1. **One Dimensional Motion**
A ball is thrown up in the air from a height $y = 0$ m at time $t = 0$ s. It passes by a certain vertical location at time $t = 0.30$ s when going up, then passes by that location again at time $t = 1.80$ s when it's coming down.

(a) What is the initial velocity of the ball?
Solution: The phrasing of the problem implies that this is a one-dimensional problem, and that the only force that the ball is feeling is the force of gravity, $F_g = mg$ pointing *down*. With that in mind, we label the "up" direction as positive, so the force of gravity with the proper sign is really $F_g = -mg$. Newton's 2nd la w also tells us that $F_g = ma_y = -mg$, so the acceleration due to gravity is $a_y = -g$.

We have three equations that we primarily use for kinematics problems. For the y dimension here (the only one we care about), $y - y_0 = v_0 t + \frac{1}{2}a_y t^2$, $v = v_0 + a_y t$, and $v^2 - v_0^2 = 2a_y(y - y_0)$. We are given times, so we probably don't want to use the last equation to figure anything out yet since it doesn't involve time t. We also don't know initial or final velocities, so we probably can't use the second equation. So look at the first equation and think a bit. A bit of thinking will give you the insight that the motion up and down from the "certain vertical location" is just a separate (smaller) one dimensional motion problem where we know the time it takes for the ball to go up and come back down again, $t = 1.80$ s – 0.30 s = 1.50 s. Here $y - y_0 = 0$ m (the ball starts out at the same vertical location as it ends 1.50 s later), and we know $a_y = -g$. We can then calculate

$$\begin{aligned} y - y_0 &= v_0 t + \frac{1}{2}a_y t^2 \\ 0 \text{ m} &= v_0(1.50 \text{ s}) + \frac{1}{2}(-9.8 \text{ m/s}^2)(1.50 \text{ s})^2 = (1.50 \text{ s})v_0 - 11.025 \text{ m} \\ 11.025 \text{ m} &= (1.50 \text{ s})v_0 \quad \Rightarrow \quad v_0 = \frac{11.025 \text{ m}}{1.50 \text{ s}} = 7.35 \text{ m/s} \end{aligned}$$

This is **not the answer for part a**. This is the velocity of the ball at the time it's at the "certain vertical location". But we know the ball took 0.30 s to get there while decelerating, so we can use the second kinematics equation above to find

$$\begin{aligned} v &= v_0 + a_y t \\ (7.35 \text{ m/s}) &= v_0 + (-9.8 \text{ m/s}^2)(0.30 \text{ s}) \\ (7.35 \text{ m/s}) &= v_0 + (-2.94 \text{ m/s}) \quad \Rightarrow \quad \boxed{v_0 = 10.29 \text{ m/s}} \end{aligned}$$

There are other ways to approach this problem that also give the right answer. For example, since the ball took 0.3 s to get up to the "certain vertical location" from the ground, it'll take that long to get back to the ground after passing that location on the way down. So the total time the ball is in the air from the ground to the ground is $t = 1.80 \text{ s} + 0.30 \text{ s} = 2.10$ s. The final velocity of the ball is negative the initial velocity of the ball, $v = -v_0$. So we can now use the second kinematics equation to find the answer quite quickly.

$$\begin{aligned} v &= v_0 + a_y t \\ -v_0 &= v_0 + (-9.8 \text{ m/s}^2)(2.10 \text{ s}) \\ 2v_0 &= 20.58 \text{ m/s} \quad \Rightarrow \quad \boxed{v_0 = 10.29 \text{ m/s}} \end{aligned}$$

(b) What is the maximum height that the ball reaches?
Solution: At the maximum height, $v = 0$ (the ball's momentarily not moving up or down). We want to find a total distance and don't know the time, so we use the third equation to find $y - y_0$, the total height:

$$\begin{aligned} v^2 - v_0^2 &= 2a_y(y - y_0) \\ (0 \text{ m/s})^2 - (10.29 \text{ m/s})^2 &= 2(-9.8 \text{ m/s}^2)(y - y_0) \\ y - y_0 &= \frac{-105.8841 \text{ m}^2/\text{s}^2}{-19.6 \text{ m/s}^2} = \boxed{5.40 \text{ m} = y - y_0} \end{aligned}$$

