THE PHYSICS OF ELECTRON STORAGE RINGS
AN INTRODUCTION

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PLUS OR MINUS ?
ALGEBRAIC SIGNS IN THE STORAGE RING EQUATIONS OF
SLAC REPORT NO. 121[*]

When I was writing SLAC Report 121, I was making the implicit assumption that the curved parts of the design orbit would always bend in the same direction. The RLA design shows that such an assumption was short-sighted. I have, therefore, reviewed SLAC-121 to see which equations may need to be changed when the design orbit has parts with a reverse curvature. Fortunately, very few changes are required. In this note, I report the adjustments that should be made in SLAC-121 so that the results will be applicable to rings of arbitrary curvature. In addition, I list other miscellaneous corrections — particularly of algebraic signs.

A. Comments on Part II of SLAC-121

In describing the design orbit, it was assumed that the direction of rotation was clockwise (looking down on the orbit). See Fig. 7. (I now feel that this was a poor choice, but that's life.) That is, the orbit was assumed to curve toward the right, while the positive direction of the horizontal (or radial) coordinate, x, was taken to the left. The positive direction of the z-coordinate, of course, defines the "upward" direction. The equations of SLAC-121 will, as we shall see, generally hold with a minimum of tinkering if we maintain the convention that the x-coordinate is taken as positive to the left of direction of travel, and if we insist that the net curvature of the design orbit shall be toward the right, while permitting that parts of the orbit may have opposite curvatures — namely, toward the left.

With these understandings, Eqs. (2.1) and (2.2) may be left as they are, but Eqs. (2.3) and (2.4) need the following comments. It is convenient to define the curvature function G(s) so that it is positive when the orbit curves to the right (toward negative x), and is negative for the opposite curvature. Equation (2.3) will give this result provided that we specify that e shall represent the electric charge, with sign, of the circulating particle. (That is, e is a negative number for electrons.) For consistency, e should be interpreted the same way in Eq. (2.4), and, also in all subsequent equations in SLAC-121.

Equation (2.5) now defines the "radius of curvature" of the orbit ρ_s as an algebraic quantity. The radius is <u>positive</u> if the center of curvature is toward negative x, and negative of the center of curvature is toward positive x.

Equation (2.6) has a typographical error; it should read:

$$-d\theta_0 = \frac{ds}{\rho_s} = G(s)ds \tag{2.6}$$

(It is intended here and later that angles in the plane of the orbit are measured with the usual convention — positive angular changes are counter-clockwise.)

With these adjustments all of the remaining equations of Part II need no change to take into account orbits that may have reverse bends. It is only necessary to keep in mind that e, G(s), ρ_s, and $K_1(s)$ are all quantities with appropriate signs, and in particular, that G(s) and ρ_s (and, of course, $K_1(s)$) may have both positive to negative values around the ring.

Notice, however, that the definition of an "isomagnetic" guide field in Eq. (2.9) intends that G(s) shall have a unique value — <u>including the sign</u> — in all bending magnets. Our conventions then dictate that $G_0 = 1/\rho_0$ is necessarily a positive number.

While I am at it, I may as well point out some careless errors of sign in Part II that are not basically related to the present discussion.

Notice that K_x, K_z, and the generic K have been <u>defined</u> to be <u>positive</u> when they are <u>defocussing</u>. See Eqs. (2.19), (2.20) and (2.31).* Equation (2.32) is then wrong — it assumes the opposite definition. So Eq. (2.32) should read

$$K < 0: \quad x = a \cos (\sqrt{-K}\, s + b)$$
$$K = 0: \quad x = a\, s + b$$
$$K > 0: \quad x = a \cosh (\sqrt{K}\, s + b) \tag{2.32}$$

Similarly, the matrices of Table I are wrong. Letting the conditions on the left of Table I stand as is (K < 0, K = 0, K > 0 in that order), the matrix elements need correction by replacing K everywhere with its negative. (Change K to -K, and -K to K.) Sorry about that.

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*Probably a bad choice. And, clearly, I was quite ambivalent, since I shifted ground and wrote some of the equations with the opposite convention.

A careless error of sign was also made in writing Eq. (2.84). Clearly, if δG is to be interpreted in the normal way as the change in G (with appropriate sign) Eq. (2.84) will follow from the immediately previous equation if it reads

$$\Delta x' = -\delta G \, \Delta s. \qquad (2.84)$$

The error made here was propagated in all subsequent equations, so all of the equations of Section 2.10 should be corrected by changing δG to $-\delta G$.

There is a typo in Eq. (2.60). The last integral should be preceeded by the factor $1/2\pi$.

B. Comments on Part III of SLAC-121

All of the numbered equations in this part are, I believe, correct as they stand — with G(s) an algebraic quantity with appropriate sign.

There are a few errors in the text. In the line above Eq. (3.5) the equation should read: $K_x = -G^2$. In the material above Eq. (3.6) δG should be replaced wherever it occurs by $-\delta G$.

C. Comments on Part IV of SLAC-121

The material of this part is OK. In particular, the integral for $\mathscr{D}$, Eq. (4.18) contains G(s) to the first power, so those parts of the orbit with reverse curvatures will (for the same sign of K_1) give an opposite contribution to the integral.

There are a few typos. In Eq. (4.13) ρ should read ρ_s. In Eq. (4.17) the large parenthesis which should preceed $1/\rho$ is broken. In Eq. (4.26) the long bar after the first z' should be an arrow ($\rightarrow$). In Eq. (4.48) the negative sign after the equal sign should be deleted.

D. Comments on Part V of SLAC-121

This part suffers considerably from the implicit assumption that the design orbit had a homogeneous (always positive) curvature. To make it apply generally to orbits with some segments of abnormal curvature the following changes are required.

Eqs. (5.3), (5.9): ω_c is a positive quantity, so in these equations ρ should be replaced by its absolute value $|\rho|$. Eq. (5.20): Replace ρ^3 by $|\rho^3|$.

It follows that the quantum excitation depends only on the magnitude of the orbit curvature. So the following changes should be made in the rest of Part V.

Eqs. (5.40), (5.41): Replace $\gamma_0^3 G$ by $\gamma_0^3 |G|$.

Eqs. (5.42), (5.44), (5.45), (5.47), (5.82), (5.83): Replace G^3 by $|G^3|$.

There are also a couple of typos. In the middle of page 129, $E_0^{2/3}$ should read $E_0^{3/2}$. In Eq. (5.71) the inner parenthesis should be squared — as in the preceding equation.

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MOST COMMONLY USED SYMBOLS

A Area of a transverse section of a stored beam; or amplitude of an oscillation.

A_{int} An effective area of colliding beams.

a Amplitude of an oscillation; invariant part of an amplitude.

α Time dilation of off-energy orbits (momentum compaction).

α_i Damping coefficients of oscillations $(1/\tau)$; $i = x, z, :.$

B Magnetic field strength.

$\beta(s)$ Betatron functions.

c Speed of light.

γ Energy in units of rest energy (E/mc^2).

D Electron number per unit area (N/A); or rate of change of radiation loss with energy (dU/dE).

δ Half of the angle of intersection of colliding beams.

E Electron energy.

E_0 Nominal energy of a stored beam.

$\mathscr{E}$ Electric field strength.

e Electric charge of an electron.

ϵ Energy deviation $(E - E_0)$.

f Frequency of revolution of a synchronous electron $(c/L; c/2\pi R)$.

$G(s)$ Curvature function of the design orbit $(1/\rho_s)$.

G_0 Curvature of the design orbit of an isomagnetic guide field $(G_0 = 1/\rho_0)$.

$\zeta(s)$ Envelope function $(\sqrt{\beta})$.

$\eta(s)$ Off-energy function.

h Beam height.

$h*$ Beam height at the intersection.

I Beam current (Nef).

J_i Partition numbers of the damping rates; $i = x, z, \epsilon$.

$K(s)$ Focussing functions of the guide field.

k Rf harmonic number (ω_{rf}/ω_r); or deviation from nominal value of a focussing function (ΔK).

L Length of the design orbit $(2\pi R; c/f)$.

ℓ Trajectory length; bunch length.

m Rest mass of an electron.

N Number of stored electrons in a beam (I/ef).

ν	Betatron number.
$\Delta\nu_0$	Beam-beam interaction parameter.
P	Radiofrequency power.
p	Electron momentum.
Q	Quantum excitation strength.
q	Rf overvoltage (eV/U_0).
R	Gross radius of the design orbit ($L/2\pi$).
r_e	Classical electron radius ($e^2/4\pi\,\epsilon_0 mc^2$).
ρ_s	Radius of curvature of the design orbit at s $\left[1/G(s)\right]$.
ρ_0	Magnetic radius in an isomagnetic guide field.
s	Azimuthal (longitudinal) coordinate.
σ	Standard deviation of a distribution.
$T(\epsilon)$	Time for a revolution.
T_0	Revolution period of a synchronous electron (L/c).
t	Laboratory time.
τ	Time displacement between an electron and the center of its bunch (y/c).
τ_i	Damping time constant of an oscillation mode (i/α_i); $i = x,z,$.
$\phi(s)$	Phase variable of betatron oscillations.
$\Phi(\tau)$	Pseudopotential energy of the longitudinal oscillations.
$U(\epsilon)$	Energy loss by radiation in one revolution.
u	Energy of a radiated quantum.
$V(\tau)$	Effective "voltage" of the rf system.
w	Beam width.
w*	Beam width at the intersection.
x	Horizontal (radial) displacement from the design orbit.
y	Longitudinal (azimuthal) displacement from a bunch center ($c\tau$)
z	Vertical (axial) displacement from the design orbit.
Ω	Angular frequency of the energy (and longitudinal) oscillations.
ω_r	Angular frequency of revolution ($2\pi/T_0$).
ω_β	Angular frequency of a betatron oscillation ($\nu\,\omega_r$).

I. AN INTRODUCTORY OVERVIEW

1.1. Opening Remarks

Electron storage rings have now come of age. With the successful operation of Adone,[1] experiments will now begin using colliding beams of electrons and positrons with energies of 1 GeV and beyond, expanding the area of storage ring research which was begun at lower energies with the pioneering instruments at Stanford, Frascati, Novosibirsk, and Orsay. Projects under way at Novosibirsk, Cambridge, Hamburg, and Stanford will soon provide stored colliding beams of electrons at even higher energies. Larger numbers of workers will be basing their research in particle physics on these instruments. Many of the physicists who will be using storage rings will not have had a part in their design and construction, and will not initially have a knowledge of their inner workings. The aim of this report is to provide for such physicists a review of the basic physical processes that determine the behavior of electron storage rings — with a particular concern for their performance as instruments for research in particle physics. Because of this aim, the material is generally not presented in the form which might be most convenient for those who will be interested in the design of storage rings. It is, rather, developed in a form intended to give the using physicist an understanding of the inherent properties of his instrument — especially those which will have an influence on his observations — and to give him some feeling for its basic limitations and its ultimate capabilities.

In the rest of this introduction I give a qualitative description of each of the basic phenomena that play a role in colliding beam storage rings, including a discussion of the factors which determine the luminosity. This first part is intended to provide a background and a vocabulary for the appreciation of the other reports[†] which describe the operating experience with existing rings and the projects for new rings. In the remaining parts I shall consider in detail the theory of those basic single particle processes which determine the ultimate limits on the performance of storage rings. A discussion of the important collective effects, which have lead to many practical difficulties in high intensity rings, is <u>not</u>

[†]Other "reports" mentioned here refer to other contributions at the Varenna Summer School of 1969.

included here, but will be found in the report of Pellegrini. At the end, I return to a discussion of the limitations on the performance of high energy storage rings, and apply the results to an illustrative example — the new Stanford design for a 2-3 GeV ring.

1.2. Basic Processes

Let me begin with a brief qualitative description of the basic processes which come into play in producing a stored electron beam.[†] (See Fig. 1.)
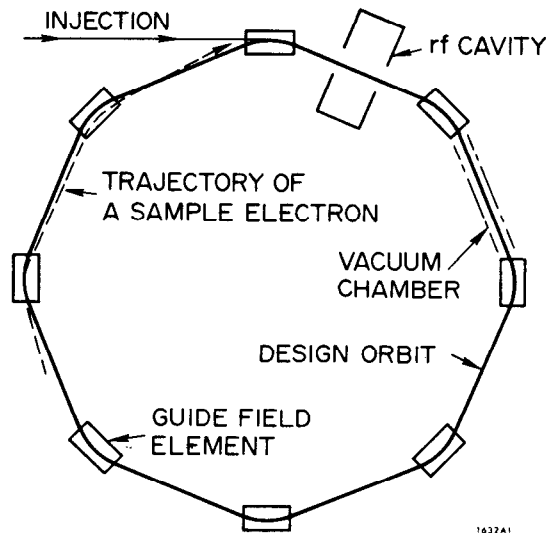
FIG. 1--Schematic diagram of an electron storage ring.

-- A short pulse of a beam of electrons is injected into a vacuum chamber embedded in a more-or-less circular magnetic guide field. The guide field leads the electrons around in more-or-less closed paths to make a stored beam.

-- The guide field has focussing properties which drive all electrons toward an ideal design orbit and cause them to execute lateral (radial and vertical) betatron oscillations about the ideal closed path.[††]

-- During each revolution an electron loses a small fraction of its energy by synchrotron radiation. For stored electrons this energy loss is compensated

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[†] I shall always speak only of electrons, since positrons are, of course, just electrons with the opposite charge.

[††] The design orbit is taken to lie in a horizontal plane.

for by a corresponding gain of energy from a <u>radio frequency cavity</u> (or from several cavities acting in concert.)

    -- The periodic accelerating field collects the electrons into circulating <u>bunches</u>, within which the individual electrons oscillate in <u>longitudinal</u> position and in <u>energy</u> relative to an ideal reference particle at the center of the bunch. The associated motions in longitudinal position and energy are called the <u>synchrotron oscillations</u>.[†]

    -- The energy loss by synchrotron radiation together with the compensating energy gain from the rf cavity gives rise to a slow <u>radiation damping</u> of all oscillation amplitudes; the trajectory of each electron tends toward that of an ideal reference particle at the center of the bunch (which moves with constant speed along the design orbit.)

    -- Radiation damping does not conserve phase density, so it is possible to inject, successively, many pulses into the neighborhood of the same ideal orbit and obtain high circulating currents from weak sources — for example with positron beams.

    -- The damping of all oscillation amplitudes is effectively arrested because of a continuous excitation of the oscillations by "noise" in the electron energy, which comes about from the fact that the synchrotron radiation is emitted in photons of discrete energy — the so-called <u>quantum fluctuations</u> of the energy loss.

    -- In stationary conditions a balance is reached between quantum excitation and radiation damping, leading to a statistically stationary distribution of the oscillation amplitudes and phases of the electrons in a bunch. The bunch then takes on the aspect of a traveling strip of ribbon which has a stationary "size" and "shape", with a Gaussian distribution of amplitudes in each of the transverse and longitudinal coordinates (see Fig. 2). (The shape of a bunch will be different at each azimuthal position because the focussing properties of the guide field vary from place to place, but in the stationary condition the bunch has the same shape at each successive transversal of any chosen azimuth.)

