Oscillators, Resonances, and Lorentzians

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1 The Driven, Damped Simple Harmonic Oscillator

Consider a driven and damped simple harmonic oscillator with resonance frequency ω_0:

$$\ddot{x}(t) + \zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = \omega_0^2 X_0 \cos(\omega t) \tag{1.1}$$

where the various terms are inserted to maintain units except the damping term ζ which is dimensionless. Assume a solution of the form

$$x(t) = x_0 \cos(\omega t - \phi)$$

as the system should eventually settle to be periodic in the driving frequency with some amplitude x_0 and phase ϕ relative to the driving phase. Then we have

$$\begin{aligned} \dot{x}(t) &= -\omega x_0 \sin(\omega t - \phi) \\ \ddot{x}(t) &= -\omega^2 x_0 \cos(\omega t - \phi) \end{aligned}$$

and the equation of motion, Eqn. (1.1), becomes

$$\begin{aligned} -\omega^2 x_0 \cos(\omega t - \phi) - \zeta\omega_0\omega x_0 \sin(\omega t - \phi) + \omega_0^2 x_0 \cos(\omega t - \phi) &= \omega_0^2 X_0 \cos(\omega t) \\ (\omega_0^2 - \omega^2) x_0 \cos(\omega t - \phi) - \zeta\omega_0\omega x_0 \sin(\omega t - \phi) &= \omega_0^2 X_0 \cos(\omega t) \end{aligned}$$

Now

$$\begin{aligned} \cos(\omega_t - \phi) &= \cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi \\ \sin(\omega_t - \phi) &= \sin(\omega t)\cos\phi - \cos(\omega t)\sin\phi \end{aligned}$$

So we then have

$$(\omega_0^2 - \omega^2) x_0 [\cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi] - \zeta\omega_0\omega x_0 [\sin(\omega t)\cos\phi - \cos(\omega t)\sin\phi] = \omega_0^2 X_0 \cos(\omega t)$$

Grouping time dependence of terms gives

$$\begin{aligned} \left[(\omega_0^2 - \omega^2) x_0 \cos\phi + \zeta\omega_0\omega x_0 \sin\phi - \omega_0^2 X_0\right] \cos(\omega_t) &+ \\ \left[(\omega_0^2 - \omega^2) x_0 \sin\phi - \zeta\omega_0\omega x_0 \cos\phi\right] \sin(\omega_t) &= 0 \end{aligned}$$

For this equation to always hold, the coefficients of each time dependent term must be zero:

$$\begin{aligned} (\omega_0^2 - \omega^2) x_0 \cos\phi + \zeta\omega_0\omega x_0 \sin\phi &= \omega_0^2 X_0 \\ (\omega_0^2 - \omega^2) x_0 \sin\phi - \zeta\omega_0\omega x_0 \cos\phi &= 0 \end{aligned}$$

The second equation gives ϕ:

$$\boxed{\phi = \tan^{-1}\left[\frac{\zeta\omega_0\omega}{(\omega_0^2 - \omega^2)}\right]} \tag{1.2}$$

To solve the first equation for x_0, note that $\tan\phi$ can be viewed as the ratio of opposite over adjacent sides of a right triangle of hypoteneuse $\sqrt{(\omega_0^2-\omega^2)^2+\zeta^2\omega_0^2\omega^2}$, so

$$\sin\phi = \frac{\xi\omega_0\omega}{\sqrt{(\omega_0^2-\omega^2)^2+\zeta^2\omega_0^2\omega^2}} \qquad \cos\phi = \frac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+\zeta^2\omega_0^2\omega^2}}$$

which then gives

$$x_0 = \frac{\omega_0^2 X_0}{\sqrt{(\omega_0^2-\omega^2)^2+\zeta^2\omega_0^2\omega^2}}$$

or

$$\boxed{x_0 = \frac{X_0}{\sqrt{(1-(\omega/\omega_0)^2)^2+\zeta^2(\omega/\omega_0)^2}}} \tag{1.3}$$

The equation for ϕ can also be written

$$\boxed{\phi = \tan^{-1}\left[\frac{\zeta}{(\omega_0/\omega-\omega/\omega_0)}\right]} \tag{1.4}$$