(c) What is the height of the "certain vertical location" that the ball passes at those times?
Solution: Here we know the time the ball takes to get to that location and its initial velocity, but we don't know a final velocity (how fast the ball is moving when it reaches the "certain vertical location"). The first kinematics equation seems to be the natural one to use since it's got distance, time, and initial velocity in it (all known quantities) and doesn't include the unknown quantity of final velocity.

$$\begin{aligned} y - y_0 &= v_0 t + \frac{1}{2} a_y t^2 \\ &= (10.29 \text{ m/s})(0.30 \text{ s}) + \frac{1}{2}(-9.8 \text{ m/s}^2)(0.30 \text{ s})^2 = \boxed{2.65 \text{ m} = y - y_0} \end{aligned}$$

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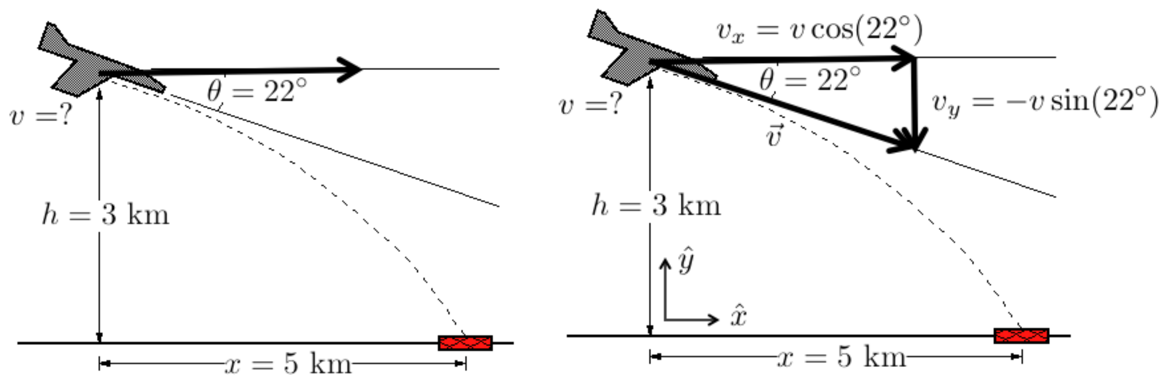
2. **Two Dimensional Motion: Plane Drop**
A plane is diving towards the ground at an angle of $\theta = 22°$ below horizontal, at an altitude of $h = 3$ km. It wants to drop a freely-falling package so it lands on the ground $x = 5$ km ahead. Neglecting air resistance and forces other than gravity, how fast must the plane be going when it drops the package for the package to land in the right place?



Solution: The figure above shows the vectors for this problem. In particular, we need to write a coordinate system (x,y in the lower left) and decompose the velocity of the plane into x and y components. Note that the vertical velocity component is negative (it points in the negative $\hat{y}$ direction, while the horizontal velocity component is positive (it points in the positive $\hat{x}$ direction). Note that these are really the **initial** components of the velocity.

Gravity is in effect, similar to the previous problem, so we know that $a_y = -g = -9.8$ m/s^2 and $a_x = 0$ m/s^2.

Recall that in projectile problems, we can usually figure out the time the projectile is in the air from the vertical motion equations if we are not given the time as part of the problem. The standard kinematic equations with $a_x = 0$ are:

$$\begin{aligned} x - x_0 = v_{0x} t \qquad & v_x = v_{0x} \qquad & v_x^2 - v_{0x}^2 = 0 \\ y - y_0 = v_{0y} t + \frac{1}{2} a_y t^2 \qquad & v_y = v_{0y} + a_y t \qquad & v_y^2 - v_{0y}^2 = 2a_y(y - y_0) \end{aligned}$$

Here we have the vertical distance $y - y_0 = -h = -3$ km (note the negative sign; the package is going down), the horizontal distance $x - x_0 = 5$ km, and the angle in the components. We don't care about the time, so we should try to eliminate it from our horizontal and vertical equations. Use the first horizontal equation to find:

$$t = \frac{x - x_0}{v_{0x}} = \frac{5000 \text{ m}}{v \cos(22°)} = \frac{5393 \text{ m}}{v}$$