[†] Often called "phase" oscillations. I prefer a different term in order to avoid the confusion which results when one wishes to speak of the phase of the "phase" oscillations.
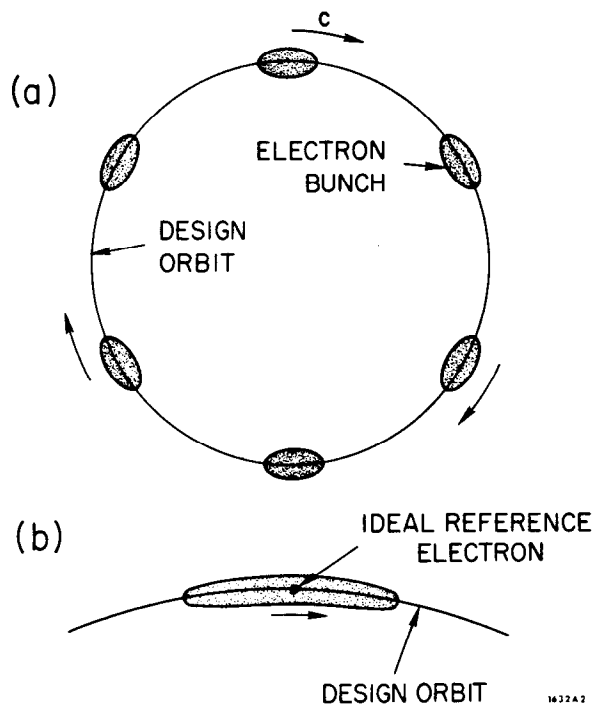
FIG. 2--Circulating bunches in a stored beam.

-- For each coordinate of an electron there is some maximum oscillation amplitude above which the electron no longer remains captured in the bunch. We may refer to the range of stable amplitude in each coordinate at its <u>aperture</u>. An electron is lost from a bunch when some disturbance increases the amplitude in any coordinate beyond the corresponding aperture limit. The aperture limit for each coordinate may be set by a physical obstacle which intercepts the electrons, or by nonlinear effects in the focussing forces which lead to unbounded trajectories for large displacements from the ideal reference electron.

-- Electrons may be lost by scattering or energy loss in collisions with molecules of the residual gas in the vacuum chamber,[†] or by a large statistical fluctuation in the quantum excitation of an oscillation amplitude.

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The basic processes considered above are the <u>single-particle</u> effects which are primarily responsible for the intrinsic properties of a stored electron beam. Until now I have considered a bunch as a collection of noninteracting electrons each of which moves as though it were alone in the storage ring. Unfortunately, life is not so simple.

[†]Scattering on the residual gas can, in principle, also modify the shape (and increase the size) of the stored bunch. But with relativistic electrons and the low chamber pressures required for long beam lifetimes this effect is generally negligible.

1.3 Collective Effects

When the number of electrons in a circulating bunch is large enough (typically, greater than 10^9 or so) interactions among the electrons of a bunch, or among bunches, become important — and have, in fact, been a serious problem in all electron storage rings. I turn now to a brief listing of the most significant collective effects.

-- The AdA- or Touschek-effect. Two electrons oscillating within a bunch may Coulomb scatter, transferring some of the oscillation energy of each electron from one coordinate to another. The new amplitudes in the second coordinate may lie outside the available aperture, or may contribute to an increase of the bunch dimensions. The Touschek-effect is generally significant only at low energies — below 1 GeV or so.

-- Coherent oscillations. Each electron in a circulating bunch produces electromagnetic fields in the vacuum chamber which influence the motion of the other stored electrons.[†] Such collective interactions among the electrons can lead to unstable coherent oscillations, in which all of the electrons of a bunch oscillate in a collective mode whose amplitude grows exponentially with time. Such coherent oscillations may involve either the transverse or longitudinal motion of the electrons and can lead to a growth of the bunch size or to the loss of electrons from the bunch.

-- Constructive interference of the radiation fields of electrons in a bunch may give rise to coherent synchrotron radiation, which can increase the energy loss of individual electrons. (This effect is not believed to be significant in the storage rings now in operation.)

<p style="text-align:center">*   *   *</p>

To get the high current densities desired in electron storage rings it is generally necessary that the coherent instabilities be suppressed or otherwise controlled. Then the remaining collective effects (which are essentially incoherent) combine with the single particle effects discussed earlier in determining the bunch dimensions. (I am assuming that the strange bunch-lengthening effect observed in many storage rings — which is, as yet, not understood — will

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[†]The direct electromagnetic interaction between two electrons of a bunch decreases as the energy squared and is generally negligible for high energy storage rings.

ultimately be explained in terms of one or another of the processes already described.)

Once one has learned how to make a high current, stored beam, it remains only to prepare two of them and arrange that they collide. Except that, unfortunately, new complications then arise.

1.4.   Two-Beam Effects

When two stored beams are made to collide — by arranging things so that the orbits of the two beams intersect, and that bunches of each beam arrive simultaneously at the intersection — the electron motions are disturbed by two-beam effects.

-- When an electron of Beam 1 passes through the intersection, it feels the strong electromagnetic field set up by Beam 2. This macroscopic field disturbs the single-particle orbits of the electrons on Beam 1, and at sufficiently large current densities, leads to what we may call a "soft" instability — one in which there is an incoherent growth of the transverse oscillation amplitudes and, therefore, of the dimensions of Beam 1. It is this effect which will, in general, set the ultimate limit on the rate of high energy interactions which can be achieved in electron storage rings.

-- The forces between the two beams will couple the coherent oscillation modes of the two beams and can produce unstable modes in the two-beam system. Also these coherent oscillations must be suppressed if successful colliding beam operation is to be achieved.

-- Close collisions between pairs of particles in the colliding beams can cause scattering or energy loss with a resultant loss of the particles from the bunch, or an increase of oscillation amplitude. Such effects are, of course, welcome; they are, after all, the collision processes the storage rings have been constructed to produce!

-- The trajectories of the stored beam can be arranged so that the counter-rotating beam collide head-on or at some small crossing angle. The magnitude of the crossing angle affects both the size of the zone of particle-particle collisions and the strength of the macroscopic beam-beam interactions; this angle plays, therefore, an important part in the performance of the storage ring.

1.5 Luminosity

Given the energy of the particles in a storage ring, the next important parameter is its <u>luminosity</u>, which is defined as the counting rate of events for a process of unit cross section. I shall complete this introduction with a brief discussion of the factors which determine the luminosity of a storage ring. The treatment is intended to serve as a basis for the following sections of this report, and also as a background for the other reports which discuss the experiences with operating rings or the designs of projected rings. Consider some particular process which can occur in the collisions of the particles in two colliding beams. (You may include, if you wish, in the definition of the "process," the requirement that certain particles be detected in certain counters.) Let σ be the cross section, for the process and R the rate of events of that kind which occur at a particular intersection region; then the luminosity $\mathscr{L}$ is defined by

$$R = \mathscr{L} \tag{1.1}$$

(If the two beams collide at more than one place around the ring, the luminosity, as used here, will refer to the events at only one of the intersections.)

Let's look now at how the luminosity is related to the properties of the stored beams. Consider first the simplest situation in which each beam contains only a single bunch and these bunches collide head-on at the intersection. (See Fig. 3.)

INTERACTION

REGION
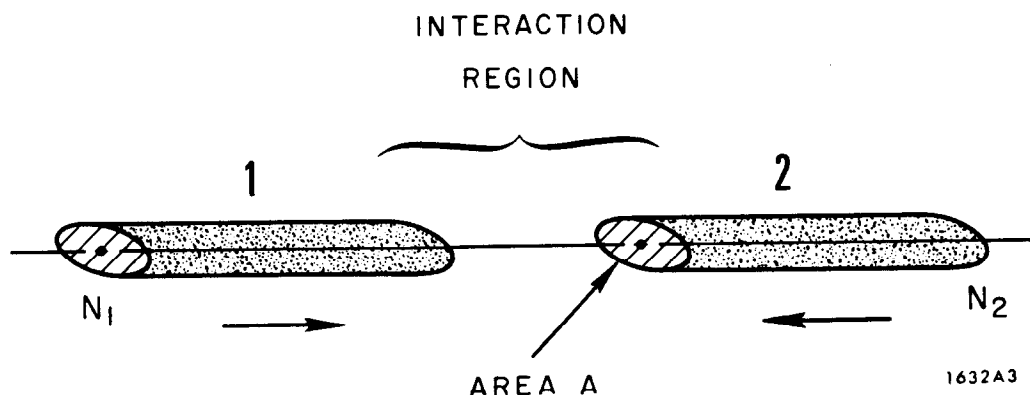
FIG. 3--Head-on collision of two bunches.

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The total number of particles in the bunch may be different for the two beams — say N_1 for one beam and N_2 for the other. Let's imagine for the moment that a bunch is a ribbon-like object with a transverse area A, and that there is a uniform density of particles inside. (For a rectangular transverse section the area A would be just the product of the width w and the height h.) Let the bunches circulate around the ring with a revolution frequency f. When a particle of Beam 1 passes through the bunch of Beam 2 the probability of an event of unit cross section is N_2/A. Since there are N_1 particles in Beam 1 and the bunches collide at the frequency f, the luminosity at the interaction region would be given by

$$\mathscr{L} = \frac{N_1 N_2 f}{A} \qquad (1.2)$$

The model of a bunch I have just used is, of course, over simplified. As described earlier, we expect a bunch to have a Gaussian distribution of particle density in each coordinate — the transverse section of the "ribbon" is a fuzzy ellipse. Suppose we let w and h stand for the width and height of the horizontal and vertical density distributions in a bunch, where by these dimensions I wish to refer to twice the root-mean-square spread of the distributions. We may then define the "area" of the bunch to be

$$A = \frac{\pi w h}{4} \qquad (1.3)$$

This area may not, however, be used directly in Eq. (1.2) because the luminosity will be obtained from the overlap integral of the two-dimensional density distributions of the two bunches. This integral just contributes a factor of 1/4, so we have, for real (Gaussian) bunches, that

$$\mathscr{L} = \frac{1}{4} \frac{N_1 N_2 f}{A} \qquad (1.4)$$

with the area defined as in Eq. (1.3).

Next consider the effect of intersection at an angle. Suppose that the trajectories of the bunches intersect with a "vertical" crossing angle as indicated schematically in Fig. 4. Now when a particle of one beam transverses the bunch of the second beam, the mean transverse particle density it sees — and therefore, the probability of interaction — depends on the projected area of the opposite bunch. One might at first think that the area A of Eq. (1.2) should simply be replaced by the static projected area which would be the product of the beam width w by an effective
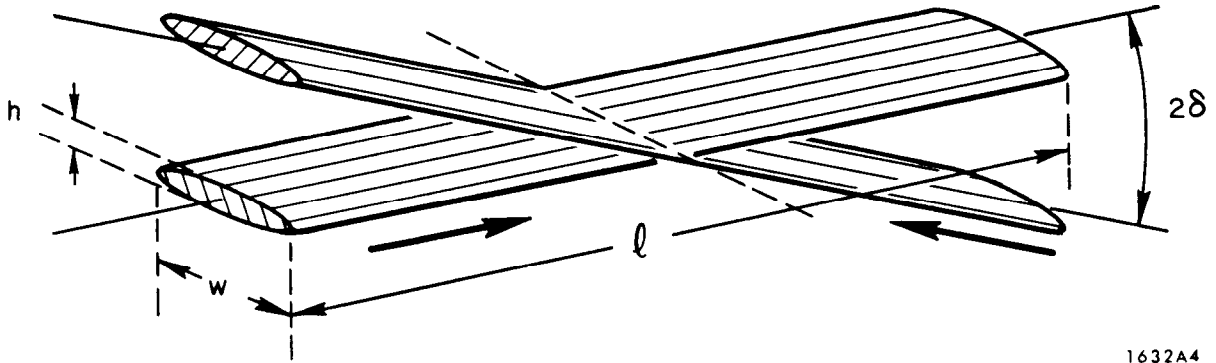
FIG. 4--Bunches colliding with a vertical crossing angle.

projected height. Suppose that the vertical thickness of the ribbon is much less than the projected height; then the latter would be just the product of the beam length ℓ and the crossing angle 2δ. However, the relative motion of the two bunches must be taken into account. When this is done, one finds that the proper projected height is <u>one-half</u> the product of the length and crossing angle, so that for an idealized ribbon-like beam we should in computing A of Eq. (1.2) take for the effective height $h_{eff} = \ell \delta$.

If we now take into account the Gaussian distribution of density of particles in the bunch in all three dimensions (using ℓ to represent twice the rms longitudinal spread), and also the fact that, in general, both the beam thickness and the crossing angle will contribute to the projected height seen at the interaction, then the luminosity is correctly given by Eq. (1.4) also for a vertical crossing angle if we use for the area A the effective projected area

$$A_{eff} = \frac{\pi}{4} \, w \, h_{eff} \quad \text{(vertical crossing)} . \tag{1.5}$$

with

$$h_{eff} = (h^2 + \ell^2 \delta^2)^{1/2} \tag{1.6}$$

It is also possible to arrange the storage ring orbits so that the beams will cross with a "horizontal" crossing angle as shown schematically in Fig. 5. Again the luminosity is given by Eq. (1.4) if for the area A one now uses the effective projected area

$$A_{eff} = \frac{\pi}{4} \, w_{eff} \, h \quad \text{(horizontal crossing)} \tag{1.7}$$
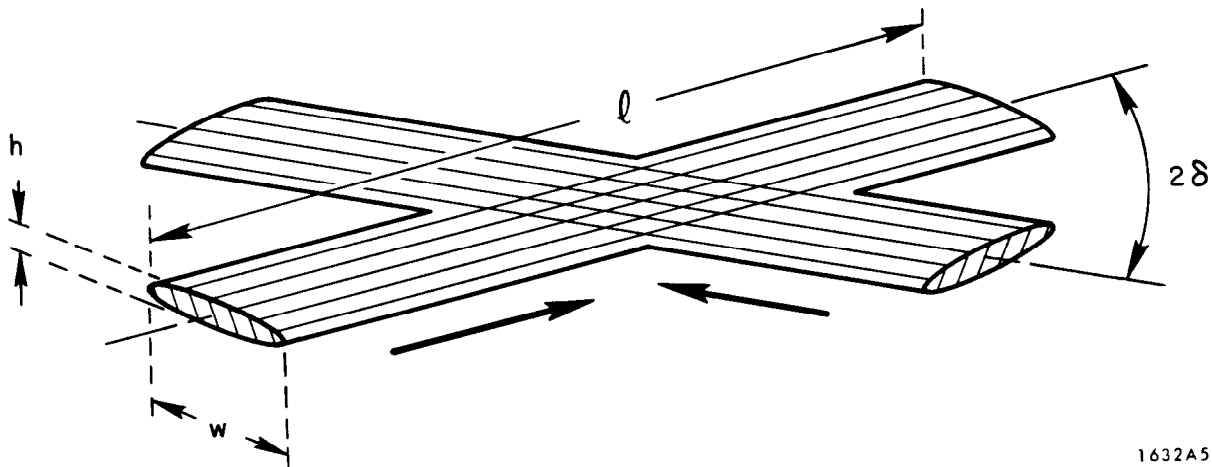
FIG. 5--Bunches colliding with a horizontal crossing angle.

with

$$w_{eff} = (w^2 + \ell^2 \delta^2)^{1/2} \tag{1.8}$$

where δ is now one-half the horizontal angle between the two beam trajectories.