This is intuitively reasonable since $\phi = 0$ if there is no damping ($\zeta = 0$). The peak of x_0 is on resonance at $\omega = \omega_0$, where $x_0/X_0 = 1/\zeta$. Indeed, this is a special term, defined as the **quality factor** Q of the resonator:

$$Q \equiv \frac{x_0(\omega=\omega_0)}{X_0} = \frac{1}{\zeta}$$

x_0/X_0 and ϕ/π are plotted in Fig. 1 for various values of Q. Note that the resonance becomes more sharply peaked and the phase crossing from 0 to π becomes sharper as Q increases. In rough terms, Q is the number of oscillations that a damped oscillator undergoes after being excited by a delta function excitation before falling to a negligible amplitude.

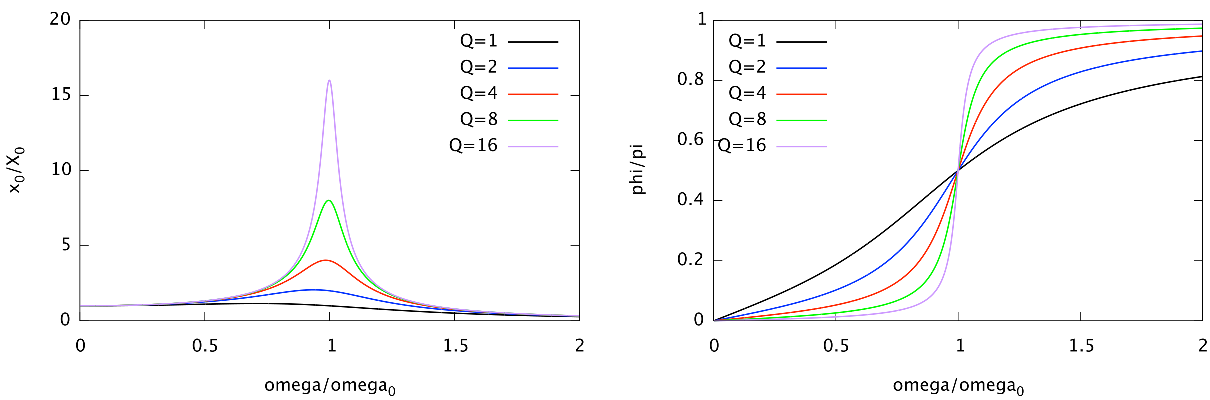

Figure 1: Resonant amplitude response (scaled to driving amplitude X_0, left) and phase response (right) vs frequency for various resonance Q.

2 Weak Damping

The most interesting case for accelerator RF structures is **weak damping**, since we generally want most of our energy to "ring" resonantly in the cavity and be available for beam acceleration, rather than being dissipated in resistive or transport losses. The highest Q values have been achieved in superconducting RF (SRF) cavities, where Q can be upwards of 10^{10-11}.

For a weakly damped oscillator ($\zeta \ll 1$ or $Q \gg 1$) near resonance ($\delta \equiv \omega - \omega_0 \ll \omega_0$), we can approximate $(\omega/\omega_0)^2 \approx (1 + 2\delta/\omega_0)$. Keeping only terms to second order in δ and ζ in Eqn. (1.3), we find

$$\frac{x_0}{x_0(\omega = \omega_0)} = \frac{\zeta}{\sqrt{4\delta^2/\omega_0^2 + \zeta^2}} = \frac{\zeta\omega_0}{\sqrt{4\delta^2 + \zeta^2\omega_0^2}} \tag{2.1}$$