Now put this into the first vertical equation, since we know $y - y_0$. The rest is "just" plugging in values and careful algebra to find v:

$$\begin{aligned}
y - y_0 &= v_{0y}t + \frac{1}{2}a_y t^2 \\
(-3000\text{ m}) &= -v\sin(22^\circ)t + \frac{1}{2}(-g)t^2 \\
(-3000\text{ m}) &= -v\sin(22^\circ)\left(\frac{5393\text{ m}}{v}\right) + \frac{1}{2}(-9.8\text{ m/s}^2)\left(\frac{5393\text{ m}}{v}\right)^2 \\
(-3000\text{ m}) &= -(2020\text{ m}) - \frac{142513800\text{ m}^3/\text{s}^2}{v^2} \\
(-980\text{ m}) &= -\frac{142513800\text{ m}^3/\text{s}^2}{v^2} \\
v^2 &= \frac{142513800\text{ m}^3/\text{s}^2}{980\text{ m}} \qquad \Rightarrow \qquad \boxed{v = 381\text{ m/s}}
\end{aligned}$$

We can convert this to miles/hr too.

$$v = 381\text{ m/s}\left(\frac{3600\text{ s}}{1\text{ hr}}\right)\left(\frac{1\text{ km}}{1000\text{ m}}\right)\left(\frac{1\text{ mile}}{1.61\text{ km}}\right) = 852\text{ mi/hr}.$$

That's pretty fast! What's the time the projectile is in the air? Use the time equation that we figured out above for this problem:

$$t = \frac{x - x_0}{v_{0x}} = \frac{5000\text{ m}}{v\cos(22^\circ)} = \frac{5393\text{ m}}{381\text{ m/s}} = \boxed{14.15\text{ s} = t}$$

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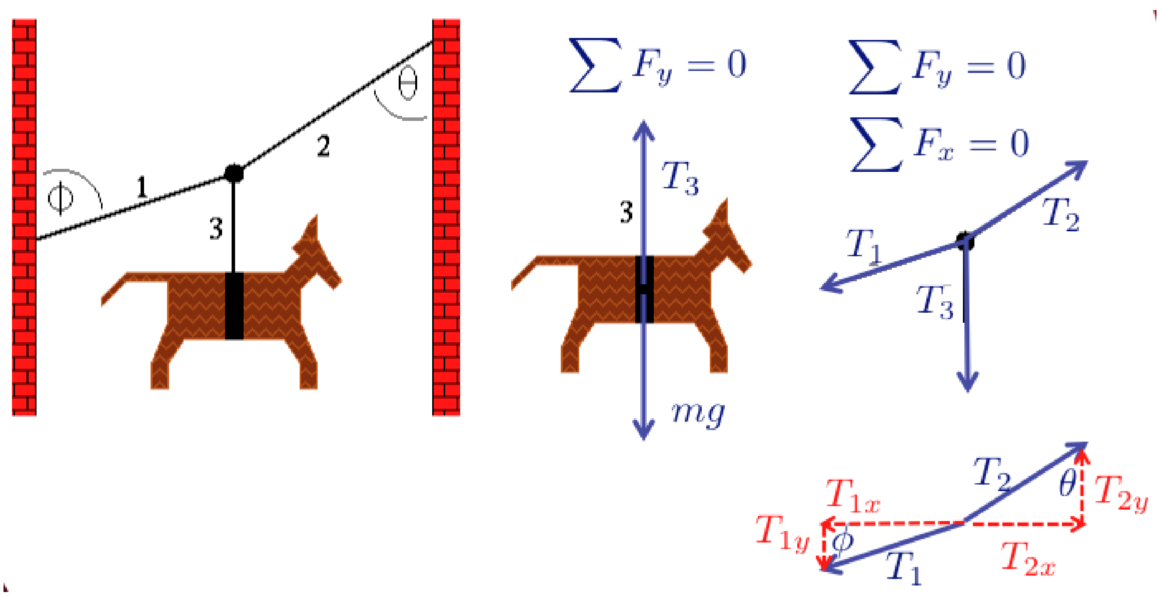
3. **Tensions in Wires: Dingo**
An Australian Dingo, mass 30 kg, is supported by 3 massless cables as shown in the following diagram. What are the tensions of each cable if the angles are $\theta = 50^\circ$ and $\phi = 70^\circ$? **Solution:** The figure