Finally we should take into account that the circulating beams may contain many separate bunches. Suppose that each beam consists of B identical bunches arranged so that each bunch of one beam encounters a bunch of the other beam as it transverses a specified region of intersection. See Fig. 6. Now let N_1 and N_2 be the total number of electrons in each beam, so that the number of electrons per bunch is N_1/B in one beam and N_2/B in the other. The contribution to the luminosity from each pair of colliding bunches is then reduced by $1/B^2$, but there are B such pairs contributing, so the total luminosity at one interaction region is reduced only by the factor $1/B$ below that for single bunch beams. I choose to include this effect by retaining Eq. (1.4) for the luminosity and absorbing the factor B into the definition of an "effective interaction area" of the intersection of the beams.

A generalized luminosity formula may then be written as

$$\mathscr{L} = \frac{f}{4} \frac{N_1 N_2}{A_{int}} \tag{1.9}$$

with

$$A_{int} = \frac{\pi}{4} B w_{eff} h_{eff} \tag{1.10}$$

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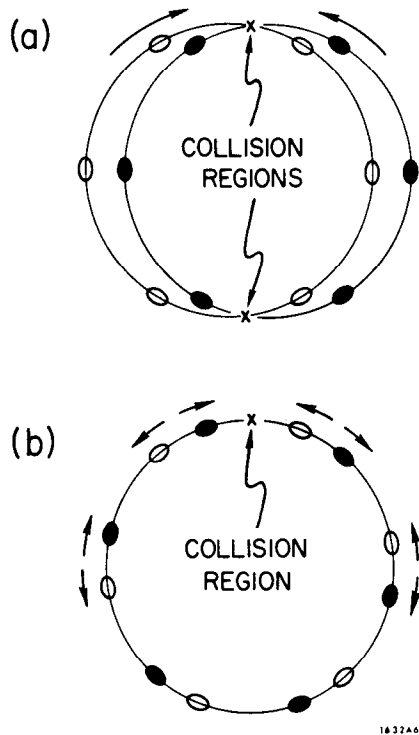
FIG. 6--Beam collision geometries with several circulating bunches.

where for w_{eff} and h_{eff} we are to use either w and h, or one of the expressions in Eq. (1.6) or Eq. (1.8) depending on whether there is a vertical or a horizontal crossing angle.

It is often convenient in the storage ring business to characterize the intensity of a stored beam in terms of its <u>electric</u> <u>current</u> rather than in terms of the number of stored particles. The current I of a beam is defined as the mean rate at which electric charge passes any chosen point on the orbit. This current is related to N by

$$I = Nef \qquad (1.11)$$

with e the electronic charge. In terms of the beam currents the luminosity becomes

$$\mathscr{L} = \frac{1}{4e^2 f} \frac{I_1 I_2}{A_{int}} \qquad (1.12)$$

In what follows I shall find it more convenient to continue to use N, the number of stored particles, as a measure of the intensity of a stored beam, although from time-to-time referring to it loosely as the "beam current."

1.6. Beam Density Limitation

Equation (1.9) would imply that the luminosity can be increased at will by increasing the number of particles in either or both beams. That the reality is otherwise was first pointed out by Amman and Ritson.[3] As the electrons of a beam traverse the interaction region their trajectories are disturbed by the macroscopic electromagnetic field generated by the collective action of the electrons of the other beam. When these disturbances reach a certain strength they influence in an essential way the properties of the stored beams. In particular, they cause a dramatic growth in the beam area and corresponding decrease in the luminosity, which more than cancels the effect of any further increase in the current. I shall consider this effect in detail later (Section 2.13); for now I shall take into account the effects of this beam-beam interaction by making the assertion that the "effective transverse particle density" of a stored beam must be, at the interaction region, no larger than a certain critical value. Specifically, the effective transverse particle density D in a bunch, defined as the ratio of the number of stored electrons N to the effective area A_{eff} (at the interaction region), must not exceed a critical value D_c. We must impose, then, the condition that

$$D = \frac{N}{A_{int}} < D_c \tag{1.13}$$

where D_c is a number which is independent of the beam current, but is given in terms of the basic ring parameters — including the beam energy.

A physical justification of Eq. (1.13) must be deferred until later. (See Section 2.12.) It may be useful for now, however, to report here a formula for D_c. I must emphasize, though, that the formulation I shall give is correct only for certain restrictions on the characteristics of the ring and its operation. Although most well-designed rings will satisfy these restrictions, the applicability of the formula should be confirmed for any particular case. With these cautionary admonitions I write that

$$D_c = \frac{2\Delta\nu_0 \gamma}{r_e \beta_V} \tag{1.14}$$

where $\Delta\nu_0$ is the traditional notation for the "maximum linear tune shift" and stands for a number approximately equal to 0.025; γ is the beam energy in units of the electron rest energy, r_e is the classical electron radius, and β_V is the usual notation for a certain function ("the vertical betatron function" which describes

the focussing properties of the magnetic guide field) evaluated at the beam intersection point. The function will be considered in detail in the next Part; for now the number β_V may be crudely described as being proportional to the "sensitivity" of the electron trajectories to a transverse perturbation applied at the intersection point — a small β_V indicating a smaller effect for a fixed perturbation. Since β_V is the only "free" parameter in Eq. (1.14), you will appreciate why so much attention is devoted to it in discussions of the designs of colliding beam storage rings.

1.7. Maximum Luminosity

The first consequence of the intensity limitation just described is that the maximum luminosity will always be reached when both stored beams are operated at the same maximum permissible current. (Requiring only that, as has been tacitly assumed until now, the two stored beams move in quite similar guide fields and so have the same area at the intersection.) Say that one beam has more current than the other, and that the numbers of electrons of the "stronger" and "weaker" beams are N_s and N_w respectively. The density limitation will then apply only to the strong beam; if its current is as large as possible, we will have that

$$\frac{N_s}{A_{int}} = D_c \tag{1.15}$$

and for the luminosity, using Eq. (1.9), that

$$\mathscr{L} = \frac{f}{4} D_c N_w \tag{1.16}$$

It is clear that the luminosity can always be increased by increasing N_w — until, of course, it becomes as large as the number of electrons in the strong beam! The maximum luminosity will always be achieved when the transverse particle density in each beam is at the limiting value.

I shall, from now on, assume that a storage ring is always operated with the same number N of stored electrons in each of the two beams. The luminosity formula, Eq. (1.9), should then be written:

$$\mathscr{L} = \frac{f}{4} \frac{N^2}{A_{int}} \tag{1.17}$$

And the maximum luminosity will be obtained when

$$\frac{N}{A_{int}} = D_c \tag{1.18}$$

I would like now to consider briefly some of the ways in which the limit on the luminosity of any particular storage ring may arise.

Case 1: The effective area of the beams at the interaction is limited below some maximum value A_{max}, and there is available sufficient beam current to reach the critical particle density.

In this case we may always fill the storage ring to the critical particle number N_c given by

$$N_c = D_c A_{int} \tag{1.19}$$

and the maximum luminosity will be

$$\mathscr{L}_1 = \frac{f}{4} D_c^2 A_{max} \tag{1.20}$$

Notice that in Case 1 the maximum luminosity does not depend explicitly on the number of stored particles available, but is, rather, directly proportional to the maximum available beam area. This behavior is usually characteristic of low-energy storage rings (or of high energy rings operated at low energy), and describes the behavior of all presently operating rings. The form of Eq. (1.20) makes clear why other reports give particular emphasis to the problem of controlling the effective area of stored beams.

I should point out that in interpreting Eq. (1.20) you should be careful to take into account the following considerations. For a given ring with fixed focussing properties and operated at a particular energy, D_c is a fixed number. If the effective area is varied without changing the focussing properties of the ring, Eq. (1.19) shows properly the dependence of the luminosity on the area. However, in comparing two rings with different focussing properties, or the same ring with different focussing conditions, it may be that both A_{int} and D_c change together and the variation of the luminosity will then not be in direct proportion to the variation of A_{int}. Such complexities will be considered in some detail in a later section.

Case 2: The number of stored particles in the beams is limited (at a given energy) to some maximum value N_{max}, and it is possible to adjust the effective area so as to reach the critical current density.

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In this case the stored beams are filled to the intensity N_{max}, and the effective interaction area is adjusted to the value

$$A_c = \frac{N_{max}}{D_c} .$$ (1.21)

The maximum luminosity which can be achieved is then

$$\mathscr{L}_2 = \frac{f}{4} D_c N_{max}.$$ (1.22)

It is proportional to the <u>first power</u> of the beam intensity and to D_c, but does not depend explicitly on the beam area. Case 2 generally applies to high energy storage rings operated in their upper energy range. Clearly, the highest possible currents are desired, and if the highest luminosity is to be achieved it is necessary always to control the area to satisfy Eq. (1.21).

> <u>Case 3</u>: The particle number is limited to some value N_0, the effective area is limited above some minimum value A_0, and their ratio is <u>less than</u> the critical density D_c.

People generally try to avoid such circumstances in the design of a storage ring, but they may occur, for example, at the very highest operating energies of some rings. Then the limit on the current density plays no role, and the maximum attainable luminosity is just

$$\mathscr{L}_3 = \frac{f}{4} \frac{N_0^2}{A_0}$$ (1.23)

It varies as the square of the available current and inversely as the minimum area.

The dependence of the luminosity on the significant parameters of the ring — particularly on the energy — is quite different in the three cases considered above, and is one of the main concerns of the remaining parts of this report. Before turning to such details, however, it will be useful to review briefly the factors which may determine the effective area of the beams at their intersection.

1.8.   Effective Interaction Area

In Eq. (1.10) the effective interaction area was defined as the product of the number of bunches <u>B</u>, with the projections of the beam width and the height — apart from the factor $\pi/4$. I wish now to consider how these factors may be determined for a "given" ring, by which I mean, here, one operated at a given energy and with all of the essential properties of the guide field held fixed.

As remarked earlier a stored bunch will, under stationary conditions, have a size set by quantum effects in the synchrotron radiation. In an ideal machine with a flat orbit such effects act directly to produce random radial oscillations and determine a "natural" or intrinsic beam width — which depends only on the electron energy and the focussing parameters. The width is typically about a millimeter. The direct quantum excitation of vertical oscillations is, on the other hand, very small, and is, in a practical machine, generally dominated by the coupling of energy from the horizontal to the vertical oscillations. Such coupling is due to the various small imperfections in a real storage ring, and it is generally presumed that the beam height can only with difficulty be made less than five to ten percent of the beam width. Often, coupling between radial and vertical oscillations may be intentionally augmented in order to increase the beam height as a way of increasing the beam area. (Such augmented coupling can be introduced by operating a ring so that there is a resonance — or near resonance — between the horizontal and vertical betatron frequencies, or by introducing a special coupling element such as a skew quadrupole.) The maximum area is reached when the oscillations in the two coordinates are effectively equal — resulting in a large increase in the beam height with only a small decrease in the beam width — so that the beam has a nearly circular cross section.†

This technique has been used to increase the luminosity obtained from rings which fall in Case 1 of the preceding section — for example the ring ACO at Orsay.

Both the width and the height of a beam can, in principle, be increased by the artificial excitation of the betatron oscillations, although attempts to do so in ACO have not lead to the expected increase of luminosity. The possibility deserves more study, because if incoherent oscillations could conveniently be excited one would be able to increase the beam area to the maximum value set by the available aperture and obtain the largest possible luminosity for rings operated at low energies (Case 1).

With beam intersection at an angle the crossing angle can be varied to obtain a desired effective beam area. With some rings, particularly with those in which

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†To be precise I should say that the number D_c may depend slightly on the form of the beam. It is generally independent of the beam dimensions for a ribbon-shaped beam but may change by a factor of 2 if the beam section is made circular. See Section 2.13.

both beams (one electron and one positron) are stored in a common magnetic guide field — as in Adone — a continuous control of A_{int} may be obtained by a continuous variation of the vertical crossing angle at the intersection.

It is, finally, clear that the effective interaction area can be adjusted by controlling B, the number of circulating bunches. Generally a ring is equipped with a radio-frequency accelerating system whose operating frequency is at some harmonic k of the rotation frequency f. Then k is also the maximum number of bunches which can be stored. By selective filling of the available bunch positions, however, the number B of stored bunches can also be made any integer less than k. So the range of possible values of B is $1 < B < k$. This opportunity for controlling A_{int} may, however, be of limited use if the selective partial filling of the bunches so reduces the filling efficiency that it decreases the total beam current that can be achieved. It may, however, offer the best alternative for achieving the optimum luminosity in high energy storage rings.

Let me close this section by emphasizing an essential feature of the beam intensity limitation. The optimum luminosity condition depends on the geometrical parameters h, w, δ, and B only through a single number, the interaction area A_{int}. All methods[†] used to achieve a particular value of A_{int} are equally valid — and there is no fundamental reason to prefer one over the other. A wide flexibility is, therefore, available in the design and operation of storage rings. It is hoped that the formulation presented here will make clear how rings adopting different approaches are to be compared.

This completes the Overview I wished to give of the physics of electron storage rings. I turn in the next part to a detailed, quantitative discussion of some of the basic phenomena which I have been able to describe only qualitatively until now.

[†]Within certain wide limits.

II. THE BETATRON OSCILLATIONS[†]

2.1.   <u>Coordinates of the Motion</u>

Electrons are held in a storage ring by the forces from the magnetic guide field. Magnets are disposed along an <u>ideal orbit</u> which is generally a smooth, roughly circular or racetrack shaped, closed curve. When the magnet currents are adjusted to any particular set of consistent values the designed fields are intended to be such that an electron of a <u>nominal energy</u> E_0, once properly started, will move forever along the ideal orbit. All other stored electrons are constrained by the guide field to move in quasi-periodic, stable trajectories in the neighborhood of the ideal orbit.[††] The nature of these stable trajectories is the subject of this part. The treatment will, however, be limited to a so-called linear approximation and will be applied only to electrons of constant energy, ignoring the effects of the radiation loss and the accelerating fields. Such effects will be taken into account later as perturbations of the idealized trajectories.

In most rings the ideal orbit lies in a plane, and I shall limit the discussion to such cases; although the extension to the more general cases is relatively straightforward. The presentation will be simplified by presuming that the plane of the ideal orbit lies horizontal.

