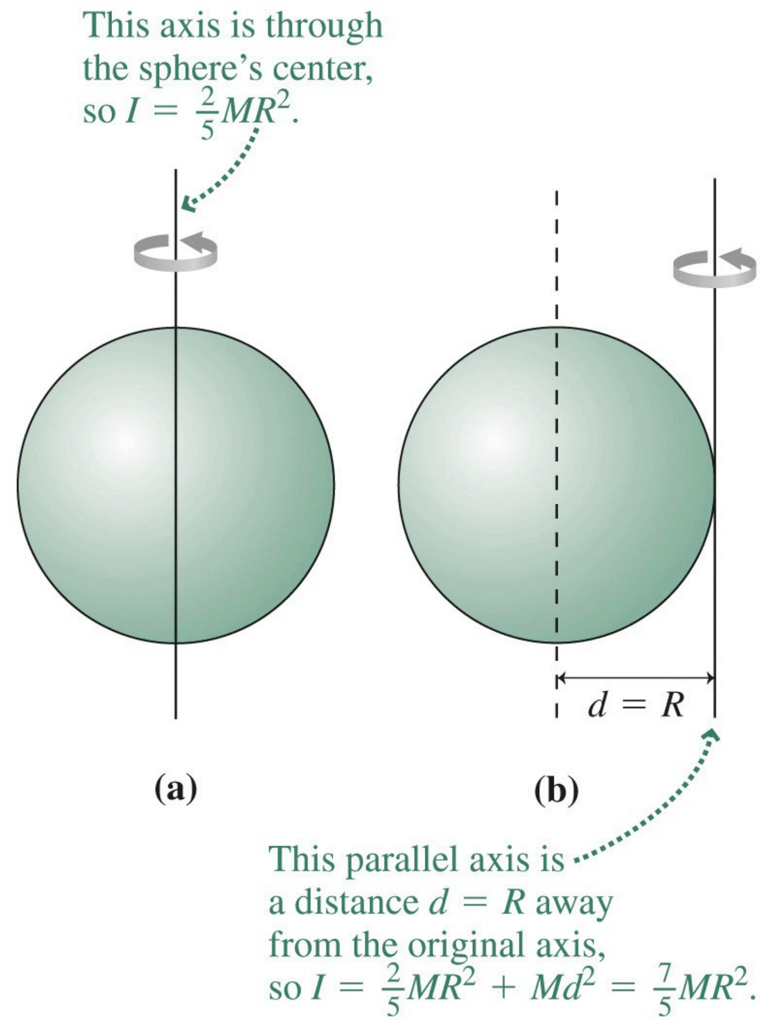
$$I = \frac{1}{12}Ma^2$$

Review: Parallel Axis Theorem

- If we know the rotational inertia I_{cm} about an axis through the center of mass of a body, the **parallel-axis theorem** allows us to calculate the rotational inertia I through any parallel axis.
- The parallel-axis theorem states that

$$I = I_{\rm cm} + Md^2$$

where d is the distance from the center-of-mass axis to the parallel axis and M is the total mass of the object.



Example: Hula Hoop Rotational Inertia

- What is the rotational inertia of a hula hoop of radius r and mass M around its edge?

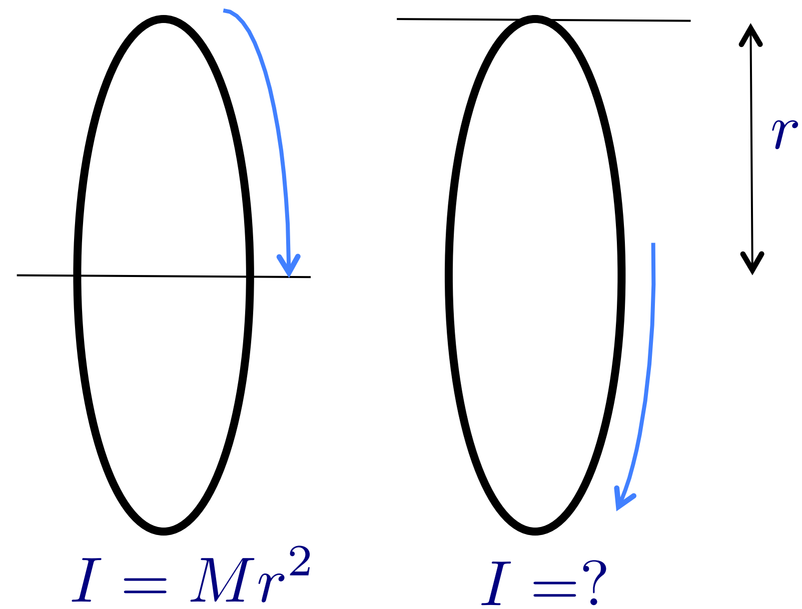
- Its center of mass is (obviously) at the center of the circle and all of its mass is at the same radius

$$I_{\text{cm}} = Mr^2$$

- The parallel axis theorem gives

$$I_{\text{edge}} = Mr^2 + Mr^2 = 2Mr^2$$

Interesting… It takes twice as much torque to turn a ring around its edge as it takes to turn around its center.



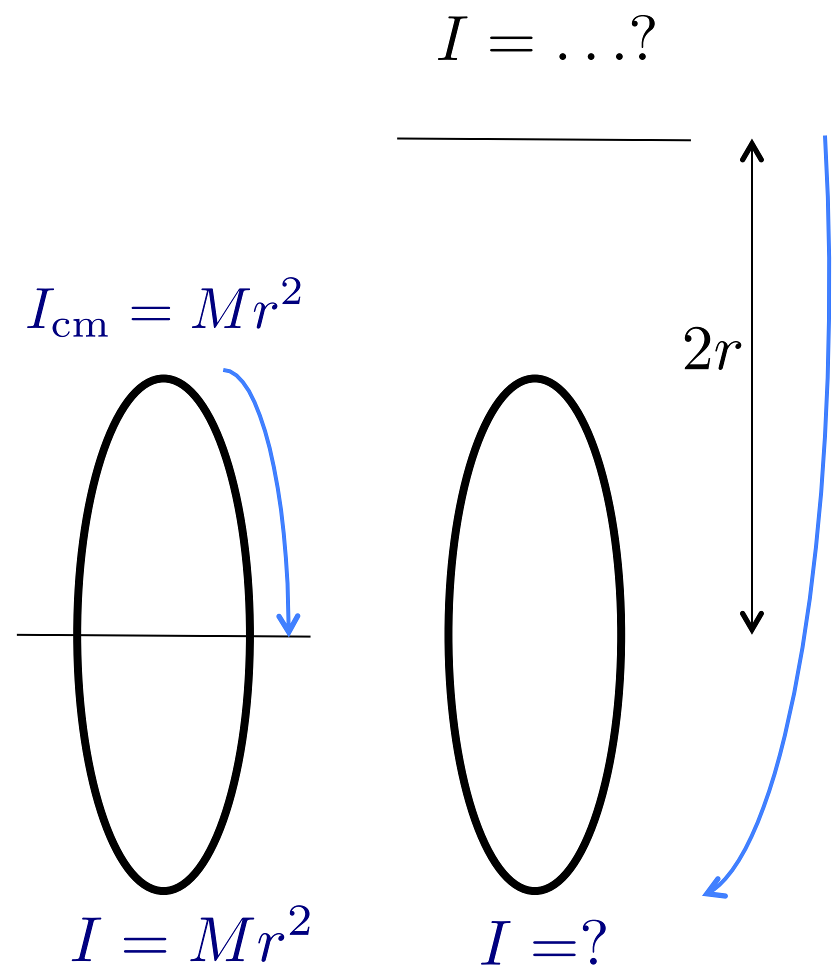
Example: Hula Hoop Rotational Inertia

- What is the rotational inertia of a hula hoop of radius r and mass M around an axis 2r from its center?