This goes to 1 as $\delta \to 0$. At $\delta = \pm\zeta\omega_0/2$, the amplitude is reduced to $1/\sqrt{2}$ of the maximum, $x_0 = x_0(\omega = \omega_0)/\sqrt{2}$. $1/\sqrt{2}$ is used because the power is proportional to this amplitude squared, and the 1/2 power point is the 3 dB attenuation point. Hence the **resonance width**, 2δ at this point, is

$$\text{resonance width} \equiv \Delta\omega = \zeta\omega_0$$

$\delta = \pm\zeta\omega_0/2$ also happen to be the points where the phase lag is $\pi/4$ and $3\pi/4$. The fractional width of the peak is

$$\boxed{\frac{\Delta\omega}{\omega_0} = \zeta = 1/Q \qquad \Rightarrow \qquad Q = \frac{\omega_0}{\Delta\omega}} \tag{2.2}$$

so for a weakly damped oscillator, the product of the resonance width and height are constant and the area under the peak stays roughly constant independent of Q. Also note that when Q is very large, the resonance widths are small. An ensemble of SRF cavities must be carefully tuned so their primary resonant frequencies align. However, small manufacturing and tuning differences between constructed SRF cavities will make it unlikely that other primary frequencies of the cavity assembly (**higher order modes** or HOMs) will be within their respective $\Delta\omega$ of each other and thus interfere. These HOMs are therefore usually treated as independent resonators except in very large SRF installations such as the ILC, where the sheer number of cavities may lead to HOM crosstalk.

Fig. 1 also suggests that the quality factor Q can be measured from the phase ϕ. In particular, the phase response of a weakly damped resonator is linear near the resonance, $\omega = \omega_0 + \delta$:

$$\phi \approx \tan^{-1}\left[\frac{\zeta}{(1 - \delta/\omega_0) - (1 + \delta/\omega_0)}\right] = \tan^{-1}\left[\frac{-\omega_0}{2\delta Q}\right] \tag{2.3}$$

Linearizing around $\phi = \pi/2$ with $\Phi = \phi - \pi/2$, $\tan\phi = -\cot\Phi \approx 1/\Phi$, we find $\Phi = 2\delta Q/\omega_0$, or

$$\boxed{\frac{d\Phi}{d\omega} = \frac{2Q}{\omega_0}} \tag{2.4}$$

As expected, the slope of the phase near resonance is large when Q is large.

As mentioned, the square of Eqn. (2.1) is proportional to the power stored in the resonant system. This equation is known as a Lorentzian function, related to the Cauchy distribution, which is typically parameterized [1] by the parameters (x_0, γ, I) as:

$$f(x; x_0, \gamma, I) = I\left[\frac{\gamma^2}{(x - x_0)^2 + \gamma^2}\right]$$

Q may be found for a given resonance by measuring the width at the 3 dB points directly, by evaluating the zero-crossing phase slope (2.4), or by fitting the Lorentzian to the power response.

3 Superconducting Cavity Power Spectrum

A typical real power spectrum for TM110 frequencies is shown in Fig. 2, where the amplitudes of all four transfer function measurements (S31, S41, S32, S42) through the cavity HOM ports