above shows the vectors for this problem with two different diagrams of various acting forces. In the middle diagram, we note that there are only two forces acting on the dingo: its weight (pointing down) and the tension T_3 (pulling up). Nothing is accelerating here (the question implies that this is a statics problem), so we know that $a_y = 0$ m/s^2 and Newton's 2nd law tells us that $\sum F_y = ma_y = 0$ for the dingo. This gives

$$\sum F_y = 0 = T_3 - mg \qquad \Rightarrow \qquad T_3 = mg = (30\text{ kg})(9.8\text{ m/s}^2) = \boxed{294\text{ N} = T_3}$$

So we have one of the tensions. How do we find the others? We have to treat the *knot* where the three ropes meet as an object and look at the forces on it. There are three tensions pulling it in different directions, as s¿een in the right force diagram above. We have to break those tensions down into x and y components, and use the same Newton's 2nd law (with $a_x = 0$ m/s^2 and $a_y = 0$ m/s^2). So here we also have $\sum F_x = 0$ and $\sum F_y = 0$. The drawing above also shows how we can break the tensions T_1 and T_2 down into x and y components to use them in the $\sum F_x = 0$ and $\sum F_y = 0$ equations. Note that the T values here are all **lengths** of the respective sides of the triangles; we'll get the signs of these vectors right in the next part.

$$\begin{aligned}
\sin\theta &= \frac{T_{2x}}{T_2} \quad\Rightarrow\quad T_{2x} = T_2 \sin\theta \\
\cos\theta &= \frac{T_{2y}}{T_2} \quad\Rightarrow\quad T_{2y} = T_2 \cos\theta \\
\sin\phi &= \frac{T_{1x}}{T_1} \quad\Rightarrow\quad T_{1x} = T_1 \sin\phi \\
\cos\phi &= \frac{T_{1y}}{T_1} \quad\Rightarrow\quad T_{1y} = T_1 \cos\phi
\end{aligned}$$

We already figured out T_3 and it's all vertical, so let's look at the vertical equation first. We note that T_3 and T_{1y} both point in the negative y direction, so we add them in as negative values, while T_{2y} points in the positive y direction so its positive:

$$\begin{aligned}
\sum F_y &= 0 \\
T_{2y} - T_{1y} - T_3 &= 0 \\
T_2 \cos\theta - T_1 \cos\phi - T_3 &= 0 \\
T_2 \cos(50^\circ) &= T_3 + T_1 \cos(70^\circ) \\
0.6428\, T_2 &= 294 \text{ N} + 0.3420\, T_1 \\
T_2 &= 457.38 \text{ N} + 0.5321\, T_1
\end{aligned}$$

We can also look at the horizontal equation, $\sum F_x = 0$:

$$\begin{aligned}
\sum F_x &= 0 \\
T_{2x} - T_{1x} &= 0 \\
T_2 \sin\theta - T_1 \sin\phi &= 0 \\
T_2 \sin\theta &= T_1 \sin\phi \\
T_2 &= (T_1 \sin\phi / \sin\theta) = T_1(\sin(70^\circ)/\sin(50^\circ)) = 1.227\, T_1
\end{aligned}$$

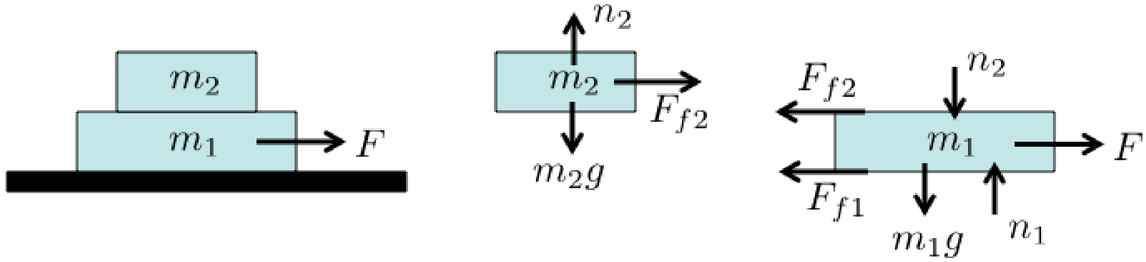
The two equations for T_2 must be equal, so we have