- Its center of mass is (obviously) at the center of the circle and all of its mass is at the same radius

$$I_{\rm cm} = Mr^2$$

- The parallel axis theorem gives…



Calculating Parallel Axis Theorem

Know $I = \frac{1}{12} ML^2$

$d = L/2$

$\Rightarrow I(\text{end}) = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2$

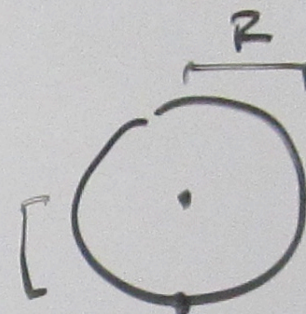
$= \frac{1}{12} ML^2 + \frac{1}{4} ML^2$

$= \frac{4}{12} ML^2 = \frac{1}{3} ML^2$

(1)

P.A.T. $\Rightarrow$

$I = I_{cm} + Md^2$



$I_{cm} = MR^2$

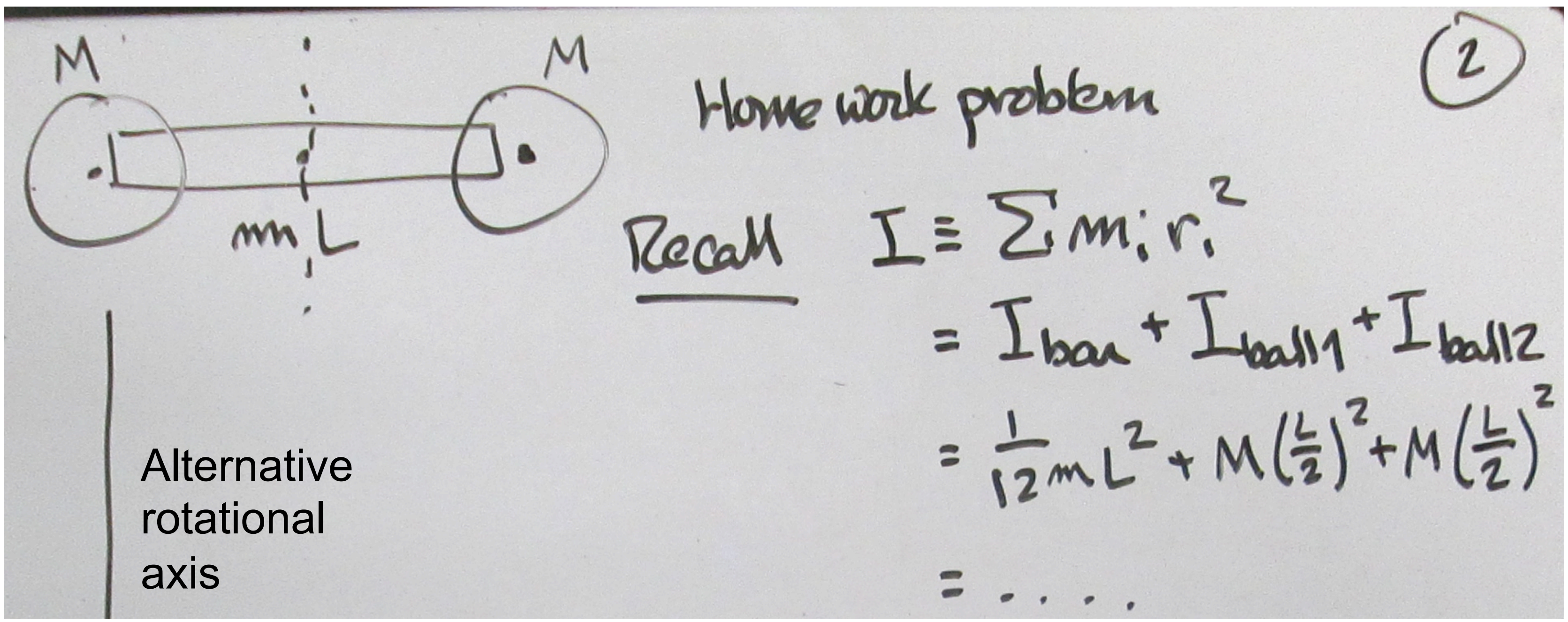
$I(\text{edge}) = MR^2 + MR^2$

$I(2R) = MR^2 + M(2R)^2$

$= 5MR^2$

Homework: Moment of Inertia

- Note that the definition of moment of inertia means that individual moments of inertia can be added together
  - For example, one of the homework problems involves a solid rod plus two additional point masses on the ends of the rod, rotating around the center of the rod



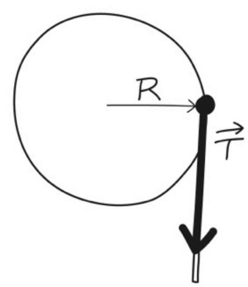
Combining Linear and Rotational Motion

- We now have the tools to do some interesting problems…
  - Bucket of mass m unrolls from a cylinder of mass M and radius R into a well.
  - What is the bucket's (linear) acceleration?

$\hat{y}$ ↓

Be careful about signs!