It is convenient to describe the motion of an electron in terms of coordinates related to the ideal orbit. The instantaneous position of an electron will be specified by (s, x, z) where s is the distance along the ideal orbit from some arbitrary reference point to the point nearest the electron, and x and z are the horizontal and vertical distances from the ideal orbit. See Fig. 7. We may call s the <u>azimuthal coordinate</u>. The horizontal and vertical displacements are, of course, the displacements locally perpendicular to the design orbit. The positive sense of s will be taken in the sense of the electron's motion, of x in the "outward" direction, and of z in the "upward" direction. I shall often refer to x as the <u>radial</u> coordinate.

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[†]Most of the ideas presented in this part will be found — although often in a different form — in the now classic paper, "Theory of the Alternating Gradient Synchrotron," by Courant and Snyder[4] or in the book, "Accelerateurs Circulaires de Particules," by Bruck.[5]

[††]I shall use consistently, the following terminology: a "trajectory" is <u>any</u> electron path; while an "orbit" is a particular trajectory which repeats itself on successive revolutions.
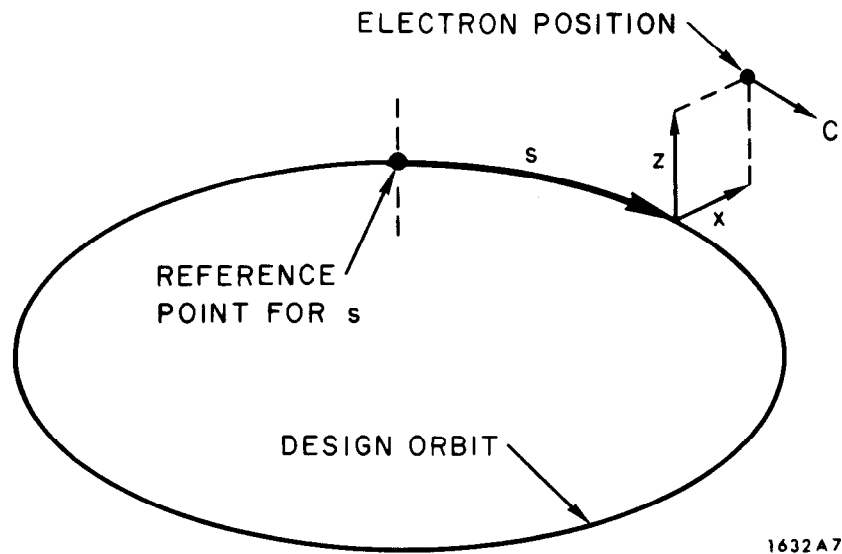
FIG. 7--Coordinates for describing the trajectories.

The coordinates x and z will be considered as "small" quantities in the sense that they are assumed to be always much less than the local radius of curvature of the trajectory, and that in considering variations of the magnetic guide field in the vicinity of the ideal orbit, only linear terms in x and z need be retained. These conditions define the linear approximation of our treatment.

Because the design orbit is a closed curve, the azimuthal coordinate s is cyclic. That is, as s increases indefinitely the location in space repeats itself, repeating each time that s increases by the circumference of the orbit. Let's call this circumference L — and refer to it as the length of this design orbit. A physical location on the azimuth may be identified by s, or by s + L, or by s + 2L, and so on. It will from time to time be convenient to use in place of L an equal quantity $2\pi R$, where R is a kind of "effective radius" of the design orbit. It is common — though strictly improper — to speak of R as the "mean" radius of the ring.

2.2. The Guide Field

The guide field is taken to be static, so the motion of an electron is determined only by the magnetic field strength $\vec{B}$ at each point of its trajectory. As the ideal orbit has been taken to be always horizontal, the field must be purely vertical

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everywhere on that orbit. I shall make here a further assumption: that the design magnetic field is ideally symmetric with respect to the plane of the ideal orbit. Taking into account all of the assumptions so far deliniated, the magnetic guide field may be characterized completely by giving just two quantities for each azimuthal position s, namely, $B_0(s)$, the magnitude of the magnetic field on the ideal orbit, and $(\partial B/\partial x)_{0s}$ the horizontal gradient of the field strength evaluated at the ideal orbit — that is, at x = 0 — for each azimuth. (Since the field is symmetric with respect to the plane of the ideal orbit $\vec{B}_0$ and $\partial\vec{B}/\partial x$ have only vertical components, and we need give only their magnitudes.) As already mentioned, the field $B_0(s)$ produces the curvature of the ideal orbit; whereas the field gradient dB/dx gives rise to the focussing forces responsible for the stable trajectories near that orbit.

The two transverse components of the magnetic field acting on an electron at (s, x, z) may now be written as

$$B_z(s,\ x,\ z) = B_0(s) + \left(\frac{\partial B}{\partial x}\right)_{0s} x; \tag{2.1}$$

$$B_x(s,\ x,\ z) = \left(\frac{\partial B}{\partial x}\right)_{0s} z. \tag{2.2}$$

The last relation follows from Maxwell's equations, which give, for fields with the symmetry imposed here, that $\partial B_x/\partial z = \partial B_z/\partial x$. And the linear approximation has, clearly, been evoked to permit dropping of any terms in the higher derivatives of the fields. The field components above are to be used to obtain the Lorentz force in the equations of motion of the electron.

Storage rings are designed to operate over a range of electron energies. This is accomplished by arranging that all magnetic fields can be varied together — being scaled in proportion to the desired operating energy. Clearly if the magnetic field on the design orbit is changed everywhere by the same factor the design orbit will again be a possible trajectory of an electron whose momentum is changed by the same factor. Varying all fields together merely changes the energy to be associated with the design orbit. For these reasons, it is convenient to specify the properties of the guide field in a manner which is independent of any selected operating energy, which is easily done by dividing all fields by a factor proportional to associated electron energy. I choose to define the (linear)

properties of the guide field by the two functions

$$G(s) = \frac{ec\,B_0(s)}{E_0} \qquad (2.3)$$

$$K_1(s) = \frac{ec}{E_0} \left(\frac{\partial B}{\partial x}\right)_{0s} \qquad (2.4)$$

where E_0 is the nominal energy, c is the speed of light, and e is the electronic charge.

Notice that these functions have a simple physical significance. We are here interested only in highly relativistic electrons for which $E = cp$; so $G(s)$ is just the inverse of the radius of curvature ρ_s of electron of the nominal energy at $x = 0$, $z = 0$.

$$G(s) = 1/\rho_s \qquad (2.5)$$

We may, then, call $G(s)$ the <u>curvature</u> <u>function</u>. The function $K_1(s)$ is the rate of change of the inverse radius with radial displacement.

The functions $G(s)$ and $K_1(s)$ may be fairly arbitrary, but must satisfy a few important constraints. First, $G(s)$ must be such that it does indeed define a closed orbit. (We may think that G <u>defines</u> the ideal orbit, or alternatively that some arbitrarily specified closed orbit defines G uniquely.) The change $d\theta_0$ in the direction of the tangent to the ideal orbit in an azimuthal interval ds is

$$d\theta_0 = \frac{ds}{\rho_s} = G(s)\,ds. \qquad (2.6)$$

The angle swept out in one revolution must be 2π; so $G(s)$ must satisfy

$$\int_0^L G(s)\,ds = 2\pi. \qquad (2.7)$$

Second, both $G(s)$ and $K_1(s)$ are necessarily periodic functions of s, because the azimuthal coordinate s in physically cyclic — returning to the same point on the orbit after one revolution. We must have that

$$G(s+L) = G(s)$$

$$K_1(s+L) = K_1(s) \qquad (2.8)$$

where L is the orbit length. Except for these constaints, $G(s)$ and $K_1(s)$ may have more or less arbitrary variations with s.

Although the guide field functions G and K_1 may, in principle, be quite general, it is often convenient to simplify the design or the operation of a storage ring by imposing certain restrictions on them. For example, most electron storage rings are designed to have the same orbit radius, say ρ_0, in all bending magnets — and with no bending at all in the intervening "straight sections" of the orbit. Such a guide field is called <u>isomagnetic.</u> The word is perhaps slightly misleading. What is intended is that the magnetic field <u>on</u> the design orbit has everywhere the same value <u>except</u> where it is zero. Then G(s) is a dichotic function, taking on either the value G_0 or zero:

$$G(s) = \left\{ \begin{array}{l} G_0 = 1/\rho_0, \text{ in magnets} \\ 0, \text{ elsewhere} \end{array} \right\} \quad \text{(isomag).} \qquad (2.9)$$

A <u>real</u> guide field cannot, of course, ever be ideally isomagnetic, since it is physically impossible to have a discontinuous magnetic field. There must always be a transition zone at the edge of a magnet in which the field goes from zero to its nominal value. The idealized isomagnetic approximation is, however, generally quite adequate for most purposes.

Although accelerators and storage rings are often built with bending magnets which have also radial gradients of the field, it is quite common nowadays to design <u>separated</u> <u>function</u> guide fields in which the focussing functions and bending functions are assigned to different magnetic elements. That is, the guide field consists of a sequence of flat bending magnets (with no gradient) and quadrupoles (with no field on the design orbit). I shall <u>define</u> a separated function guide field as one for which the functions G(s) and $K_1(s)$ are only separately different from zero. So that we have the condition that

$$G(s) K(s)_1 = 0 \quad \text{(sep funct).} \qquad (2.10)$$

One note of caution. It is sometimes convenient to design bending magnets whose pole faces are rectangular. With such a magnet, the design orbit must enter or leave the magnet at other than a right angle to the pole edge. (See Fig. 8.) Even if the magnet is "flat" (no radial gradient in the magnet) there will be radial gradients at the edges, where the field is not zero. Equation (2.10) is not satisfied at the edges, and a guide field constructed of such rectangular magnets — together with quadrupoles — would not strictly satisfy my definition of "separated function," although they are often referred to as such. Such guide fields may however, still be isomagnetic.

BENDING
MAGNET

ψ

DESIGN ORBIT

$G(s)$

s

$K(s)$

s

1632A8
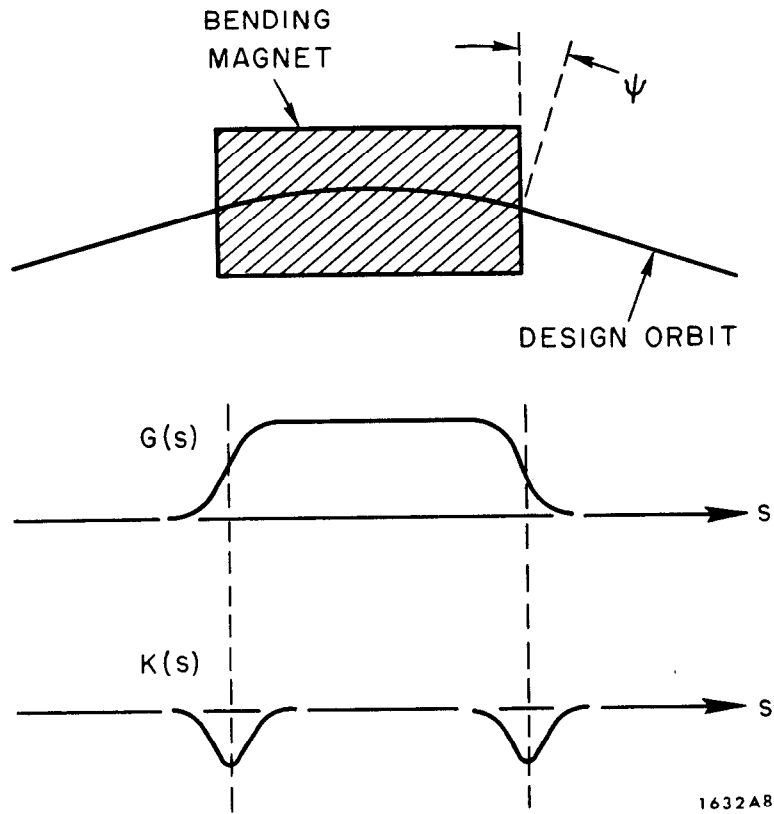
FIG. 8--Guide field with a rectangular magnet.

2.3. Equations of Motion

I would like now to write the equations of motion of an electron that is moving on a trajectory near the design orbit, and with an energy near, but not necessarily at, the design energy. I shall describe the energy of the electron in terms of the deviation ε from the design energy E_0:

$$\varepsilon = E - E_0 \tag{2.11}$$

In keeping with our linear approximation I shall keep terms only to first order in the "small" quantities x, z, and ε. Rather than using time as the independent variable it will be more convenient to use the azimuthal coordinate s. Derivatives with respect to s will be indicated by the "prime" ('); for example, $x' = dx/ds$.

Let's begine with the radial motion. Think of an electron that is at x and moving with the slope x'. See Fig. 9. The slope x' is the angle between the direction of motion of the electron and the tangent to the design orbit. Suppose

- 23 -
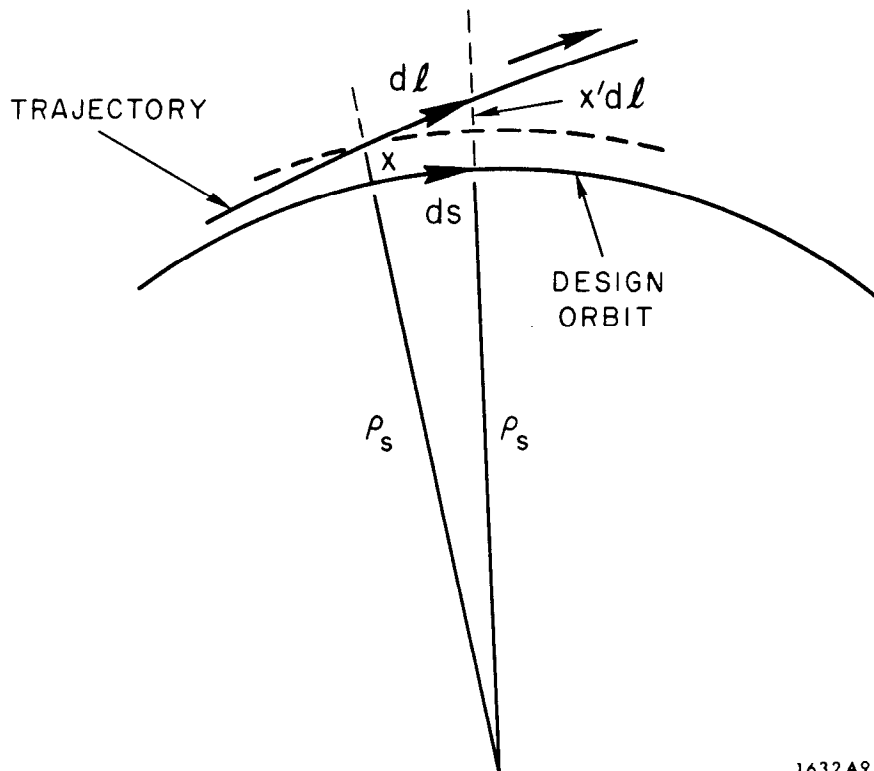
FIG. 9--Electron trajectory near the design orbit.

we call θ_0 the angle between the tangent and some arbitrary reference direction and θ the angle made by the trajectory with the same reference direction. Then $x' = \theta - \theta_0$; and

$$x'' = \frac{d(\theta - \theta_0)}{ds} \qquad (2.12)$$

The derivative of θ_0 is, we have seen, just $-1/\rho_s = -G(s)$. But what is $d\theta/ds$? The radius of curvature of the trajectory is

$$\rho = \frac{E}{ecB}, \qquad (2.13)$$

and in a path element $d\ell$ of the trajectory the change in angle is

$$d\theta = -\frac{d\ell}{\rho} = -\frac{ecB}{E} d\ell \qquad (2.14)$$

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Next, notice that so long as the angle x' is small — as I shall always assume[†] —
a path element $d\ell$ of a trajectory at x is related to the corresponding change in
s by

$$d\ell = \frac{\rho_s + x}{\rho_s}\, ds = \left(1 + \frac{x}{\rho_s}\right) ds = (1 + Gx)ds. \qquad (2.15)$$

Next, we may write for B

$$B = B_0 + \frac{\partial B}{\partial x}\, x = \frac{E_0}{ec}(G + K_1 x). \qquad (2.16)$$

Putting these two into (2.14) — together with $E_0 + \epsilon$ for E — and keeping only
first order terms, we find that

$$d\theta = \left\{- G - \left(G^2 + K_1\right)x + G(\epsilon/E_0)\right\} ds.$$

And so we get from (2.12) that

$$x'' = -\left(G^2 + K_1\right)x + G(\epsilon/E_0) \qquad (2.17)$$

The corresponding equation for the vertical motion is easier to derive; you
can easily see that

$$z'' = K_1 z \qquad (2.18)$$

Notice that with our linearized approximation, the motions in x and z are independent.