$$\begin{aligned}
1.227\, T_1 &= 457.38 \text{ N} + 0.5321\, T_1 \\
0.6946\, T_1 &= 457.38 \text{ N} \\
&\boxed{T_1 = 658.5 \text{ N}}
\end{aligned}$$

and then

$$T_2 = 1.227\, T_1 = 1.227(658.5 \text{ N}) = \boxed{808 \text{ N} = T_2}$$

4. **Friction: Slipping while Pulling**
Consider two blocks being accelerated across the floor by a constant force F pulling on the bottom, larger block. The larger block has mass $m_1 = 10$ kg, while the smaller block has mass $m_2 = 3$ kg. The coefficient of kinetic friction between the larger block and the floor is $\mu_k = 0.5$, while the coefficient of static friction between the two blocks is $\mu_s = 0.75$. What is the maximum acceleration of the blocks just before the top one slips? What is the maximum force F that can be exerted before the top block slips?



Solution: This is a significantly more complicated forces problem. With multiple objects and acceleration, we must treat each object individually and apply Newton's 2nd law to each in x and y directions to solve everything.

Consider the upper block first. It doesn't have any other blocks resting on top of it, so it should be easier to diagram. There are three forces acting on this block: the force of gravity (acting down), the normal force from the block beneath it (acting up), and the force of static friction F_{f2} (which points to the right). The frictional force must point to the right because the block is accelerating to the right, and that's the only horizontal force acting on this block! We then can see

$$\sum F_y = m_2 a_y = 0 = n_2 - m_2\, g \qquad \Rightarrow \qquad n_2 = m_2\, g$$

$$\sum F_x = m_2 a_x = F_{f2} = \mu_s n_2 = \mu_s m_2 g \qquad \Rightarrow \qquad a_x = \mu_s g = (0.75)(9.8 \text{ m/s}^2) = \boxed{7.35 \text{ m/s}^2 = a_x}$$

Here I have used the fact that the frictional force F_{f2} is the coefficient of static friction μ_s times the normal force n_2 acting on the top block. This is the maximum acceleration that the force of static friction can accelerate the top block.

Now consider the bottom block, which is substantially more complicated. It has the weight of the top block acting on it as a normal force from above (n_2 is the Newton's third law "equal and opposite reaction" to the n_2 that was pushing *up* on the top block). It has its own normal force acting on it from below, and its weight acting on it. So the total of all vertical forces gives

$$\sum F_y = m_1 a_y = 0 = n_1 - n_2 - m_1\, g = n_1 - (m_1 + m_2)g \qquad \Rightarrow \qquad n_1 = (m_1 + m_2)g$$

Horizontally, there are also three forces acting on it: the pulling force F and two frictions, one from the top block (F_{f2}, the Newton's third law "equal and opposite reaction" to the F_{f2} that is accelerating the top block), and one from the floor ($F_{f1} = \mu_k n_1 = \mu_k (m_1 + m_2)g$). We can then write down Newton's 2nd law in the horizontal direction:

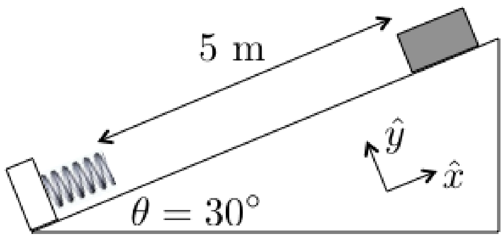
$$\begin{aligned}
\sum F_x &= m_1 a_x = F - F_{f2} - F_{f1} = F - \mu_s m_2 g - \mu_k (m_1 + m_2)g \\
F &= m_1 a_x + \mu_s m_2 g + \mu_k (m_1 + m_2)g \\
&= (10 \text{ kg})(7.35 \text{ m/s}^2) + (0.75)(3 \text{ kg})(9.8 \text{ m/s}^2) + (0.5)(13 \text{ kg})(9.8 \text{ m/s}^2) \\
&= \boxed{159.25 \text{ N} = F}
\end{aligned}$$

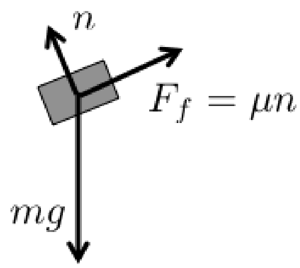
5. **Frictional Inclined Plane**

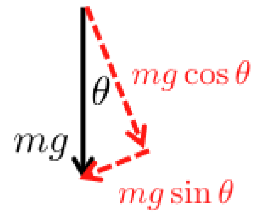
A block of mass $m = 5.0$ kg is released and slides down an incline ($\theta = 30°$) where the coefficient of kinetic friction is $\mu_k = 0.3$. The block goes 5.0 m and hits an ideal spring with $k = 500$ N/m. Assume that friction is negligible while the block is being acted upon by the spring.