bucket



cylinder

$$\text{Cylinder}: \sum \tau = I\alpha$$

$$I_{\text{cylinder}} = \frac{1}{2}MR^2$$

$$\tau = TR\sin\theta = TR$$

$$\alpha = a/R$$

$$\Rightarrow TR = \left(\frac{1}{2}MR^2\right)\frac{a}{R}$$

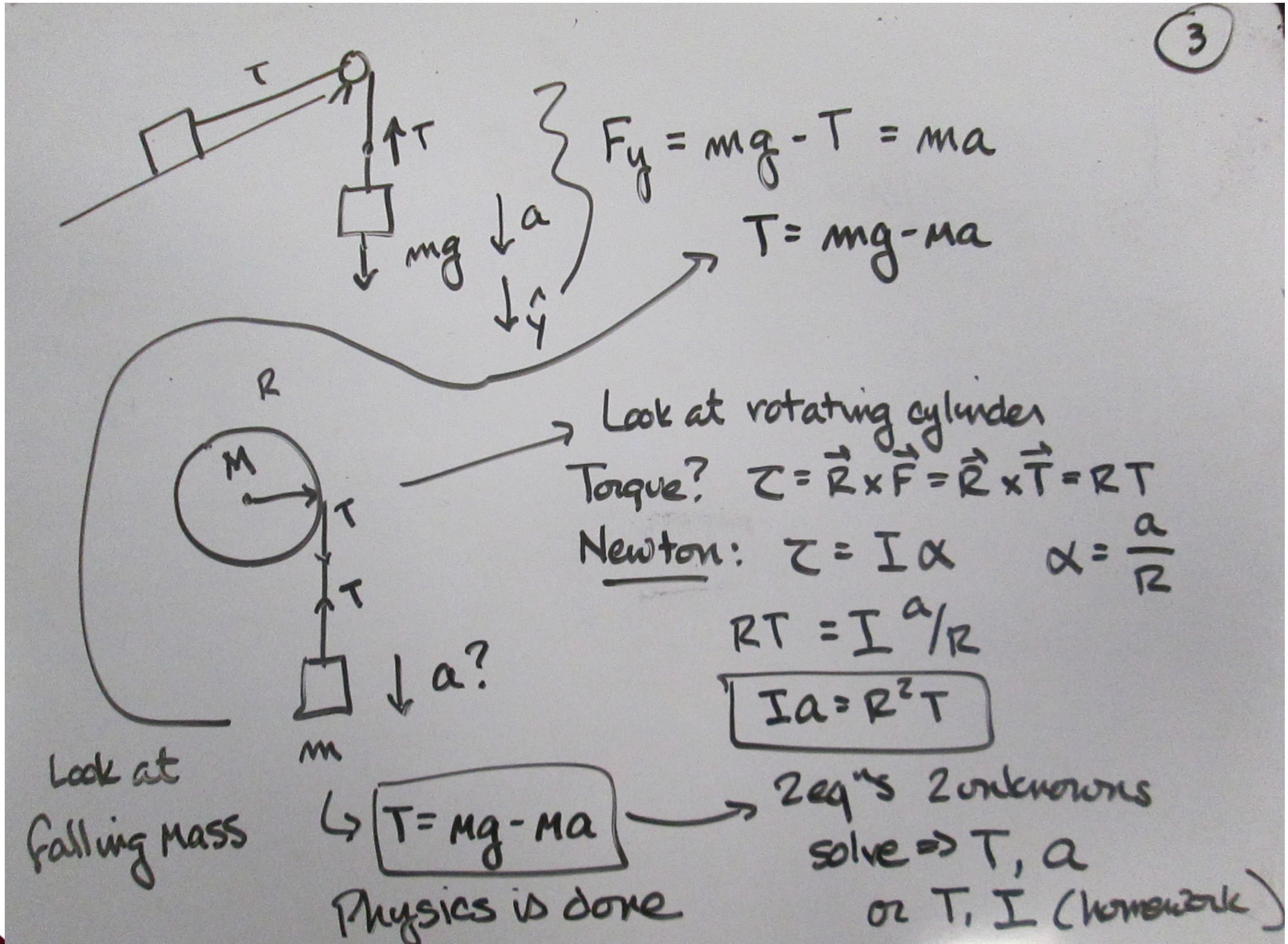
$$\Rightarrow \quad T = Ma/2$$

$$\text{Bucket}: \quad \sum F = mg - T = ma$$

$$mg - Ma/2 = ma$$

$$a = \frac{mg}{M/2 + m}$$

Similarity to Former Inclined Plane Problems

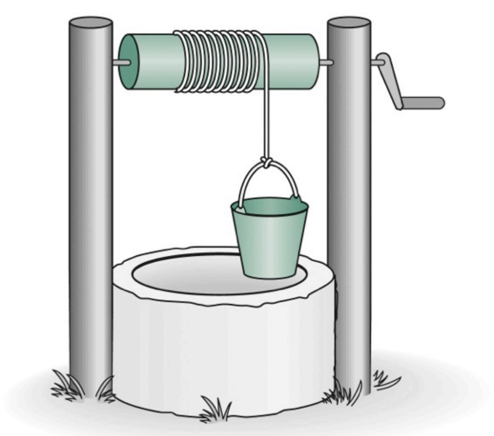


Combining Rotational and Linear Dynamics

- In problems involving both linear and rotational motion:
  - **IDENTIFY** the objects and forces or torques acting.
  - **DEVELOP** your solution with drawings and by writing Newton's law and its rotational analog. Note physical connections between the objects.
  - **EVALUATE** to find the solution.
  - **ASSESS** to be sure your answer makes sense.

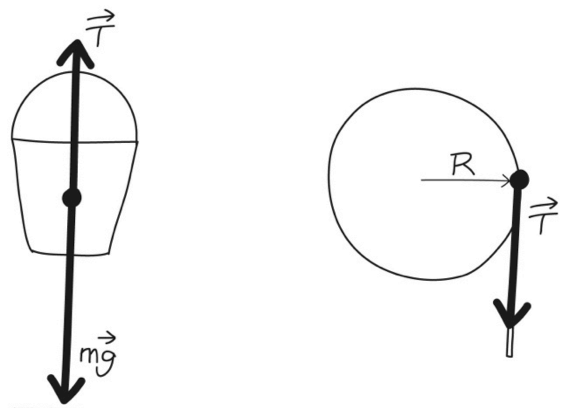
A bucket of mass *m* drops into a well, its rope unrolling from a cylinder of mass *M* and radius *R.*

What's its acceleration?



Free-body diagrams for bucket and cylinder

Rope tension $\vec{T}$ provides the connection



Newton's law, bucket:

$$F_{net} = mg - T = ma$$

Rotational analogy of Newton's law, cylinder:

$$RT = I a / R$$

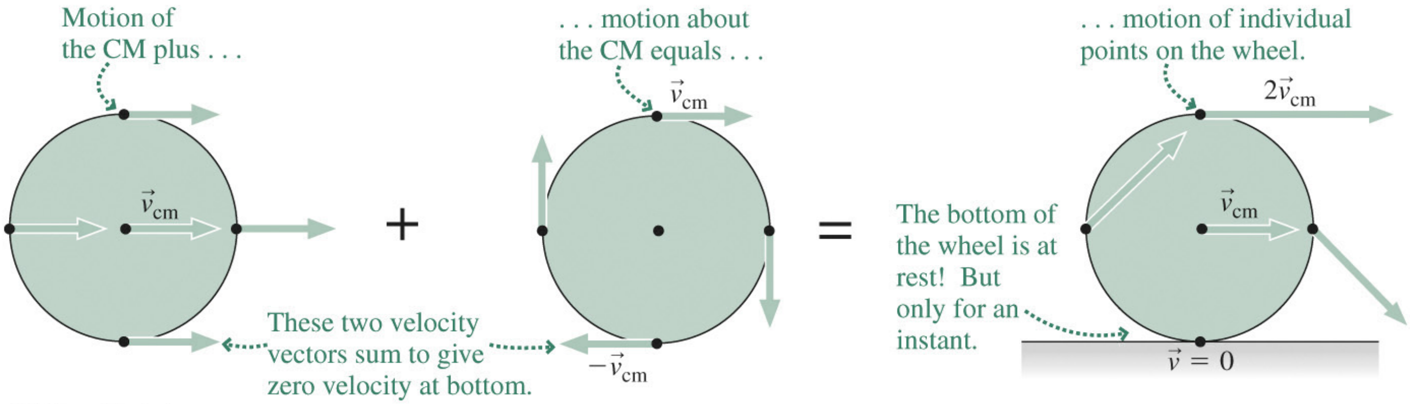
Here $I = \frac{1}{2} MR^2$

Solve the two equations to get

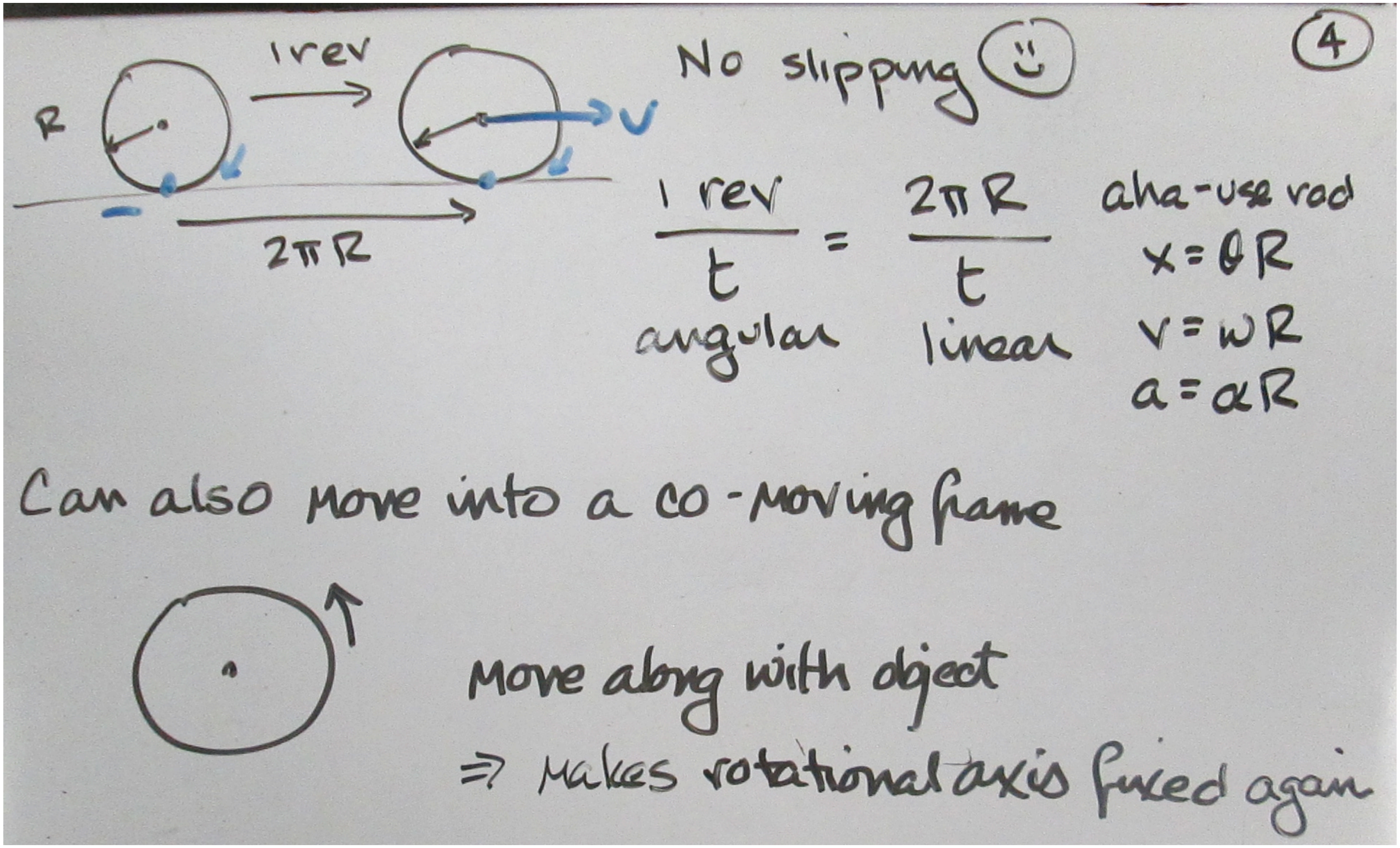
$$a = \frac{mg}{m + \frac{1}{2} M}$$

Rolling Motion

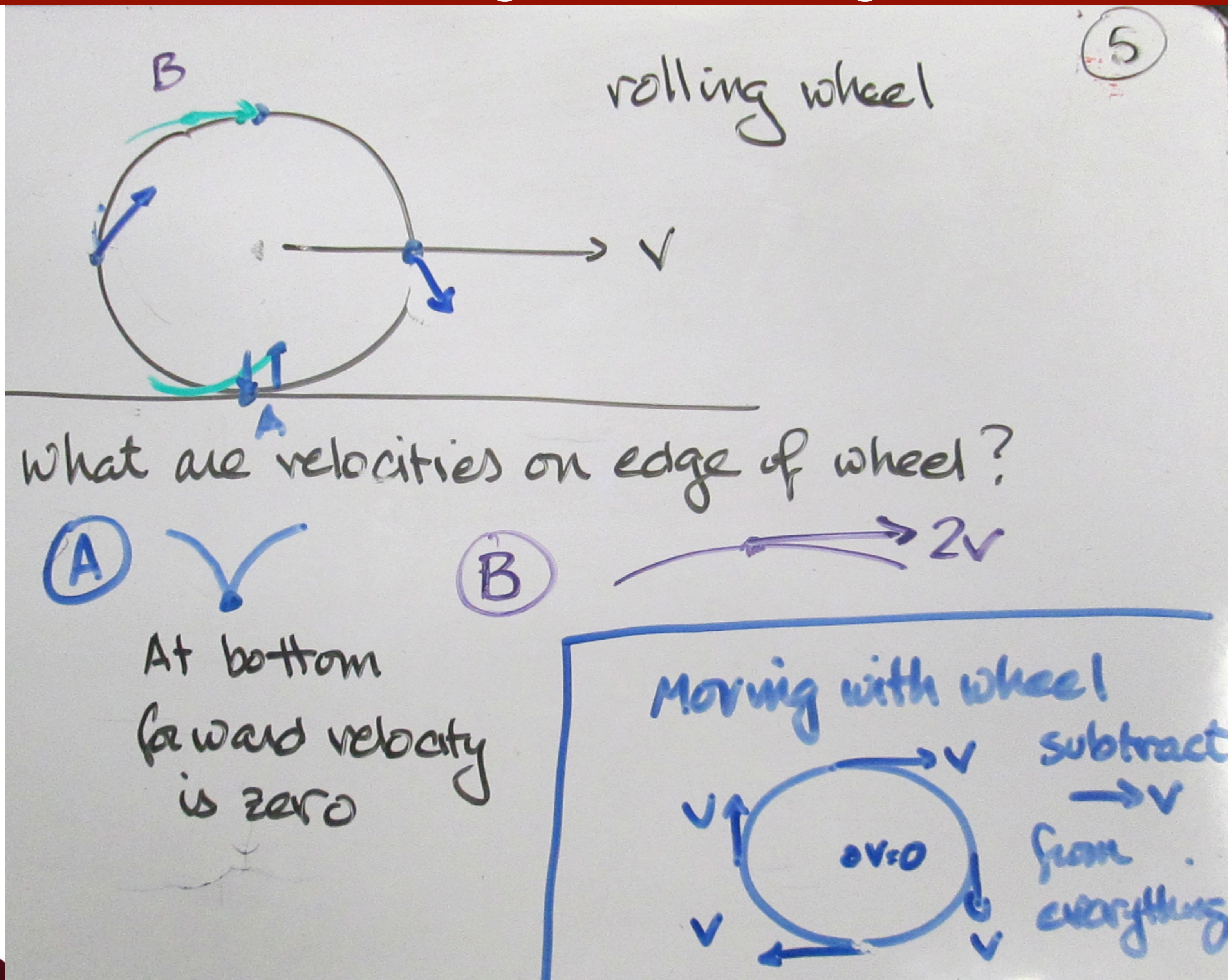
- Rolling motion combines translational (linear) motion and rotational motion.
  - The rolling object's center of mass undergoes translational motion.
  - The object itself rotates about the center of mass.
  - In true rolling motion, the object moves without slipping and its point of contact with the ground is instantaneously at rest.
  - Then the rotational speed ω and linear speed v are related by $v = \omega R$, where R is the object's radius.