For our consideration of the electron trajectories, I would like to use the
standard form:

$$x'' = K_x(s)x + G(s)\frac{\epsilon}{E_0}, \qquad (2.19)$$

and

$$z'' = K_z(s)\, z, \qquad (2.20)$$

[†]More rigorously, x' will turn out to be proportional to x with a proportionality
factor which is of the order of ρ. So long as we keep only terms to first order
in x/ρ, Eq. (2.15) follows.

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with the definitions:

$$K_x(s) = -G^2(s) - K_1(s);\qquad\qquad(2.21)$$

$$K_z(s) = + K_1(s)\qquad\qquad(2.22)$$

The term corresponding to G^2 is missing from K_z because of our assumption that the design orbit lies in a plane.[†] Storage rings are most often "strong focussing." For such rings G^2 is generally much smaller than K_1, so that K_x and K_z are approximately equal and have opposite signs.

The equation for the motion in z looks like the equation of a classical oscillator (force proportional to displacement) with a variable "restoring-force" coefficient — the function $K_z(s)$. The equation in x is similar except that it has, in addition, a varying "driving term" which is proportional to the energy deviation ϵ. In useful guide fields the solutions are indeed oscillatory, and describe the lateral oscillations — including the so-called betatron oscillations — of the electron trajectories. These oscillations result from the focussing properties of the guide field which are characterized by focussing functions K_x and K_z. As we shall see later, the function G(s) enters as well in the energy focussing properties of the guide field.

It is important to remember that all of the focussing functions are necessarily periodic in s, the minimum period being one revolution of the ideal orbit; that is, for both K_x and K_z (as well as for G)

$$K(s+L) = K(s),\qquad\qquad(2.23)$$

where L is the length of the ideal orbit. For convenience in construction — as well as in design — storage rings generally have also an inner periodicity. That is, they are made up, at least in part, of sequences of identical magnetic cells, each cell consisting of a prescribed set of magnets and quadrupoles. Then in a certain span of s, the focussing functions will satisfy

$$G(s+\ell_c) = G(s)$$

$$K(s+\ell_c) = K(s)\qquad\qquad(2.24)$$

[†] G^2 is a centrifugal force term, and a corresponding term would appear in the z-motion if the orbit had hills and valleys.

where ℓ_c is the cell length. Note, however, that while Eq. (2.23) is true for the _actual_ fields of a ring — since when s is increased by L, the electron returns to the same physical point in the ring — the cell periodicity is a _design_ property and will not be strictly true for the actual fields (due to construction imperfections).

It will be useful to have in mind some illustrative example of a guide field. Let's take the design of the proposed SLAC ring.[6] In it, most of the ring would consist of a repetition of a standard cell, each of which occupies about 1/16 of a full circle. I show in Fig. 10, the nature of the focussing functions over a part
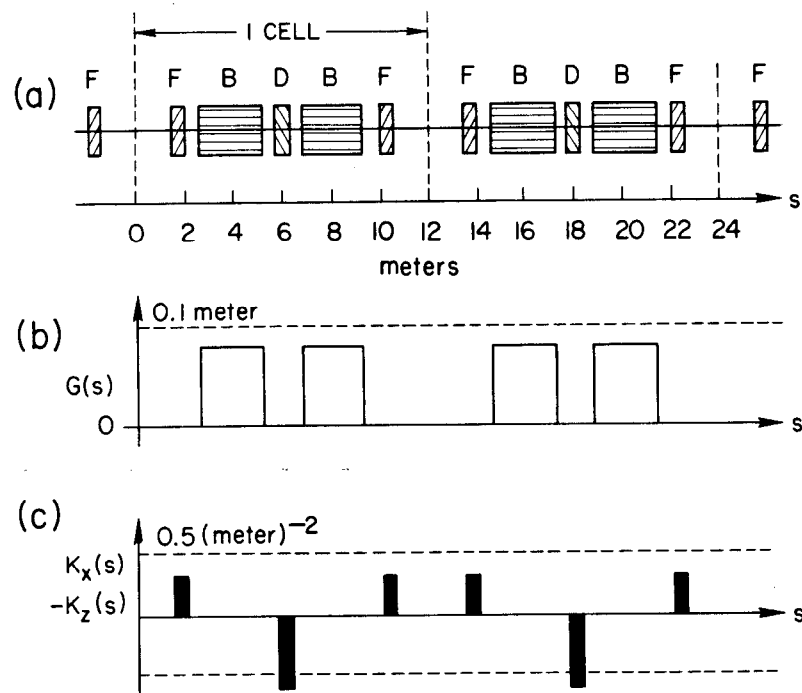
FIG. 10--Magnet lattice and focussing functions in the normal cells of a particular guide field.

of the ring, comprising two such cells. Part (a) of the figure shows the layout of bending magnets and quadrupoles. The bending magnets designated B, have a uniform field (dB/dx = 0); the quadrupoles have no field on the design orbit ($B_0 = 0$) and are designated F or D (for focussing or defocussing in the radial motion) depending on whether their gradients are positive or negative. The other parts of the figure give the focussing functions G, K_x, and K_z.

2.4. Separation of the Radial Motion

It is conceptually convenient to separate the radial motion into two parts, one part being a displaced, closed curve, which is the _equilibrium orbit_ for electrons

of the displaced energy, and the other part being the free transverse oscillation about this orbit. Suppose we write for x

$$x = x_\epsilon + x_\beta \tag{2.25}$$

then certainly Eq. (2.19) is satisfied if <u>both</u> of the following equations are true:

$$x_\epsilon'' = K_x(s) \, x_\epsilon + G(s) \, \frac{\epsilon}{E_0} \tag{2.26}$$

$$x_\epsilon'' = K_x(s) \, x_\beta \tag{2.27}$$

We may make the decomposition unique by requiring that $x_\epsilon(s)$ be a <u>single-valued function</u> at each physical azimuth; that is, that x (s) be a function which is periodic in s with period L. It is then clear that $x_\epsilon(s)$ is a possible (and in fact, unique) closed orbit for an electron of energy $E_0 + \epsilon$ (with $x_\beta = 0$), and that the general radial motion will consist of the sum of the displacement of this new equilibrium orbit and a <u>free</u> <u>betatron</u> <u>oscillation</u> x_β which satisfies Eq. (2.27).

The displacement x_ϵ is proportional to the energy deviation ϵ. Let's write

$$x_\epsilon(s) = \eta(s) \, \frac{\epsilon}{E_0} \, . \tag{2.28}$$

Now $\eta(s)$ is the single-valued function which satisfies

$$\eta'' = K_x(s)\eta + G(s). \tag{2.29}$$

And the total displacement from the ideal orbit can be written

$$x = \eta(s) \, \frac{\epsilon}{E_0} + x_\beta \tag{2.30}$$

I shall call $\eta(s)$ the <u>off-energy function</u>; it is a unique particular solution of Eq. (2.29) (because of the required periodicity) and is therefore, a function which characterizes the total guide field. It will be studied in more detail later on.

2.5. Betatron Trajectories

Equations (2.20) and (2.27) describe the free vertical and radial <u>betatron</u> <u>oscillations</u>. With the approximations made, the motions in the two coordinates are independent. Since the two equations have the same mathematical form — although the functions $K_z(s)$ and $K_x(s)$ will generally be different — let's take as the representative form

$$x'' = K(s) \, x \tag{2.31}$$

which is the same as Eq. (2.27) with the subscripts β and x suppressed. (With $K(s) = K_x(s)$, Eq. (2.31) will describe the radial betatron oscillation of an electron of the nominal energy E_0; and with z substituted for x and with $K(s) = K_z(s)$, it will describe the vertical motion.)

The focussing function K(s) is a prescribed function — the storage ring design specifies its value at each azimuthal position. If the position and slope (x and x') of an electron are given for some azimuth, the subsequent motion is uniquely determined. It can in fact be determined by a direct numerical integration of Eq. (2.31). Generally, however, the guide field is constructed of magnetic segments, in each of which K(s) may be taken as a constant so that the integration can be made algebraically for each segment and the motion can be pieced together from such solutions. Depending on whether the value of K is positive, zero, or negative in a particular segment of s, the motion in x will have one of the forms

$$K > 0: \quad x = a \cos\left(\sqrt{K}\, s + b\right)$$

$$K = 0: \quad x = a s + b$$

$$K < 0: \quad x = a \cosh\left(\sqrt{-K}\, s + b\right) \tag{2.32}$$

where $\underline{a}$ and $\underline{b}$ are constants in each segment — and may be determined from the values of x and x' at the entrance to the segment. (Since K is everywhere finite, x and x' must both be everywhere continuous — and, in particular, at the boundary between the two segments.)

As an illustration suppose we consider the motion for a K(s) like that shown for $K_x(s)$ in Fig. 10. Two possible trajectories are shown in (b) of Fig. 11. The first one is a trajectory which starts at s_0 with a unit displacement ($x_0 = 1$) but no slope ($x_0' = 0$); and the second starts at s_0 with zero displacement but with a unit slope ($x_0' = 1$). Each of them is made up of pieces described by one of the functions in (2.32). There are, of course, an infinite number of possible trajectories, depending on the initial conditions at s_0; but the two shown are of particular interest. The first one is called (for $\underline{any}$ chosen s_0) the "cosine-like" trajectory associated with s_0 and is designated $C(s, s_0)$; the other one is the "sine-like" trajectory $S(s, s_0)$.

(a)

(b)
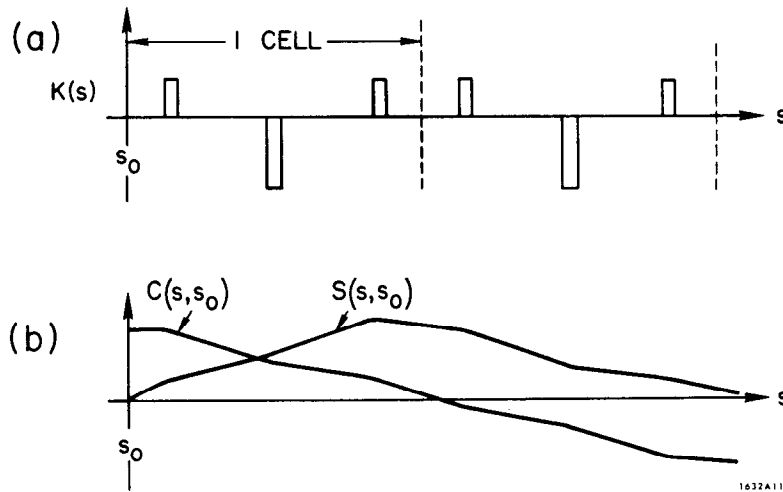
FIG. 11--Focussing function K(s) and two trajectories: the cosine-like
trajectory and the sine-like trajectory for the starting azimuth s_0.

The detailed form of C and of S will depend on the reference azimuth s_0. They
are in general, not periodic functions, even though K(s) is. For a ring with stable
trajectories, C and S are bounded oscillatory functions which have a different shape
on each successive revolution of the ring; although they are "quasi-periodic" in
the sense that after some number of revolutions they will lie very close (or in
some hypothetical cases even exactly on) the trajectory of an earlier revolution.

Now since Eq. (2.31) is linear in x, any linear combination of C and S will
also describe a possible trajectory; and more particularly, all possible trajectories
can be described by such a linear combination. That is, for any trajectory

$$x(s) = C(s, s_0)x_0 + S(s, s_0)x'_0 \qquad (2.33)$$

and

$$x'(s) = C'(s, s_0)x_0 + S'(s, s_0)x'_0 \qquad (2.34)$$

where C' and S' are the derivations of C and S with respect to s; and x_0 and x'_0
are the value of x and x' at s_0.

It is often convenient to write the last two equations in a matrix notation.
Let's let $\underset{\sim}{x}(s)$ stand for the "vector" whose components are x(s) and x'(s);

$$\underset{\sim}{x}(s) = \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} \qquad (2.35)$$

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Then we may write that

$$\underset{\sim}{x}(s) = \underset{\sim}{M}(s, s_0) \underset{\sim}{x}(s_0) \tag{2.36}$$

in which $\underset{\sim}{M}$ is the transfer matrix to s from s_0, which depends only on the properties of the guide field between the two azimuths. Its elements can be written in terms of the cosine-like and sine-like functions:

$$\underset{\sim}{M}(s, s_0) = \begin{bmatrix} C(s, s_0) & S(s, s_0) \\ C'(s, s_0) & S'(s, s_0) \end{bmatrix} \tag{2.37}$$

The transfer matrix for any span of s can often be conveniently found in terms of the matrices for segments of the span, since for any s_1 between s_0 and s,

$$\underset{\sim}{M}(s, s_0) = \underset{\sim}{M}(s, s_1) \underset{\sim}{M}(s_1, s_0) \tag{2.38}$$

The matrix for a segment which extends from s_1 to $s_2 = s_1 + \ell$ with a constant K is given in Table I for the three cases; $K < 0$, $K = 0$, and $K > 0$. They may be derived from the equations in (2.32).

The transfer matrix method is useful when designing a ring, or in looking at special problems such as the initial trajectories at injection. It does not, however, provide the most convenient description of the general nature of the trajectories of stored electrons. For many purposes another method of describing the trajectories is more useful. It may be called the "pseudo-harmonic" description.