(a) How far does the block compress the spring?

(b) How far does the block go back up the plane on the first rebound?







Solution: This is a classic conservation of energy problem, where we are comparing a few different situations: the initial picture, the bottom picture (where the block is compressing the spring), and the bounce picture (where the block is at its highest point once it's bounced back up the plane). What energies do we need to include? We need to include gravitational potential energy ($PE_g = mgh$) since the height changes, energy lost to friction (sliding along a length L is $PE_f = \mu_k nL$), and spring potential energy ($PE_s = \frac{1}{2}kx^2$). We do *not* need to include kinetic energy since the block isn't moving in any of our pictures.

The energy lost to friction has a normal force n in it. If we draw the forces on the block, we can find that $\sum F_y = 0$ gives that $n = mg\cos\theta$. So the energy lost to friction by sliding down a length L is $PE_f = \mu_k mgL\cos\theta$.

If we set $h = 0$ at the spring, then the block's height h above this point is related to its distance up the plane L. The block's height above this point is $h = L\sin\theta$, so its gravitational potential energy is $PE_g = mgh = mgL\sin\theta$.

After the bounce, the energy lost to friction is a little more complicated. Part of it is the total energy that was lost while sliding down the length L, which is $\mu_k mgL\cos\theta$ as found above. But we also lose energy sliding *up*. If the final distance back up the plane that we bounce is L_{bounce}, then we also lose energy $\mu_k mgL_{\text{bounce}}\cos\theta$. So the total energy lost to friction in the bounce picture is $PE_f = \mu_k mg(L + L_{\text{bounce}})\cos\theta$.

We can now tabulate the energies for these various pictures and add them up to use conservation of energy principles. Here x is the amount that the spring is compressed in the "bottom" picture.

picture	PE_g	PE_f	PE_s	**Total**
initial	$mgL\sin\theta$	0	0	$mgL\sin\theta$
bottom	0	$\mu_k mgL\cos\theta$	$\frac{1}{2}kx^2$	$\mu_k mgL\cos\theta + \frac{1}{2}kx^2$
bounce	$mgL_{\text{bounce}}\sin\theta$	$\mu_k mg(L + L_{\text{bounce}})\cos\theta$	0	$mgL_{\text{bounce}}\sin\theta + \mu_k mg(L + L_{\text{bounce}})\cos\theta$

Conservation of energy means that all items in the "Total" column are equal. Setting the initial and bottom pictures equal, we have

$$\begin{aligned}
mgL\sin\theta &= \mu_k mgL\cos\theta + \frac{1}{2}kx^2 \\
\frac{1}{2}kx^2 &= mgL(\sin\theta - \mu_k\cos\theta) \\
x^2 &= \frac{2mgL(\sin\theta - \mu_k\cos\theta)}{k} \\
x &= \sqrt{\frac{2mgL(\sin\theta - \mu_k\cos\theta)}{k}} \\
x &= \sqrt{\frac{2(5.0\text{ kg})(9.8\text{ m/s}^2)(5.0\text{ m})(\sin(30°) - (0.3)\cos(30°))}{500\text{ N/m}}} = \boxed{0.485\text{ m} = x}
\end{aligned}$$