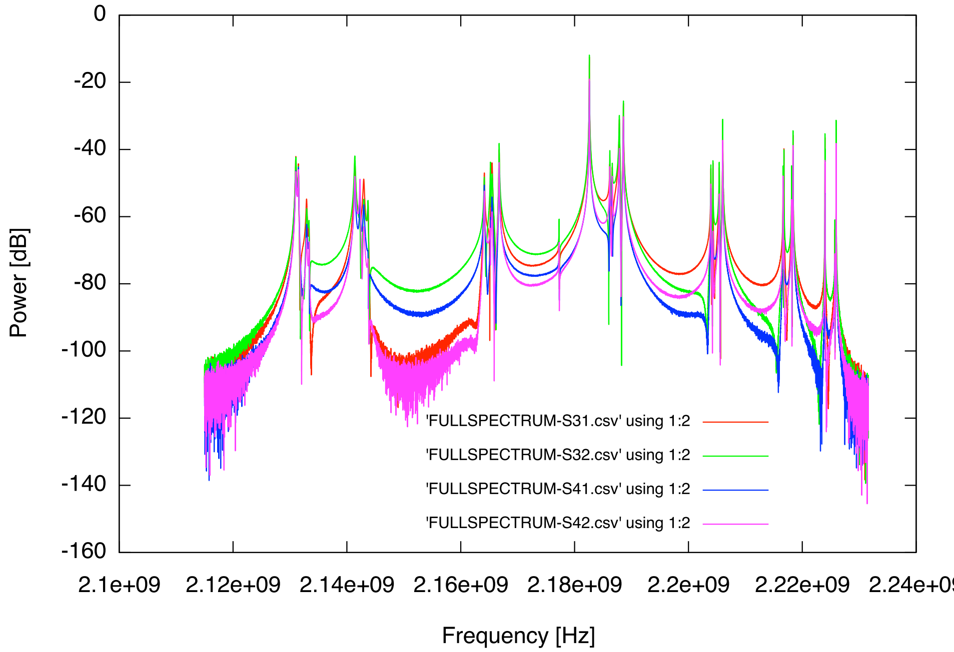

Figure 2: Plot of real part of power spectrum for the R100-8 cavity TM110 mode, for various transfer functions.

were acquired. There are 7 "groups" of peaks together, which correspond to the seven main dipole TM110 modes of the CEBAF C100 RF cavity.

The highest Q HOMs are those with the tallest, narrowest peaks, since Q is a measure of resonance sharpness. For illustration purposes, we zoom in on the HOMs at the furthest right of Fig. 2 in Fig. 3. Here you can see that there are multiple resonances that cross-talk near 2.224 GHz, but the resonance near 2.226 GHz is relatively isolated. The center of the resonance is ω_0=2.22594 GHz, and the resonance width is about $\Delta\omega \approx 1$ kHz, so $Q \approx 2.2 \times 10^6$.

Q for HOMs is very important for a phenomenon known as **beam breakup** in linear and recirculating linear accelerators. In beam breakup, particular HOMs in a cavity can be excited by passing beam, which then in turn excites oscillations in later bunches, which then excites the HOM in a feedback loop. The length of time that an HOM resonates in the cavity depends on its Q (or damping), so BBU is usually avoided by designing SRF cavities so the HOM Qs are kept below some design threshold.

4 Loaded vs Unloaded Q, Fill Time

To now, we have considered properties of our forced, damped harmonic oscillator on its own. However, RF cavities (and oscillators) need to be driven by an external force coupled to the oscillator. For RF cavities, this is usually provided through a transmission line carrying electromagnetic power from a klystron through a waveguide. Fig. 4 shows an electric schematic of such an inductive coupling, from Conte and MacKay.

Z_{eff} is the effective impedance as seen from the input line. Often this input line has its own impedance of 50 Ω so we usually try to avoid reflected power by matching $Z_{\text{eff}} = 50\,\Omega$. Because the impedances of the line and cavity must be matched, the power dissipated in the cavity (proportional to the damping, or inversely proportional to Q) must be the same as