TABLE I

Transfer Matrices for Segments of Constant K

$$K < 0: \quad \underset{\sim}{M}(s_2, s_1) = \begin{bmatrix} \cos \sqrt{K}\,\ell & \dfrac{1}{\sqrt{K}} \sin \sqrt{K}\,\ell \\ -\sqrt{K} \sin K\ell & \cos \sqrt{K}\,\ell \end{bmatrix}$$

$$K = 0: \quad \underset{\sim}{M}(s_2, s_1) = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}$$

$$K > 0: \quad \underset{\sim}{M}(s_2, s_1) = \begin{bmatrix} \cosh \sqrt{-K}\,\ell & \dfrac{1}{\sqrt{-K}} \sinh \sqrt{-K}\,\ell \\ \sqrt{-K} \sinh \sqrt{-K}\,\ell & \cosh \sqrt{-K}\,\ell \end{bmatrix}$$

$$\ell = s_2 - s_1$$

The general solutions of Eq. (2.31) can be written as

$$x(s) = a \, \zeta(s) \cos \left\{ \phi(s) - \vartheta \right\} \tag{2.39}$$

where $\zeta(s)$ and $\phi(s)$ are specially defined functions of s with certain convenient properties and a and ϑ are constants ("initial conditions") which determine a particular trajectory.[†] Specifically, if we define[††]

$$\phi(s) = \int_0^s \frac{d\bar{s}}{\zeta^2(\bar{s})} \tag{2.40}$$

so that

$$\phi'(s) = \frac{1}{\zeta^2} \, , \tag{2.41}$$

and if we define $\zeta(s)$ to be that positive valued, analytic function which satisfies

$$\zeta'' = K(s)\zeta + \frac{1}{\zeta^3} \, , \tag{2.42}$$

then, as you can show by direct substitution, the x(s) of Eq. (2.39) satisfies the differential equation (2.31).

Following tradition, I shall choose generally to deal rather than with $\zeta(s)$, with its square — which is universally written as $\beta(s)$. With this translation Eq. (2.39) gets replaced by

$$x(s) = a \sqrt{\beta(s)} \cos \left\{ \phi(s) - \vartheta \right\} \tag{2.43}$$

with

$$\phi(s) = \int_0^s \frac{d\bar{s}}{\beta(\bar{s})} \tag{2.44}$$

and

$$\beta(s) = \zeta^2(s) \tag{2.45}$$

so that $\sqrt{\beta(s)}$ is the function defined by Eq. (2.42).

Given the focussing function K(s) for a storage ring, the function $\beta(s)$ is uniquely determined; it can therefore, serve as an alternate "representation" of

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[†]Note that ϑ does <u>not</u> refer to the value of $\phi(s)$ at s = 0 since $\phi(0) = 0$.

[††]To avoid confusion when writing indefinite integrals over s, I shall write the integration variable or $\bar{s}$.

the focussing characteristics of the ring. Notice, however, that while K(s) is given in terms of the local properties (at each s) of the guide field, the function β(s) — or ζ(s) — depends on the total configuration of the ring. On the other hand once β(s) is known, K(s) can be immediately obtained from its local derivatives by Eqs. (2.42) and (2.45). But it is β(s) which reveals more directly the significant characteristics of the trajectories of stored electrons.

It is possible to have guide fields that do not result in stable (that is, bounded) trajectories. For such fields β(s) is not defined. But such fields can hardly be said to form a "storage" ring; so we are not interested in them here. Although they may be of interest to the storage ring designer — as something to be avoided!

I will return later to a discussion of how β(s) is related to the guide field properties; but it will be more useful to look first at the qualities of the trajectories described by the pseudo-harmonic solutions described by Eq. (2.43).

Don't forget that all of the discussion of this section applies equally to vertical as well as to radial motion. A ring is therefore, described by the two functions β_x and β_z (or ζ_x and ζ_z), which are derived from the two focussing functions K_x and K_z. It follows that the phase function ϕ(s) is also different for motion in x and in z.

2.6. Pseudo-Harmonic Betatron Oscillations

We have seen that the betatron oscillations — in either x or z — are described by a pseudo-harmonic oscillation whose representative form is

$$x(s) = a \sqrt{\beta} \cos(\phi - \vartheta) \qquad (2.46)$$

$$\phi = \int_0^s \frac{d\bar{s}}{\beta} \qquad (2.47)$$

where β is a given function of s, which we may call the betatron function, and a and ϑ are constants of the particular trajectory.[†] The two equations above describe completely the path taken by the electron. To get a complete picture of the motion of the electron in the coordinate x we must only add the fact that the

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[†]You may be wondering why I have not adopted the imaginary exponential notation for representing the betatron oscillation. Such a representation is, of course, possible; but in the present instance the inconvenience seems generally to outweigh the convenience gained.

electron travels always at the speed c of light. For many purposes it is adequate
to take that the azimuth of the electron varies simply as

$$s = s_0 + ct \quad \text{(approx)} \tag{2.48}$$

You should remember, however, that this is only an approximation which neglects
terms that are the order of x/ρ_s, where ρ_s is the radius of curvature of the design
orbit. The correction terms to Eq. (2.48) will be looked at in Section 3.2.

The betatron function describes completely the lateral focussing properties
of the guide field. By its nature the betatron function must be always positive-
definite; it has a "wave-like" character, and in a well-designed ring it will wander
not too far (say a fraction of an order-of-magnitude) from its mean value. It
might typically look like the curve (a) in Fig. 12. The definition of $\beta(s)$ constrains
it to be periodic in s with the period L;

$$\beta(s + L) = \beta(s). \tag{2.49}$$

It has a unique value at each physical azimuth. If the guide field has a higher
rotational symmetry — being composed of two or more identical periods — β will
have the same symmetry. A guide field which produces the β of Fig. 10(a) would
have sixteen identical cells in its circumference. Note, however, that a local
periodicity in the focussing functions G(s) and K(s) in only a part of the guide field
will not, in general, give rise to a corresponding local periodicity in β. It will
do so only when the local periodicity is repeated all the way around the ring to
produce a true rotational symmetry.

As the electron travels around the ring it executes a lateral oscillation which
is not harmonic — nor periodic. The motion is a kind of distorted sine-wave with
a varying amplitude $(a\sqrt{\beta})$ which is modulated in proportion to the root of the beta-
tron function, and with a "phase" $(\phi - \vartheta)$ which advances with s at a varying rate
proportional to $1/\beta$. The nature of the motion is illustrated in parts (b) and (c) of
Fig. 12. The two segments of trajectory shown correspond to the same $\underline{a}$, but to
different starting phases.

Suppose that we chose some initial $\underline{a}$ and ϑ, and follow the trajectory for many
successive revolutions. We would get a path such as the one shown in part (d) of
Fig. 12, where, for convenience, I have superimposed all of the successive revolu-
tions on the same segment of s. (Or, if you wish, since s is a cyclic variable, I
am plotting s(modulo L) instead of s.) The picture gives some idea of what we
would see if we watched a single stored electron circulating in a ring.
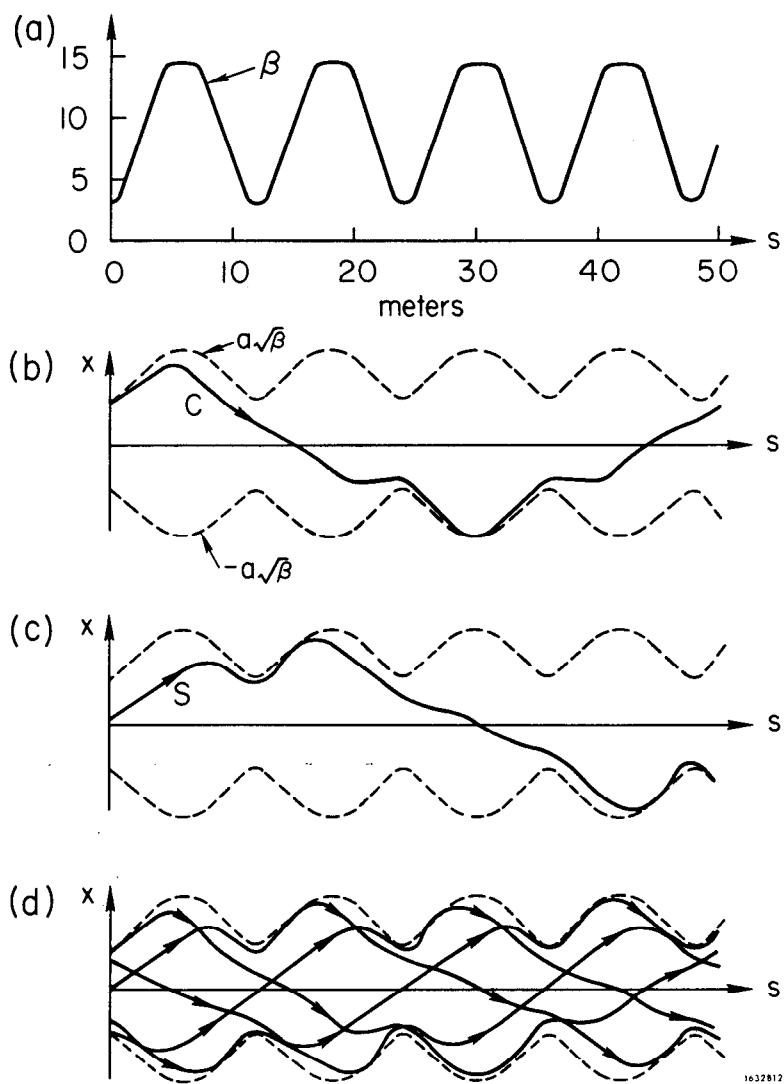
FIG. 12--(a) Betatron function. (b) Cosine-like trajectory for s=0.
(c) Sine-like trajectory for s=0. (d) One trajectory on
several successive revolutions.

One important feature of the betatron motion is evident in Fig. 12(d) — at each physical azimuth the displacement x of a circulating electron lies always below a limiting extreme value X(s) obtained by setting $\cos (\phi - \vartheta) = 1$; namely

$$X(s) = a \sqrt{\beta(s)} \qquad (2.50)$$

The complete trajectory of a stored electron will fall forever within an <u>envelope</u> defined by $\pm X(s)$. And it follows that the aperture required to contain an electron with a given oscillation amplitude varies around the ring as X(s). The ratio of the envelope width at two locations s_1 and s_2 is, of course, just

$$\frac{X_2}{X_1} = \left(\frac{\beta_2}{\beta_1} \right)^{1/2} \qquad (2.51)$$

At each physical azimuth a stored electron may generally be expected to pass frequently with a displacement near the maximum.

Let's look now at the <u>slope</u> of the betatron trajectory, $x' = dx/ds$. Taking the derivative of Eq. (2.46), we may write

$$x' = - \frac{a}{\sqrt{\beta}} \sin (\phi - \vartheta) + \frac{\beta'}{2\beta} x \qquad (2.52)$$

The first term comes from the changing phase; and the second from the variation of β.

Notice that the zeros of x' — and therefore, the peak values of x — do <u>not</u> occur when $\cos(\phi - \vartheta)$ is 1. Rather they are reached when

$$\tan (\phi - \vartheta) = \beta'/2 \qquad (2.53)$$

which means at

$$\cos(\phi - \vartheta) = \left\{ 1 + \frac{\beta'^2}{4} \right\}^{-1/2} \qquad (2.54)$$

If the peak of a particular cycle of an oscillation occurs at some s, the peak displacement then will be

$$x_{peak} = a \sqrt{\beta} \left[1 + \frac{\beta'^2}{4} \right]^{-1/2} \qquad (2.55)$$

See. Fig. 13.

In a classical harmonic oscillation the amplitude is an invariant of the motion. Its square is proportional to the energy of the oscillator, and can be expressed as a quadratic function of the instantaneous position and velocity. The corresponding invariant of the pseudo-harmonic oscillator is the constant <u>a</u>.
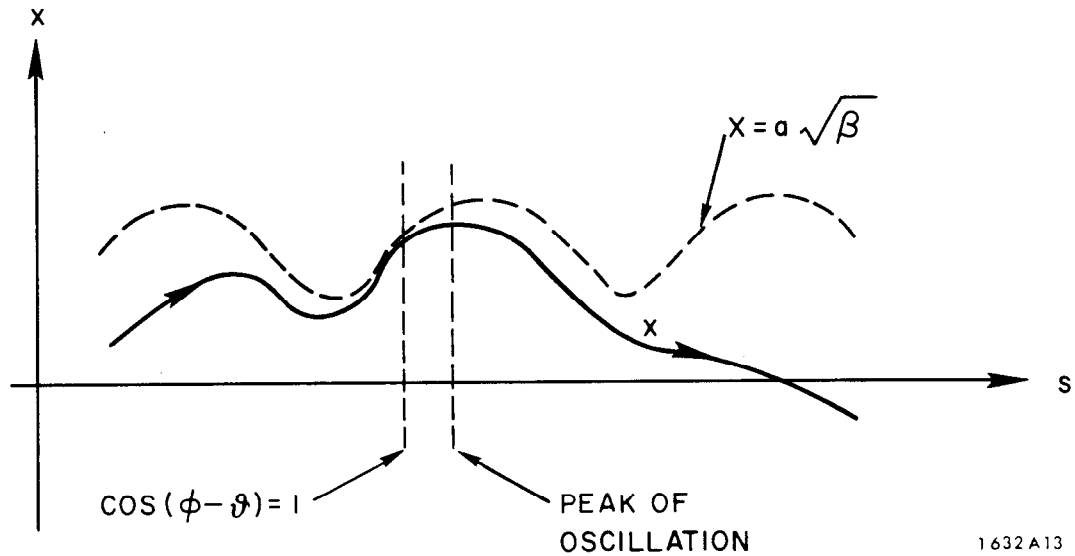
FIG. 13--The maximum of a particular cycle of a betatron oscillation.

If we isolate the cosine and sine terms in Eqs. (2.46) and (2.52), square them, and add, we can relate a^2 to x and x'. We find that

$$a^2 = \frac{x^2}{\beta} + \beta \left(x' - \frac{\beta'}{2\beta} x \right)^2 . \tag{2.56}$$

If we know x and x' at any azimuth, say s_1, $\underline{a}$ can be found and all subsequent displacements can be expressed by

$$x = \frac{1}{\sqrt{\beta_1}} \left\{ x_1^2 + \left(\beta_1 x_1' - \frac{x_1 \beta_1'}{2} \right)^2 \right\}^{1/2} \sqrt{\beta} \cos(\phi - \theta) \tag{2.57}$$

The phase constant ϑ must also be determined from x and x'. It can be obtained from

$$\tan(\phi_1 - \vartheta) = - \frac{\beta_1 x_1'}{x_1} + \frac{\beta_1'}{2} , \tag{2.58}$$

where $\phi_1 = \phi(s_1)$.