We can also set the initial and bounce picture total energies equal and solve for L_{bounce} to find

$$\begin{aligned}
mgL\sin\theta &= mgL_{\text{bounce}}\sin\theta + \mu_k mg(L + L_{\text{bounce}})\cos\theta \\
L\sin\theta &= L_{\text{bounce}}\sin\theta + \mu_k(L + L_{\text{bounce}})\cos\theta \\
&= L_{\text{bounce}}\sin\theta + \mu_k L\cos\theta + \mu_k L_{\text{bounce}}\cos\theta \\
L\sin\theta - \mu_k L\cos\theta &= L_{\text{bounce}}(\sin\theta + \mu_k\cos\theta) \\
L_{\text{bounce}} &= \frac{L(\sin\theta - \mu_k\cos\theta)}{\sin\theta + \mu_k\cos\theta} \\
&= \frac{(5.0\text{ m})(\sin(30^\circ) - (0.3)\cos(30^\circ))}{\sin(30^\circ) + (0.3)\cos(30^\circ)} \\
&= \boxed{1.58\text{ m} = L_{\text{bounce}}}
\end{aligned}$$

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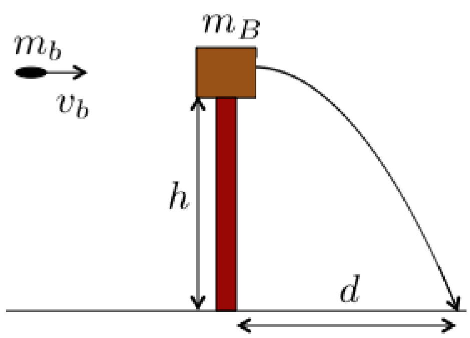
6. **Bullet Speed from Collision**
To measure the speed of a bullet, the following situation is set up. A wooden block with mass $m_B = 0.50$ kg is put on top of a fencepost with a height of $h = 1.5$ m. The bullet (mass $m_b = .010$ kg) strikes the block from a perfectly horizontal direction and remains embedded in it. The block is measured to fall $d = 1.6$ m from the base of the post. How fast was the bullet going?



Solution: This is an interesting problem in that it combines several concepts: collisions (including conservation of momentum) and projectile motion/kinematics. We can work backwards (doing the projectile motion part first since we know how far the block/bullet moved and fell), or we can work forwards, doing the collision part first. I'll work through the collision part first here.

The problem states that the bullet stays embedded in the block, so this is an inelastic collision (a "splat"), and energy is not conserved. So we cannot use conservation of (kinetic) energy to figure out how fast the block/bullet are moving after the collision. We are left with the conservation of momentum. We look at two pictures: the initial picture as the bullet is approaching the block, and the final picture just after the bullet has lodged in the block and both are moving together with horizontal velocity v_{0x}. I call it this because it will be the initial velocity for the projectile motion part of the problem. There is no vertical motion between these two pictures so we can treat everything using only horizontal components.

The total horizontal momentum in the initial picture is just the momentum of the bullet, and it is equal to the final total horizontal momentum which includes the masses of both the bullet and the block:

$$p_x = m_b v_b = (m_b + m_B)v_{0x} \qquad \Rightarrow \qquad v_{0x} = \left(\frac{m_b}{m_b + m_B}\right) v_b$$

Now we shift pictures and view this bullet/block combo as initially moving with this horizontal velocity and figure out its projectile motion using standard projectile motion kinematics equations. I use a standard coordinate system with $+\hat{x}$ pointing to the right and $+\hat{y}$ pointing to the right; then acceleration is just gravity pointing down, $a_y = -g$. We figure out the time it takes to fall first. Note that it's falling down (in the negative direction) so $y - y_0 = -h$, and $v_{0y} = 0\,\text{m/s}$.

$$\begin{aligned} y - y_0 &= v_{0y}t + \frac{1}{2}a_y t^2 \\ -h &= (0) + \frac{1}{2}(-g)t^2 \\ t &= \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(1.5\,\text{m})}{9.8\,\text{m/s}^2}} = 0.55\ s \end{aligned}$$

Now we can use the initial horizontal velocity we figured out above to calculate how far it went horizontally; that's just $x - x_0 = d$, and there is no horizontal acceleration:

$$\begin{aligned} x - x_0 &= v_{0x}t \\ d &= v_{0x}t \\ d &= \left(\frac{m_b}{m_b + m_B}\right) v_b t \\ v_b &= \frac{d(m_b + m_B)}{m_b t} = \frac{(1.6\,\text{m})(0.51\,\text{kg})}{(0.01\,\text{kg})(0.55\,\text{s})} = \boxed{148\ \text{m/s} = v_b} \end{aligned}$$