Rolling and Rotating



Rolling and Rotating

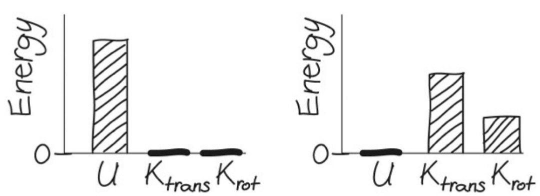


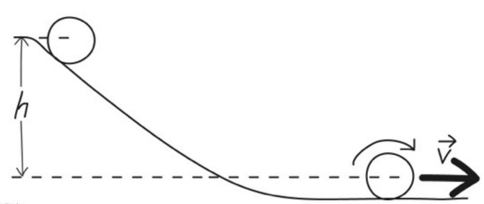
Rotational Energy

- A rotating object has kinetic energy $K_{\mathrm{rot}} = \frac{1}{2} I\omega^2$ associated with its rotational motion alone.
  - It may also have translational kinetic energy: $K_{\mathrm{trans}} = \frac{1}{2} Mv^2$.
- In problems involving energy conservation with rotating objects, both forms of kinetic energy must be considered.
  - For rolling objects, the two are related:
    - The relation depends on the rotational inertia.

Example: A solid ball rolls down a hill. How fast is it moving at the bottom?

Energy bar graphs





Equation for energy conservation

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$
$$= \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{7}{10}Mv^2$$

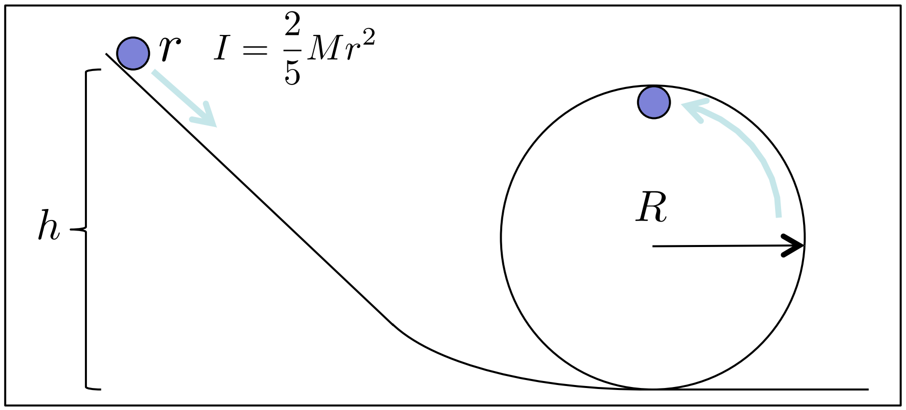
Solution:

$$v = \sqrt{\frac{10}{7}gh}$$

Energy Conservation Example

A rework of a problem I tried on the board in class…

Consider the loop the loop problem, now with a solid ball rolling down a track and barely making it around the loop. What is the minimum height that the ball needs to roll from (without slipping) to make it around a loop of radius R?



Rotating without slipping: $\omega = \frac{v}{r}$

Solid uniform ball: $I = \frac{2}{5}Mr^2$

Barely make it through the loop:

$$a_{\text{centrip}} = \frac{v^2}{R} = g \quad \Rightarrow \quad v^2 = Rg$$

$$r^2\omega^2 = v^2 = Rg$$

State	$KE_{translation}$	$KE_{rotation}$	PE_g
Top of ramp	0	0	Mgh
Top of loop	$\frac{1}{2}Mv^2$	$\frac{1}{2}I\omega^2$	$Mg(2R)$

Conservation of energy:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + 2MgR$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{5}Mr^2\omega^2 + 2MgR$$

$$gh = \frac{Rg}{2} + \frac{Rg}{5} + 2gR \quad \Rightarrow \quad h = 2.7R$$

Watch: https://www.youtube.com/watch?v=4fCFxD_Ud9E

Rotational Work

- A tangential force applied to a rotating body does work on it.

Work done by a torque τ_z

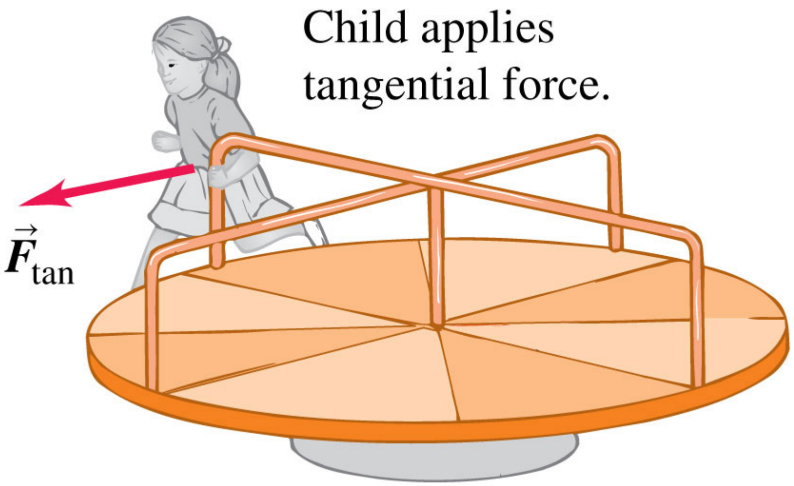
$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta$$

Upper limit = final angular position

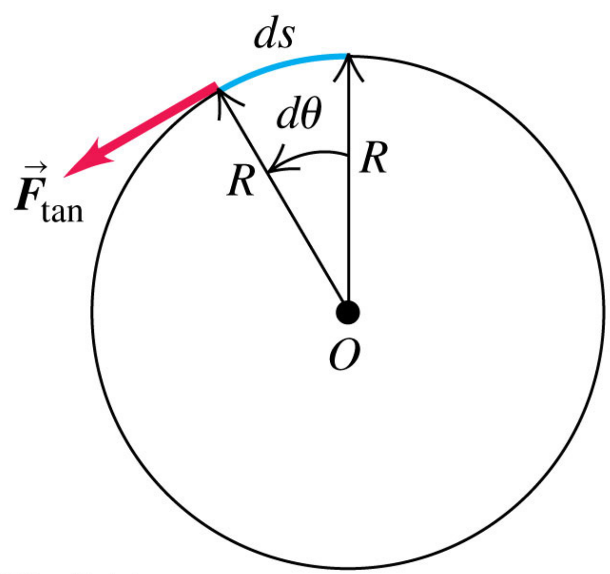
Integral of the torque with respect to angle