We are often interested only in the maximum value X(s) which can be reached at any physical azimuth on any subsequent revolution. This maximum is independent of ϑ and is obtained by replacing the cosine factor of Eq. (2.57) by 1:

$$X(s) = \frac{1}{\sqrt{\beta_1}} \left\{ x_1^2 + \left(\beta_1 x_1' - \frac{x_1 \beta_1'}{2} \right)^2 \right\}^{1/2} \sqrt{\beta(s)} \tag{2.59}$$

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If β were everywhere comparable — say not too far from some typical value β_n — then the ensuing peak amplitude which would result from a sudden lateral displacement δx would be about equal to δx, and the amplitude which would result from a sudden lateral impulse that changed the slope by $\delta x'$ would be about proportional to $\beta_n \delta x'$. Generally, we may expect that the amplitudes which result from disturbances to the trajectory will be less the smaller is β. Indeed, we may consider that $1/\beta$ is a measure of the "strength" of the lateral focussing, and that small values of β are generally desirable. More will be said on this subject later.

2.7. The Betatron Number ν

As an electron makes one complete revolution of a storage ring starting at some azimuth, say s_0, its oscillation phase $(\phi - \vartheta)$ advances by

$$\int_{s_0}^{s_0 + L} \frac{ds}{\beta} \; .$$

Because of the periodicity of β, this integral is the same for all s_0: in any complete revolution the phase increases by the same amount. This phase advance is an important parameter of a storage ring and is usually written as $2\pi\nu$ (although in Europe often $2\pi Q$); and ν is called the betatron number. We have the definition

$$\nu = \frac{1}{2\pi} \int_s^{s+L} \frac{ds}{\beta} = \frac{1}{2\pi} \int_0^L \frac{ds}{\beta} = \oint \frac{ds}{\beta} \; . \tag{2.60}$$

(I shall use from now on the complete integral symbol $\oint$ to indicate any integral around the whole ring.) The betatron numbers for the two oscillation coordinates x and z — written as ν_x and ν_z — are generally different, being derived from the two betatron functions β_x and β_z. Both ν_x and ν_z are typically not-too-large numbers near, but not at a quarter integer — such as 2.78 or 5.15. Other ways of calculating ν will be treated in Section 2.10.

Although the betatron trajectory is a contorted aperiodic oscillation, if we sit at some particular azimuth and observe the successive passages of a stored electron we find that the displacement follow a simple sinusoidal law. Suppose we pick our observation point at s_0, and let the successive passages past this azimuth be identified by the index j = 0, 1, 2, 3, ... Also let ϕ_0 be the phase at the zero-th passage. On each successive passage the phase will increase by $2\pi\nu$; at the j-th passage the phase will be

$$2\pi\nu j + \phi_0 \; .$$

and the displacement will be

$$x_j = a \sqrt{\beta_0} \cos(2\pi\nu j + \phi_0) \qquad (2.61)$$

The "amplitude" factor $(a\sqrt{\beta_0})$ is a constant $(\beta_0 = \beta(s_0))$, so the displacement, as sampled each revolution, varies as the sampling of a simple sinusoidal oscillation.

Since the time for each revolution is constant,[†] namely L/c, we may also write that the time t_j of the j-th passage is

$$t_j = \frac{L}{c} j;$$

or that

$$2\pi j = \omega_r t_j , \qquad (2.62)$$

where

$$\omega_r = 2\pi \frac{c}{L} , \qquad (2.63)$$

is the (angular) frequency of revolution of the electron. Then (2.61) becomes, for any fixed s,

$$x_s(t_j) = a \sqrt{\beta(s)} \cos(\nu\omega_r t_j + \phi_{0s}) \qquad (2.64)$$

When observed at a particular azimuth the lateral motion is indistinguishable from a sampled simple harmonic oscillation at the frequency $\nu\omega_r$ — generally called the betatron frequency.

Looking at Eq. (2.61) we can see the justification for the statement of the preceding section that at each azimuth we may sooner-or-later expect to see x take on its maximum value $X(s) = a\sqrt{\beta(s)}$. Unless ν is an integer, or better, unless the difference between ν and an integer is a simple fraction — which is not likely to be exactly true for a real storage ring — the phase (modulo 2π) at successive passages of any fixed point will "walk" through a large number of values between 0 and 2π before repeating itself. And the displacement will sooner-or-later take on its peak value X at, or near, each azimuth.

Perhaps the most important significance of the betatron number ν of a storage ring is related to the existance of disturbing resonances which appear if ν takes

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[†]Neglecting a small correction proportional to x.

on certain values. For example, if ν were an integer, the betatron oscillation would ideally become quite periodic — repeating itself each revolution. However, the smallest imperfection in the guide field (and there will surely be at least one!) will act as a perturbation which is synchronous with the oscillation frequency. A synchronous perturbation leads to a resonance excitation of the oscillations and an exponential growth of the amplitude. There will be no stable oscillation. (Said in another way, the betatron function of the actual machine may not be defined.) As we shall see later, other resonances will occur also at half-integral values of ν; or if nonlinear effects are taken into account when the difference between ν and an integer is any simple fraction.

Resonances must, of course, be avoided in both the radial and vertical betatron oscillations. It turns out that resonances of some kind may occur when ν_x and ν_z satisfy

$$m\nu_x + n\nu_z = r, \tag{2.65}$$

where m, n, and r are integers. Significant effects are generally observed only for low-order resonances, that is, those for which m, n, r take on the small values 0, 1, 2, 3. The operating point of a storage ring is specified by giving both ν_x and ν_z and must be chosen to avoid the serious resonances. The resonance relation (2.65) defines a set of lines in a (ν_x, ν_z) diagram. Some of them are shown in Fig. 14, where a possible operating point is also indicated. For one particular set
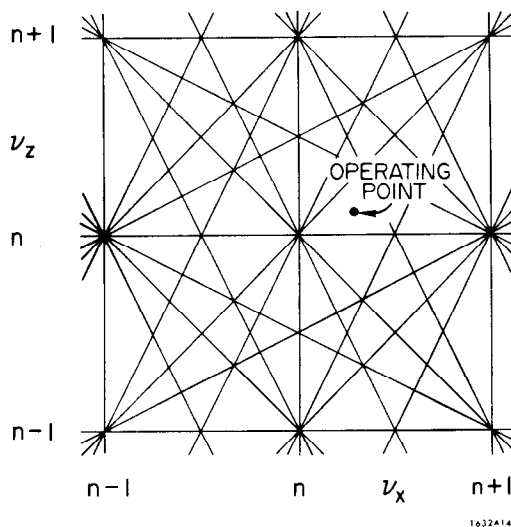
FIG. 14--Lower order resonance lives on a ν_x, ν_z diagram.

of resonances, namely when ν_x is equal to ν_z or when their difference in integer, there will be strong coupling between the horizontal and vertical oscillations. At such a resonance our assumption of completely independent oscillations is no longer valid and the motion will be more complicated. Sometimes a storage ring may intentionally be operated on or near such a coupling resonance in order to increase the amplitude of vertical oscillations by feeding them energy from the radial oscillation.

To stay clear of dangerous resonances it is necessary that the actual operating point remain fairly close to the chosen one — as is clear from Fig. 14. We may expect that magnet imperfections will generally cause shifts of ν which are proprotional to ν itself. A storage ring with a large betatron number is likely to be a "touchy" machine. This is one of the reasons that designers tend to choose ν values between about 2 and 6.

2.8. An Approximate Description of Betatron Oscillations

For many purposes it is convenient — and sufficient — to approximate the betatron motion by a simple harmonic oscillation. Consider the oscillation

$$x = A \cos(s/\lambda - \vartheta) \qquad (2.66)$$

where λ is a constant (the "reduced wavelength"). One complete oscillation is completed while s advances by one wavelength $2\pi\lambda$. It is clearly convenient to think of the pseudo-harmonic oscillation of Eq. (2.46) as merely a "sine"-wave with a locally varying reduced wavelength — if we may ignore the amplitude variation. And, so long as β doesn't vary too wildly, we might expect to have a reasonable approximation to the actual motion if we use the form of Eq. (2.66) with a suitable choice for λ. Suppose we define the number β_n to be that constant which would give the same phase change (2.47) in one revolution as does the actual β. That is, we define β_n by

$$\oint \frac{ds}{\beta} = \frac{L}{\beta_n} \qquad (2.67)$$

and call it the typical value of β. Then the oscillation

$$x = A \cos(s/\beta_n + \theta) \qquad (2.68)$$

will — with $A = a \sqrt{\beta_n}$ — agree with the actual trajectory at least once each revolution; and, in particular, will on the average stay in phase with the true oscillation.

I show in Fig. 15 one of the trajectories of Fig. 12 together with the approximation obtained from Eq. (2.68); for many purposes the approximation is quite adequate.[†]
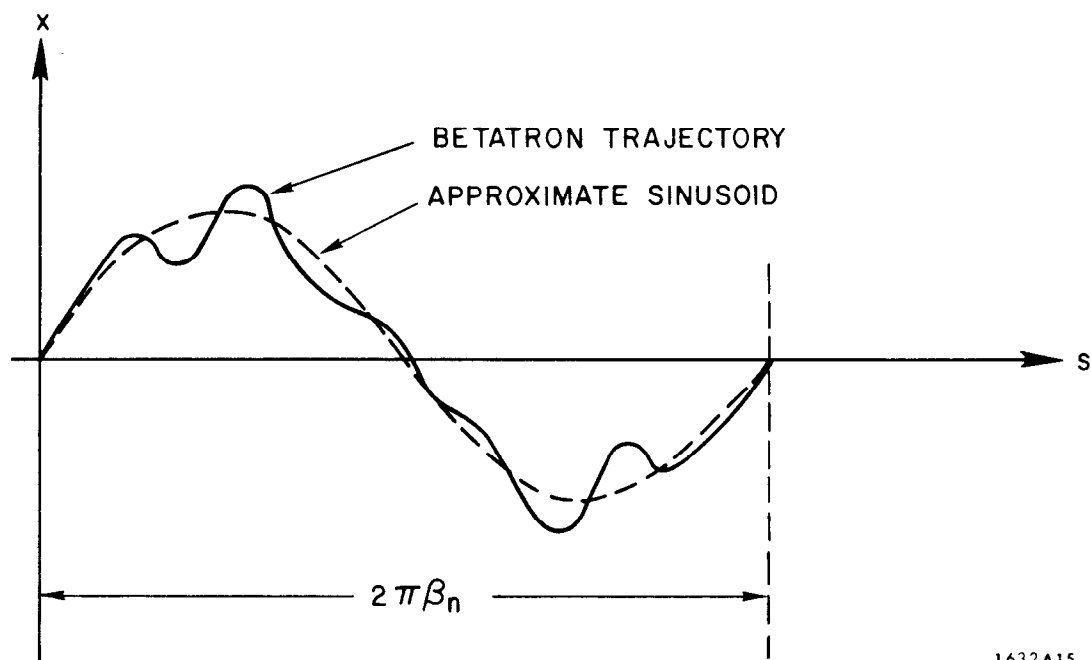
FIG. 15--Approximation to the betatron trajectory.

It is convenient to remember that, by (2.67), $1/\beta_n$ is just the average around the ring of $1/\beta$:

$$\frac{1}{\beta_n} = \left\langle \frac{1}{\beta} \right\rangle \tag{2.69}$$

Recalling the definition of ν from Eq. (2.60) we may also write

$$\frac{L}{\beta_n} = 2\pi\nu \tag{2.70}$$

People commonly define the <u>gross radius</u> R of a ring by

$$2\pi R = L \tag{2.71}$$

[†]When using the approximation of Eq. (2.68), it becomes generally convenient to revert to the usual exponential representation of the cosine function.

then β_n is most simply defined by

$$\beta_n = \frac{R}{\nu} \qquad (2.72)$$

(Note that β_n is <u>not</u> equal to the average of β although it may not be much different if the undulations of β are not too large.)

The time-variation of the approximate trajectory of Eq. (2.68) is simply written as

$$x = A \cos (\nu \omega_r t - \theta) \qquad (2.73)$$

as follows by using (2.78) together with the definition of ω_r as c/R. The angular frequency $\nu \omega_r$ is usually referred to as the "betatron frequency" and will be written ω_β. Notice that when the approximate trajectory is observed only as it passes some one fixed azimuth, its time variation is indistinguishable from the actual one — compare Eq. (2.64).

The approximate description of this Section is not only entirely adequate for many calculations of storage ring effects, but indeed provides the only tractable approach for an analysis of some of the coherent effects which involve large numbers of stored electrons.

2.9.   Nature of the Betatron Function

The storage-ring designer is very much occupied with finding a magnet design which will provide a suitable betatron function $\beta(s)$. And a user may generally expect that along with the design of the ring will come a plot of that function. I do not wish here to go into the intricate matter of the techniques for arriving at a design which will yield a "good" $\beta(s)$, but rather I would like only to give some idea about how $\beta(s)$ may be evaluated for a given set of magnet parameters.

What is a desirable form for $\beta(s)$? We have already seen at the end of Section 2.7 that at least in certain respects, small values of β (strong focussing) are desirable — provided that β is reasonably uniform. Unfortunately, small β's can only be obtained with alternating gradient focussing which tends to give reasonably large undulations to β. Also, smaller β's imply larger values of ν, which may, as remarked in Section 2.8, lead to greater difficulties from resonances. One normally tries to arrive at a β whose typical value in most of the ring is some fraction (1/2 to 1/6) of the mean radius R, and which does not have too extreme an undulation.

In Section 2.6 the betatron function $\beta(s)$ was defined as that single-valued continuous function whose square root $\zeta(s)$ satisfies

$$\zeta'' = K(s)\zeta + \frac{1}{\zeta^3} \tag{2.74}$$

where $K(s)$ is the magnet focussing function. The typical modern storage ring is made up of a chain of segments in each of which the function $K(s)$ has a constant value, positive, negative, or zero. An example was given in Fig. 11(a), with the corresponding $\beta = \zeta^2$ shown in Fig. 12(a).

The requirement that $\zeta(s)$ be periodic, together with the nonlinear term $1/\zeta^3$, gives a unique specification — including the scale. The function $\zeta(s)$ is the "eignefunction" of Eq. (2.74) and because of the nonlinearity, there is no arbitrary normalization of the amplitude.

From a dimensional argument we would expect that ζ should scale as $|K|^{-1/4}$, or that β should scale $|K|^{-1/2}$. (Recalling that $1/\beta$ is like the "frequency" of an oscillator we might expect it to go as the root of the "restoring force constant".) For a given geometry of the field, such a scaling law is roughly true. It is in fact, strictly true if the scale length of the focussing geometry is scaled as $|K|^{-1/2}$, as would generally be true for a well-designed guide field.