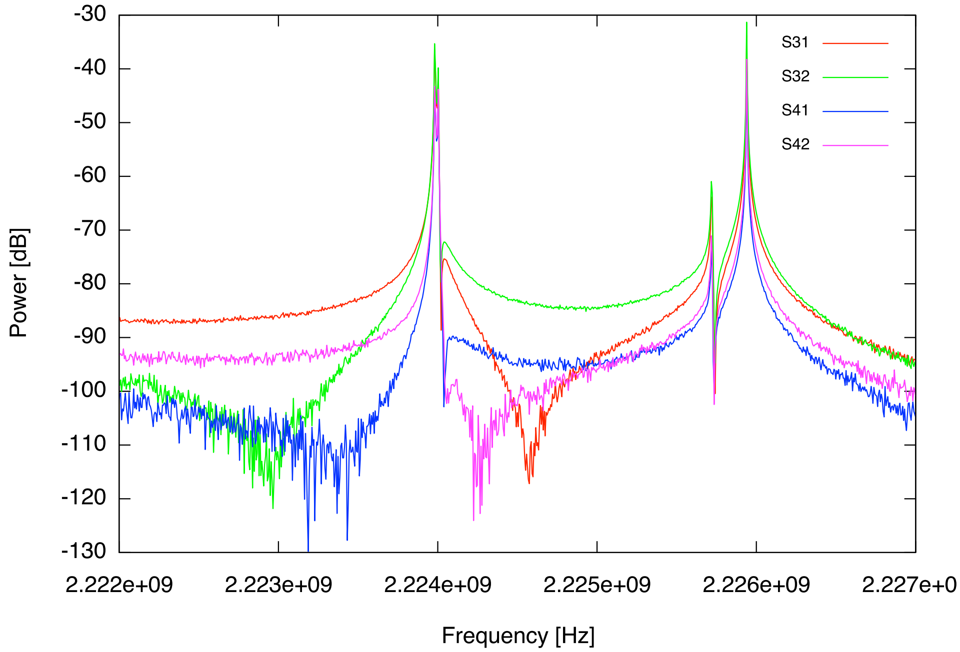

Figure 3: Zoom of real part of power spectrum for the R100-8 TM110 mode, for various transfer functions.

dissipated in the power source, so we often speak of a **loaded Q**,

$$\boxed{Q_L = \frac{Q}{2}\,.} \tag{4.1}$$

as well as an **unloaded Q** for the bare cavity.

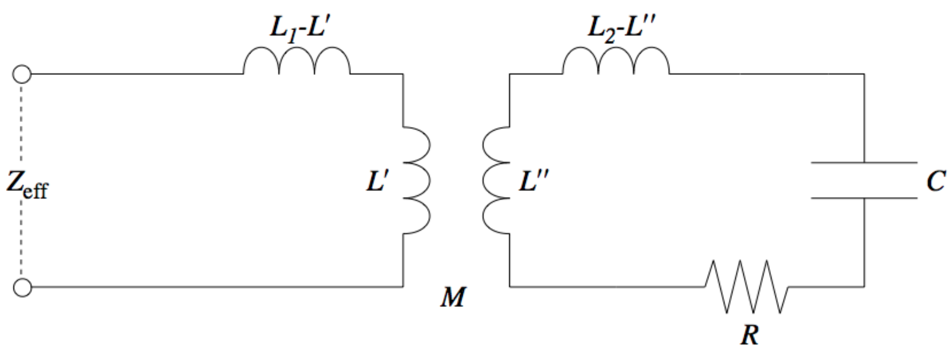

Figure 4: A transmission line on the left coupled to an RF cavity of modeled impedance L_2, resistance R, and capacitance C on the right through an inductor loop with mutual inductance $M = \sqrt{L'L''}$, from Conte and MacKay Fig. 9.5.

We have to turn on and off our cavities. How long does it take to reach a reasonably steady state? Solving the driven harmonic oscillator in the case where the driving turns on as a step function gives

$$x(t) = X_0 Q \left(1 - e^{-\omega_0 t/2Q}\right) \sin(\omega_0 t) \tag{4.2}$$

The characteristic **rise and fall times** (or filling time) of the cavity, τ, is found where the exponent is 1/2, or where

$$\tau = \frac{Q}{\omega_0} = \frac{2Q_L}{\omega_0} \tag{4.3}$$

Note that there is a typo on p. 203 of Conte and MacKay in equation (9.95) for this expression. The response of $Q = 16$ and $Q = 64$ cavities is shown in Fig. 5. It takes about

Q oscillations of the on-frequency driving term to bring the oscillator up to full amplitude. Usually a step function isn't used because the back-voltage from the cavity will be large and may trip the driving RF source.