Lower limit = initial angular position

(a)

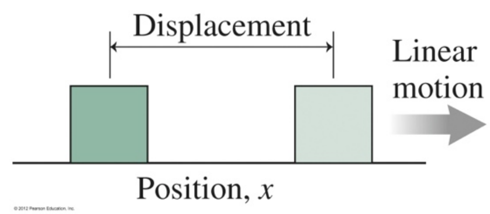


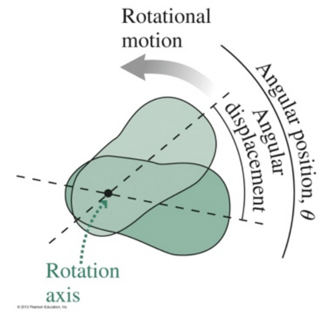
(b) Overhead view of merry-go-round



Summary

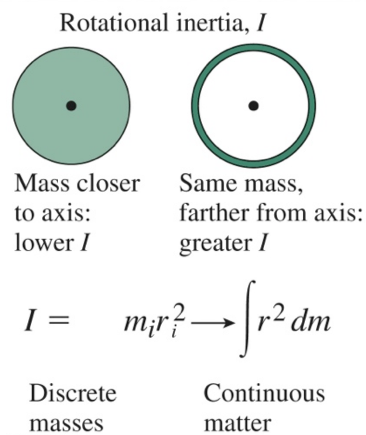
- Rotational motion in one dimension is exactly analogous to linear motion in one dimension.
  - Linear and angular motion:

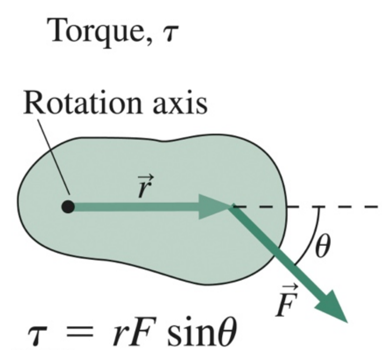




  - Analogies between rotational and linear quantities:

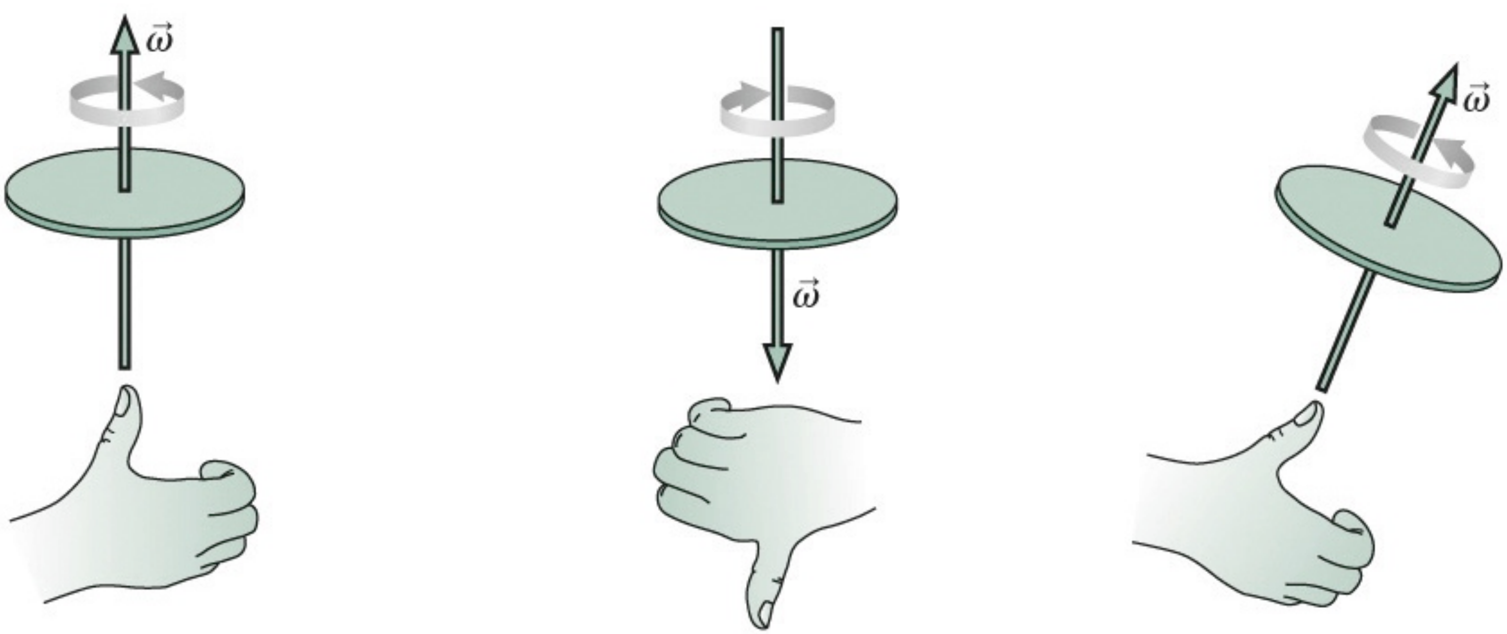
Linear Quantity or Equation	Angular Quantity or Equation	Relation Between Linear and Angular Quantities
Position x	Angular position θ	
Speed $v = dx/dt$	Angular speed $\omega = d\theta/dt$	$v = \omega r$
Acceleration a	Angular acceleration α	$a_t = \alpha r$
Mass m	Rotational inertia I	$I = \int r^2\, dm$
Force F	Torque τ	$\tau = rF\sin\theta$
Kinetic energy $K_{trans} = \frac{1}{2}mv^2$	Kinetic energy $K_{rot} = \frac{1}{2}I\omega^2$	
Newton's second law (constant mass or rotational inertia):		
$F = ma$	$\tau = I\alpha$	





Direction of the Angular Velocity Vector

- The direction of angular velocity is given by the **right-hand rule.**
  - Curl the fingers of your right hand in the direction of rotation, and your thumb points in the direction of the angular velocity vector

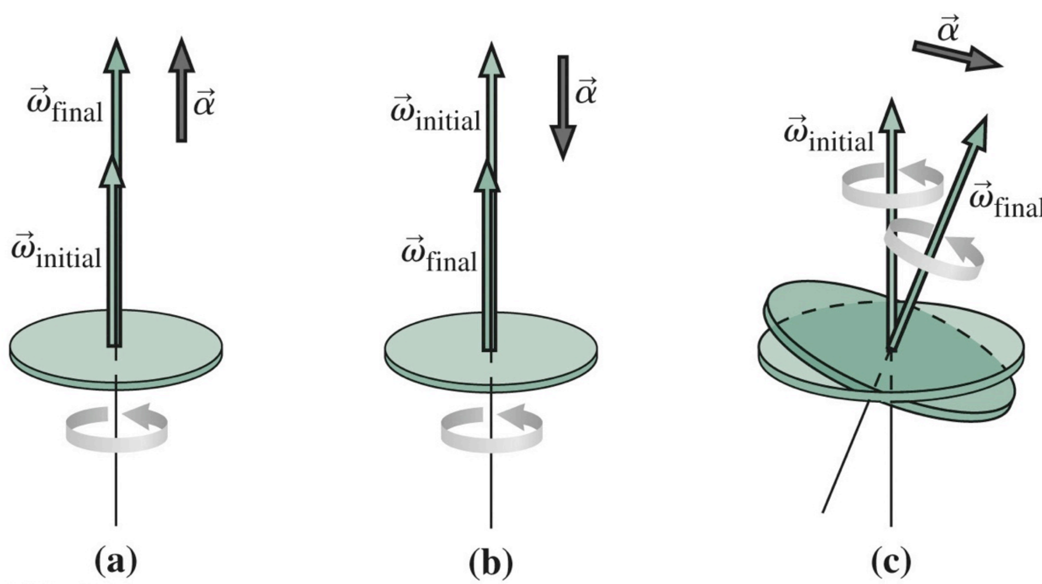


Direction of the Angular Acceleration

- Angular acceleration points in the direction of the change in the angular velocity $\Delta\vec{\omega}$:

