# Wed Jan 25 Computer Lab: Nonlinear Dynamics

## 1 Octupoles and Fourth Order Resonances

Consider an otherwise linear lattice with a single octupolar nonlinearity:

$$H = (2\pi Q_x)J_x + (2\pi Q_y)J_y + V_4(x, y; s) \tag{1}$$

where $V_4(x, y; s) = \frac{1}{24}O(s)(x^4 - 4x^2y^2 + y^4)$ and $O(s) = -b_3(s)/B\rho$.

1. Expand $V_4$ in action-angle coordinates, $x = \sqrt{2J_x}\cos\phi_x$ and $y = \sqrt{2J_y}\cos\phi_y$. What resonances does the octupole drive?
2. There are some terms that will arise that have no angular dependence. What physical significance do these terms have?
3. Expand into Fourier harmonics and assume that the tune is near the $4\nu_x = l$ resonance. Find the fixed points of this resonance; which are stable fixed points and which are unstable?
4. **Bonus:** Calculate the resonance island widths and frequency of small oscillations around the stable fixed points in terms of the octupole strength.

## 2 Computer Lab: Nonlinear Dynamics

In a Java-enabled web browser, go to the URL `http://www.toddsatogata.net/2011-USPAS/Java/`. Within this directory are some simple Java programs and tutorials that are meant to demonstrate aspects of accelerator nonlinear dynamics. You will usually have to type in a number of turns to track and hit return before starting to click in areas to launch particles. Todd should demonstrate at the start of the lab.

## 3 The Henon Map

`http://www.toddsatogata.net/2011-USPAS/Java/henon.html`
Open the Henon map Java example in the URL above and play with it a bit. Clicking in the black area at the center of the screen will "launch" particles to be tracked through a simple motion that is similar to a linear ring with a single sextupole. The normalized coordinates $(x, x')$ are displayed once per iteration of the map in a Poincaré section. This is among the simplest of nonlinear maps, a rotation with a kick, so it has been extensively studied by dynamicists. Set the number of iterations to 1000; this will track 1000 "turns" for each launched particle.

1. Set $b_2 = 0$ and track a particles for several different tune values $Q$. Why does the motion always appear pretty much the same? What happens when you set the tune to a low-order rational number, like 0.25?

2. Keep $Q = 0.25$ and start raising $b_2$. Around what values of $b_2$ do you start seeing something new happening for large-amplitude particles? Are the tunes for these particles increasing or decreasing from 0.25?

3. Set the tune near $Q = 1/3$ and raise the sextupole strength to about $b_2 = 0.2$. Compare the motion to motino that we discussed in class. What happens when you move the tune to the other side of $Q = 1/3$, and why?

4. Set the tune to $Q = 0.252$ and gradually raise $b_2$ from near zero by increments of 0.2 from 0–2. How does the phase space change? Bonus: what order in sextupole strength drives the observed resonance?

5. Set the sextupole strength to $b_2 = 0.5$ and find the separatrix. Change the number of iterations to 100 and track near and away from the separatrix. From this can you infer the period of the particle motion around the separatrix?

## 4 The Henon Map with Damping

`http://www.toddsatogata.net/2011-USPAS/Java/henon3.html`
In electron rings, there is synchrotron radiation damping, so particles tend to damp to the closed orbit. This can be included in the nonlinear dynamics simulation, along with sources of noise, to show how particles can damp into resonance islands instead of onto the closed orbit.

1. Set the tune $Q$ to 0.254 and $\delta$ to 0.001. Launch a few particles to observe that you have damping turned on. Here the damping delta is very strong.

2. Gradually raise $b_2$ from near zero by increments of 0.2 from 0–2. How does the phase space change now? Note that the resonance islands can become attractors even in the presence of relatively strong damping, so they are really stable fixed points.

## 5 Octupoles and Decapoles

`http://www.toddsatogata.net/2011-USPAS/Java/odo.html`
We have looked at sextupoles, so what about octupoles and decapoles? This exercise will also introduce Poincaré plots in action-angle coordinates instead of the normalized $(x, x')$ coordinates we have been using. There is also *tune modulation* in this simulation that we should turn off; set the number of iterations to about 200, q to 0, Tm to 1, b1b to 0, and b1q to 0.

1. Set the tune $Q = 0.193$, and $b_3$ and $b_4$ (octupole and decapole strengths) to zero. Tracking particles now gives horizontal lines, where the horizontal axis is phase (going from 0 to $2\pi$), and the vertical axis is action $J$.

2. Set the octupole and decapole strengths to 0.1. How has the phase space changed? Locate the fifth-order resonance islands. How do they change when you change the octupole and decapole strengths? The detuning (or location of the islands) is driven to first order in the octupole strength, while the resonance strength itself ($5Q_x$) is driven to first order in the decapole strength.

## 6 Chirikov Resonance Overlap and Beam-Beam

`http://www.toddsatogata.net/2011-USPAS/Java/beamtune.html` Open the beam-beam map Java example in the URL above and play with it a bit. This is a simplified model of the beam-beam interaction of two beams as we will discuss in class; you can think of it as a nonlinear kick from an oncoming Gaussian beam. Keeping the tune at $Q = 0.331$, set the beam-beam tune shift $\xi$ to zero, and click within the black area to launch particles and produce Poincaré plots. With no nonlinearity, these are all horizontal lines, consistent with constant action.

1. Produce phase space plots by gradually increasing $\xi$ by 0.001. With What $\xi$ do you start to see resonance islands?
2. Vary $\xi$ up to about 0.03. How do the resonance island locations and widths seem to scale with $\xi$? What other harmonics of phase space distortion appear at small, medium, and large amplitudes?
3. As $\xi$ gets even larger, you will see more and more resonance islands appear at small amplitudes. These islands remain small, but at some point $\xi$ is large enough that resonances start to overlap, and stochastic motion occurs consistent with the Chirikov overlap criterion. Experimentally find the lowest value of $\xi$ where stochasticity occurs to two significant figures. (Hint: It's between 0 and 1.)