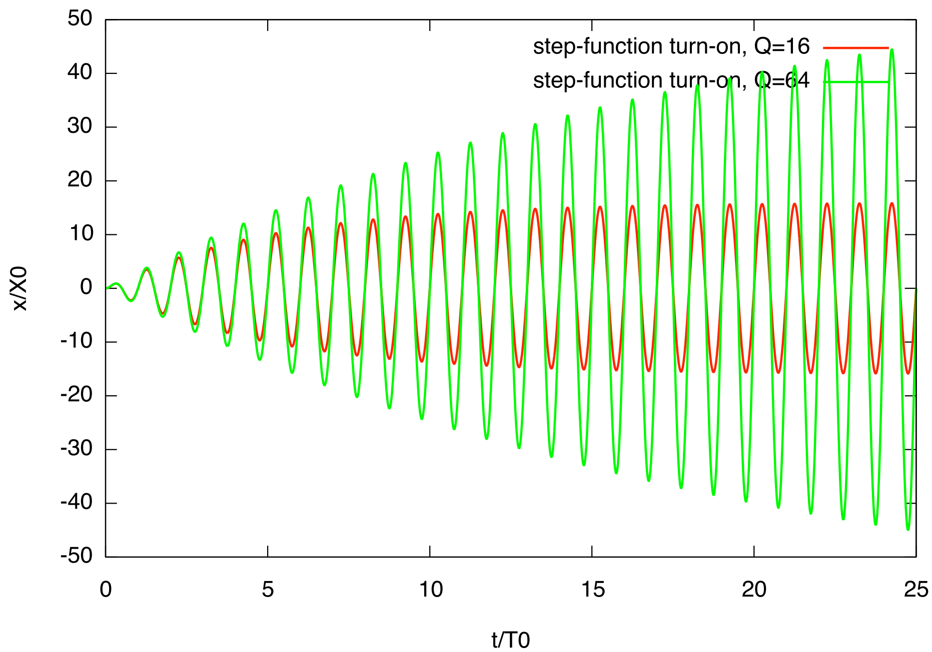

Figure 5: Driven damped harmonic oscillator transient response to a step-function turn-on with Q=16 and Q=64. Note that it takes more RF cycles to fill a high Q cavity than a lower Q cavity.

5 Transit Time and Shunt Impedance

For a standing wave accelerating RF structure (the most common type of structure for RF cavities), consider the electric field across a gap length g as roughly constant and varying with time with the cavity frequency ω_0:

$$E_s = E_0 \cos \omega_0 t \tag{5.1}$$

The RF wavelength is $\lambda = 2\pi c/\omega_0$. The total energy gain when a particle crosses the cavity gap is then

$$\Delta E = E \int_{-g/2}^{g/2} \cos\left(\frac{\omega_0 s}{\beta_r c}\right) ds = e\, E_0\, g\, T_{\rm tr} = e\, V_0 \tag{5.2}$$

$$T_{\rm tr} = \frac{\sin(\pi g/\beta_r \lambda)}{(\pi g/\beta_r \lambda)} \tag{5.3}$$

where $\beta_r = v/c$ is the relativistic beta and $V_0 = E_0 g T_{\rm tr}$ is the effective voltage of the gap. $T_{\rm tr}$ is called the **transit time** across the RF cavity, and it is directly related to how efficient the RF cavity is at converting voltage to particle acceleration. One can make $T_{\rm tr}$ approach 1 by lowering the gap size g for a given cavity, but lowering the gap size too far can lead to sparking between RF electodes. For may modern DTLs, $T_{\rm tr}$ is optimized to be about 0.8 to 0.85.

Shunt impedance is used to describe the relationship between available input power and achievable voltage. This is the voltage at which the input power to the cavity equals the losses in the cavity. Per Conte and MacKay (9.99), this relationship looks like a classical circuit relationship:

$$Z_{\text{shunt}} = \frac{V_{\text{dc}}}{\langle P_{\text{loss}} \rangle} \tag{5.4}$$

where V_{dc} is the voltage that a DC accelerator would require to provide the given beam power, and $\langle P_{\text{loss}} \rangle$ is the excess power delivered to the accelerating structure that is not delivered to the beam.

This has some interesting consequences, such as a need for multi-cell cavities. If we have a single-cell cavity with shunt impedance Z_{shunt}, then we can maintain a voltage V with power

$$P_{\text{single cavity}} = V^2/R_{\text{shunt}} \tag{5.5}$$

to the cavity. However, if the system has N cells, each cell can provide gradient and we only need a proportional power of

$$P_{\text{multicell}} = N(V/N)^2/R_{\text{shunt}} = P_{\text{single cavity}}/N \tag{5.6}$$

There is less stored energy in each cavity, so we cannot accelerate as much beam current. We also need more space for cavities at a given frequency with multiple cells. However, the scaling is still such that high gradients in single passes require multiple cells, hence the proliferation of SCRF structures that have upwards of 5, 7, or even 9 cells.

References

[1] http://en.wikipedia.org/wiki/Cauchy_distribution.