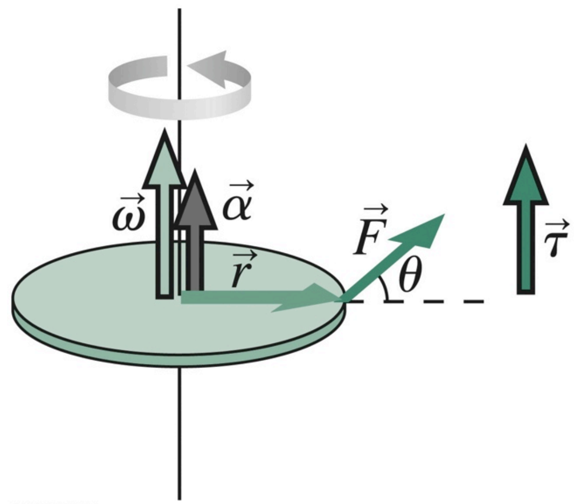
$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$$

  - The change can be in the same direction as the angular velocity, increasing the angular speed.
  - The change can be opposite the angular velocity, decreasing the angular speed.
  - Or it can be in an arbitrary direction, changing the direction and speed as well.

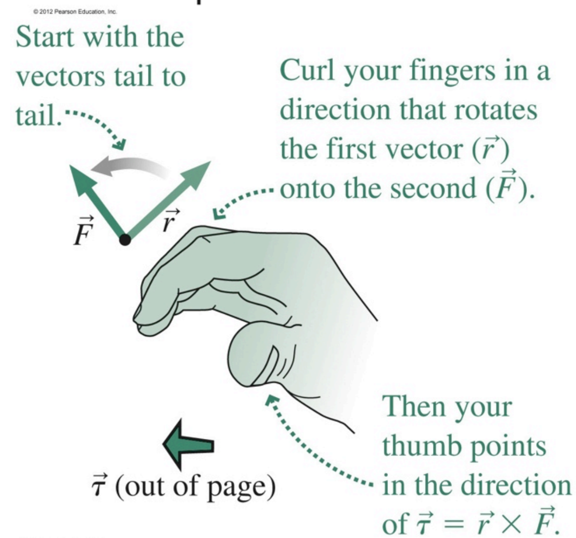


Direction of the Torque Vector

- The torque vector is perpendicular to both the force vector and the displacement vector from the rotation axis to the force application point.
  - The magnitude of the torque is $\tau = rF\sin\theta$.



– Of the two possible directions perpendicular to $\vec{r}$ and $\vec{F}$, the correct direction is given by the right-hand rule.



– Torque is compactly expressed using the **vector cross product**:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Angular Momentum

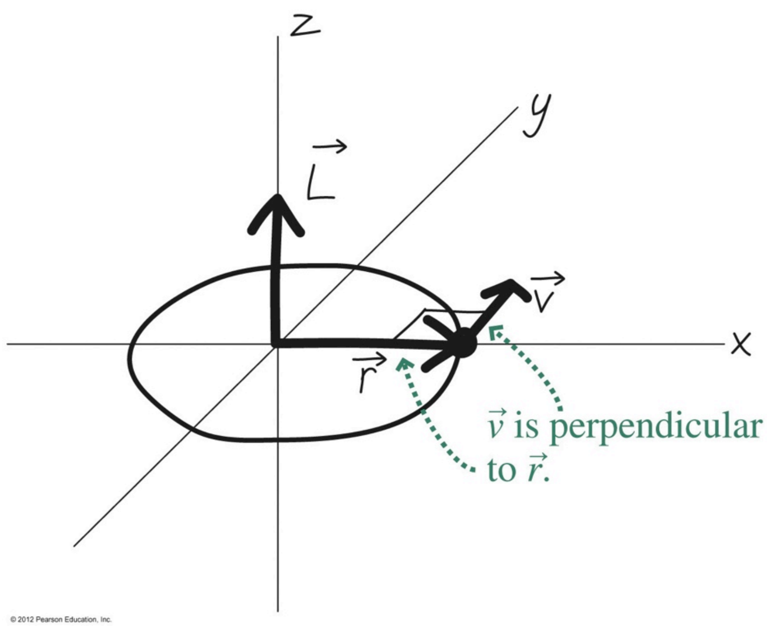
- For a single particle, angular momentum $\vec{L}$ is a vector given by the cross product of the displacement vector from the rotation axis with the linear momentum of the particle:

$$\vec{L} = \vec{r} \times \vec{p}$$

  – For the case of a particle in a circular path, $L = mvr$, and $\vec{L}$ is upward, perpendicular to the circle.

  – For sufficiently symmetric objects, $\vec{L}$ is the product of rotational inertia and angular velocity:

$$\vec{L} = I\vec{\omega}$$

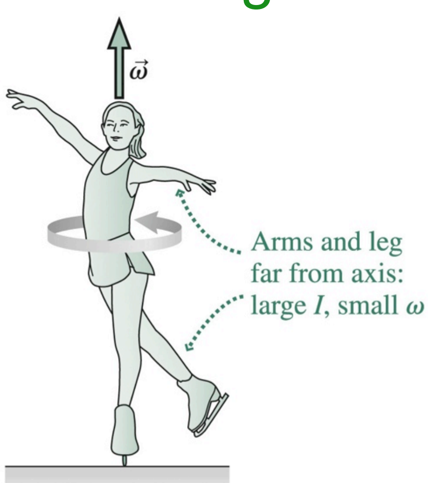


Newton's Law and Angular Momentum

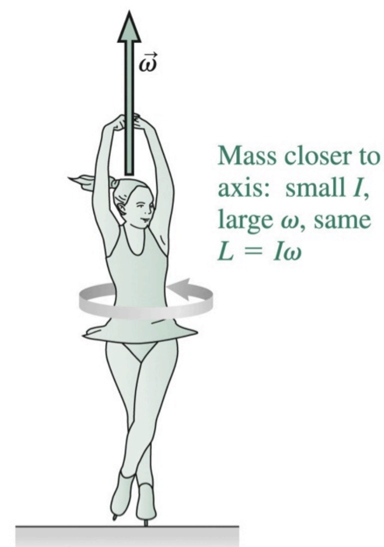
- In terms of angular momentum, the rotational analog of Newton's second law is

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

  - Therefore a system's angular momentum changes only if there's a non-zero net torque acting on the system.
  - If the net torque is zero, then angular momentum is conserved.
    - Changes in rotational inertia then result in changes in angular speed:

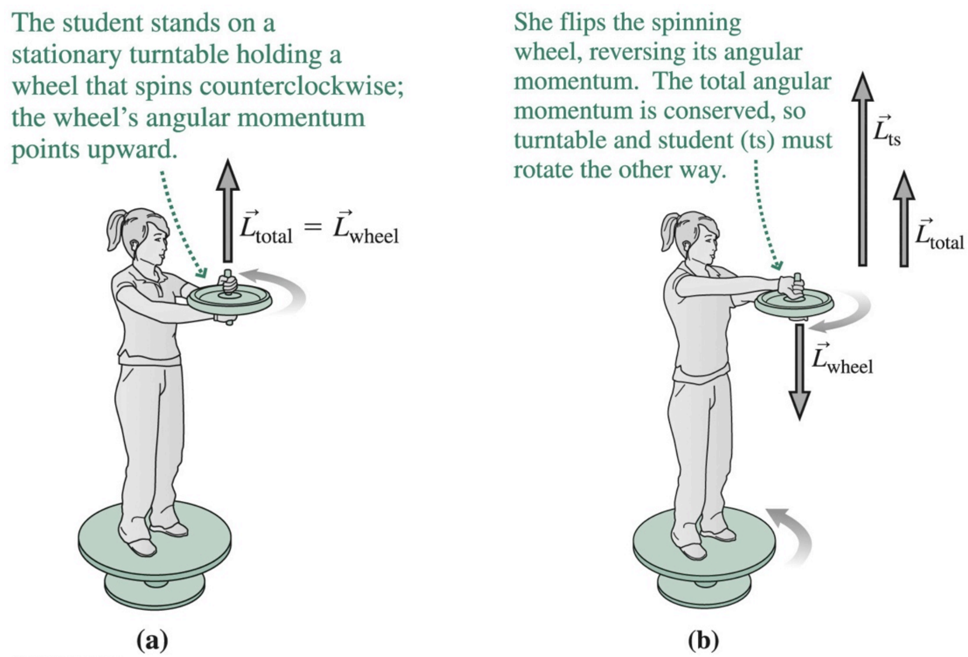


The skater's angular momentum is conserved, so her angular speed increases when she reduces her rotational inertia.



Conservation of Angular Momentum

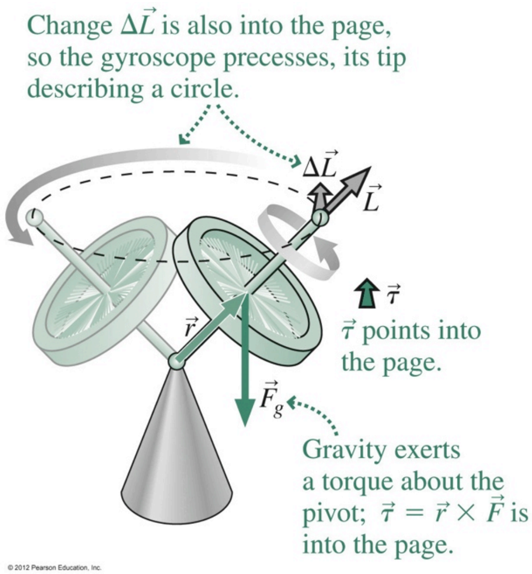
- The spinning wheel initially contains all the system's angular momentum.
- When the student turns the wheel upside down, she changes the direction of its angular momentum vector.
- Student and turntable rotate the other way to keep the total angular momentum unchanged.



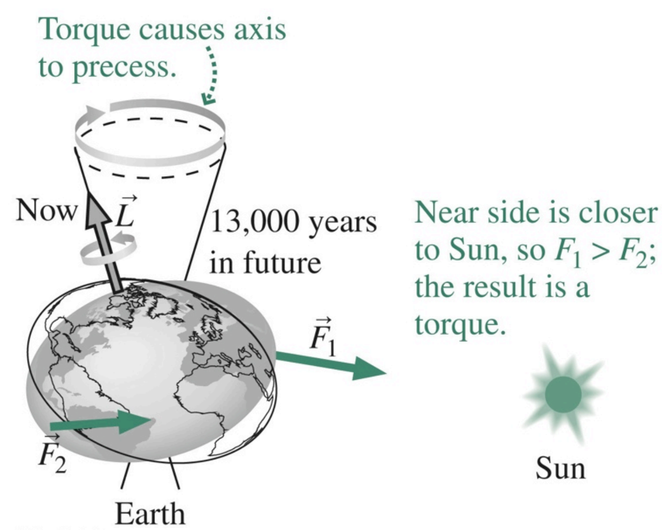
Precession

- Precession is a three-dimensional phenomenon involving rotational motion.
  - Precession occurs when a torque acts on a rotating object, changing the direction but not the magnitude of its angular momentum vector.
  - As a result the rotation axis undergoes circular motion:

Precession of a gyroscope

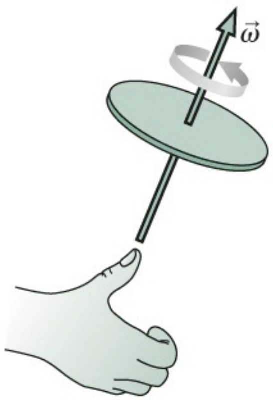


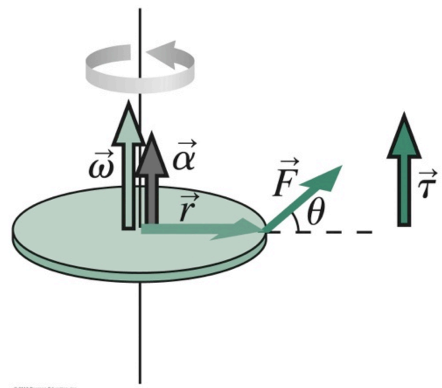
Precession slowly changes the direction of Earth's rotation axis

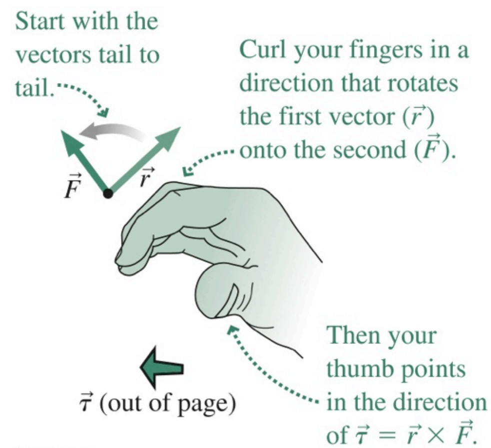


Summary

- Angular quantities are vectors whose direction is generally associated with the direction of the rotation axis.
  - Specifically, direction is given by the right-hand rule.
  - The vector cross product provides a compact representation for torque and angular momentum.







  - Angular momentum is the rotational analog of linear momentum: $\vec{L} = \vec{r} \times \vec{p}$; with symmetry, $\vec{L} = I\vec{\omega}$.
  - In the absence of a net external torque, a system's angular momentum is conserved.