In a region of s where $K(s)$ is a constant Eq. (2.74) has just the form of the one-dimensional equation of motion of a particle being acted on by a linear "restoring" force $-K\zeta$ and a "repulsive core" force $1/\zeta^3$. Or if you prefer, of a particle which moves with a potential energy proportional to

$$-K\zeta^2 + \frac{1}{\zeta^2} \ .$$

(The second term is much like a "centrifugal barrier"!) The shape of the effective potential is shown in Fig. 16 for $K > 0$, $K = 0$, and $K < 0$. In any region where $K \geq 0$ the acceleration in ζ (the displacement of the model particle) is always positive; and ζ is driven always toward larger values — or, of course, turned around if it has an initial velocity toward the origin of ζ. For positive K the driving force is, at large ζ, proportional to the size of K. On the other hand, in any region where $K < 0$ there will be a nice stable potential well, and when ζ is large, there is always a "force" driving it toward the origin.
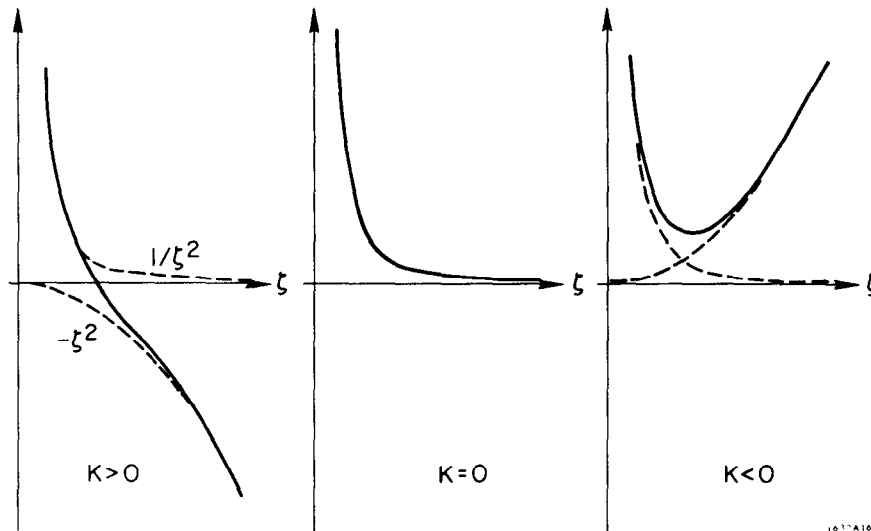
FIG. 16--Effective "potential" functions for ζ.

It is also qualitatively apparent that there may exist "stable" solutions for which $\zeta(s)$ enters a region of $K > 0$ moving inward (toward the origin) and is turned around by the "repulsion" only to be sent back inward again by the "attractive" force in a later region where $K < 0$. For a periodic $K(s)$ like that in Fig. 17(a), we must
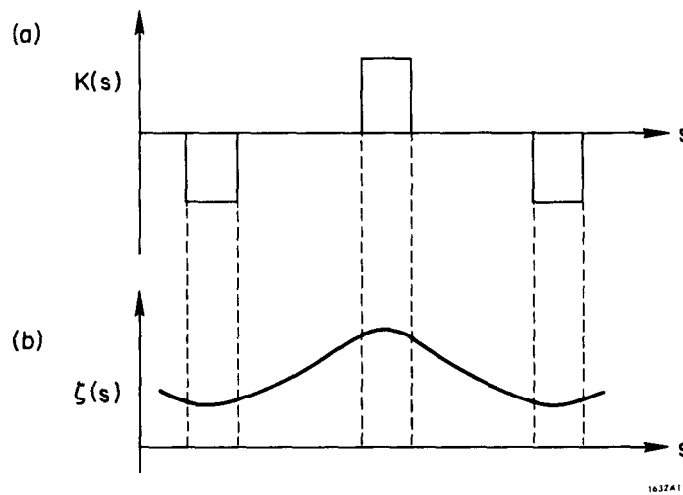
FIG. 17--Form of the function $\zeta(s)$ with a periodic focussing function $K(s)$.

expect a solution $\zeta(s)$ like the one sketched in part (b). The solution exhibits an important general characteristic of the function $\zeta(s)$: its maxima occur in focussing sections — one where $K < 0$ — and its minima occur in defocussing or neutral sections — where $K \geq 0$.

It is also clear that, for a given spacing of the segments of different K values, if the magnitude of K is scaled upward, the amplitude of the undulations will grow rapidly larger. Less apparent — and left as a point to ponder — is the fact that as the scale of K is increased a situation will be eventually reached for which a "stable" — i.e., periodic — solution no longer exists for $\zeta(s)$. So the strength of the focussing (magnitude of K) and the element spacing must be adjusted together in playing with a storage ring <u>lattice</u> (a word used to indicate the geometry of the segments).

You may be tempted to wonder: "Why not just have K < 0 at all s? Clearly the stability of $\zeta(s)$ is then guaranteed." But don't forget that when K is negative for one coordinate of lateral motion in the storage ring — say x — then the K for the other coordinate — z — is positive, and <u>vice versa</u>. Recall Eqs. (2.21) and (2.22). The need for an alternating gradient is clear.

It should also be now apparent that the undulations of $\zeta(s)$ — and so also of $\beta(s)$ — will be "out-of-phase" in the two coordinates x and z, the ζ for one being at its maximum where the ζ for the other is at its minimum.

The out-of-phase behavior is quite general — even for rather complicated focussing lattices — although it will not generally be true that the $\zeta_x(s)$ and $\zeta_z(s)$ are entirely similar in shape. In Fig. 18 I show the two functions ζ_x and ζ_z for the periodic lattice of Fig. 10.
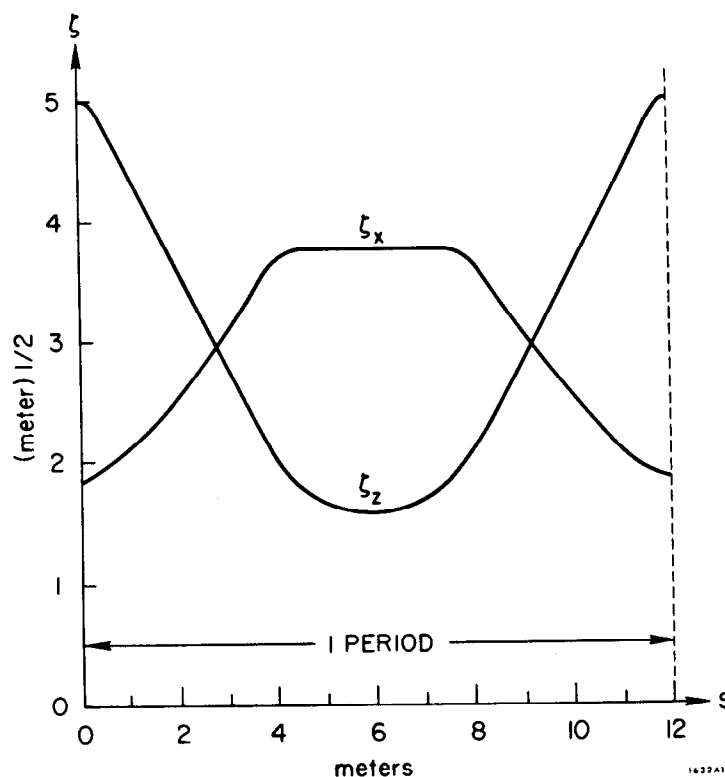
FIG. 18--The functions ζ_x and ζ_z for the guide field of Fig. 10.

It is instructive to relate the betatron function $\beta(s)$ to the sine-like trajectories which were defined in Section 2.6. The sine-like trajectory $S(s, s_0)$ associated with the azimuth s_0 is that trajectory which starts at s_0 with zero displacement and unit slope. It can be expressed in terms of the pseudo-harmonic oscillation Eq. (2.46) by setting $a = \sqrt{\beta(s_0)}$ and $\vartheta = \pi/2 - \phi(s_0)$;

$$S(s, s_0) = \sqrt{\beta(s_0)\beta(s)} \, \sin \int_{s_0}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \qquad (2.75)$$

(You can check that $S(s_0, s_0) = 0$, and $S'(s_0, s_0) = 1$.) Now consider what happens if we follow this sine-like trajectory for one complete revolution — that is to $s = s_0 + L$. The integral becomes, by Eq. (2.60) just $2\pi\nu$. Because of the periodicity of the betatron function, $\beta(s_0 + L) = \beta(s_0)$. So

$$S(s_0 + L, s_0) = \beta(s_0) \sin 2\pi\nu, \qquad (2.76)$$

which — since ν is independent of s_0 — I can also write as

$$\beta(s) = \frac{S(s+L, s)}{\sin 2\pi\nu} \qquad (2.77)$$

The betatron function at s is, within a constant, just the displacement after one revolution of the sine-like trajectory which starts at s. See Fig. 19.
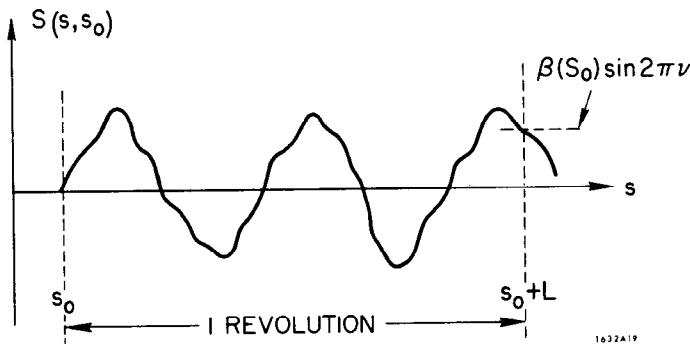
FIG. 19--Relation between $S(s, s_0)$ and $\beta(s_0)$.

We now have another prescription for finding $\beta(s)$. One needs only calculate directly the sine-like trajectory after one revolution, starting at each s. The displacements obtained are proportional to $\beta(s)$. There is left only to determine the "normalization" factor $1/2\pi\nu$. Using the definition of ν, Eq. (2.60) together

with Eq. (2.77), you can see that ν can be obtained as the solution of the transcendental equation

$$\frac{2\pi\nu}{\sin 2\pi\nu} = \oint \frac{ds}{S(s+L, s)} \qquad (2.78)$$

So, given $S(s+L, s)$ for all s, we can determine uniquely $\beta(s)$.

The calculation of $S(s+L, s)$ can be carried out by a straightforward numerical integration of the equations of motion. Or, for a piece-wise-uniform guide field, it is conveniently obtained by using the matrix method described in Section 2.5. Recalling Eq. (2.37), the sine-like trajectory from s_0 to $s_0 + L$ is just the upper right element of the transfer matrix $\underline{M}(s, s_0)$ for the complete machine starting at each azimuth s_0.

It can also be shown — but I shall not stop to do so — that ν can be obtained from the trace of the matrix for the complete ring. Namely

$$\cos 2\pi\nu = \frac{1}{2} \text{Tr } \underline{M}(s+L, s) = \frac{1}{2} \Big\{ C(s+L, s) + S'(s+L, s) \Big\} \qquad (2.79)$$

where C is the cosine-like function. So if C and S' are calculated as well as S, ν can be determined and Eq. (2.77) will give $\beta(s)$ directly.

Although I have thought it more convenient to write the differential equation for β in terms of its square root ζ, one can, of course, write one in terms of β directly. Equation (2.74) can be rewritten as

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1 \qquad (2.80)$$

The form is clearly less convenient. I may, however, use it to make the following observations.

In a field free segment of the guide field $K(s) = 0$, and the solution to Eq. (2.80) is

$$\beta = \beta_0 \left\{ 1 + \frac{(s - s_0)^2}{\beta_0^2} \right\} \qquad (2.81)$$

where s_0 and β_0 are suitable constants. If β has a minimum in the field free segment then β_0 and s_0 are the values of β and s at the minimum. Generally, the intersection of two colliding beams occurs at a symmetry point where β must be a minimum. Then Eq. (2.81) gives the form of $\beta(s)$ in the vicinity of the intersection. Its form is illustrated in Fig. 20. Notice that the coefficient of the quadratic term is just the inverse of the value of β at the minimum — the smaller is β_0, the more rapid the increase of β with increasing distance from the minimum.
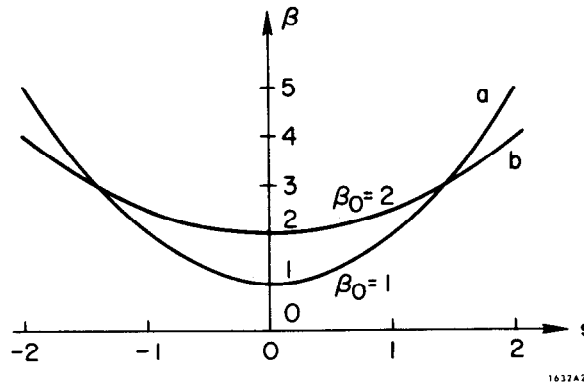
FIG. 20--Variation of β near the minimum that occurs in a long
field free region.

Finally, observe that in a segment in which K is large and β' is not, Eq. (2.80)
can be approximated by

$$\beta'' = -2K\beta \qquad (2.82)$$

Then β is a sinusoid or an exponential depending on the sign of K.

2.10.   Disturbed Closed Orbits

Until now I have considered the trajectories of electrons in a prescribed
guide field. I wish next to consider the following question: Suppose we have
analyzed the electron trajectories for this prescribed field; how will the trajectories
be different if there are small deviations of the fields from the assumed prescrip-
tion? In our linear approximation, the prescribed — or nominal — guide field was
specified by giving its value on the ideal orbit and its radial derivative. See
Section 2.3. Also, it was assumed that the field at the ideal orbit was everywhere
vertical. I wish now to inquire about the effects of small deviations from the
nominal field. If the vertical magnetic field at the ideal orbit differs from its
nominal value, or if there is some small horizontal field, the lateral accelerations
will be different from what is necessary to keep an electron on the design orbit.
The deviations of the field at the design orbit will be called field errors. Changes
in the fields which cause the focussing functions K_x and K_z to differ from their
nominal values will, for convenience, be called gradient errors.[†]

[†]Although field errors will, through the term G^2 in Eq. (29) also change K_x, such
effects are not usually important.

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When there are field errors, the design orbit is no longer a possible trajectory. If the errors are small, however, there will be another closed curve which is a possible orbit for an electron of the nominal energy. I shall call this trajectory the _disturbed_ _closed_ _orbit_. The general trajectory will execute betatron oscillations with respect to this disturbed closed orbit. And the form of the betatron oscillations will be determined by the modified focussing function. That is, if we continue to let x represent the displacement from the original design orbit, we may write

$$x = x_c + x_\beta,$$ (2.83)

where x_c is the displacement of the disturbed closed orbit from the ideal one, and x_β is the "free" betatron oscillation about the disturbed closed orbit.

If the closed orbit displacements are small, our assumed linearity of the field variations means that the betatron oscillations are the same with respect to the disturbed closed orbit as they would be with respect to the design orbit. We may therefore, consider separately the distortions of the closed orbit caused by field errors and the disturbances to the betatron oscillations caused by gradient errors. And we may interpret Eq. (2.83) as a superposition of the closed orbit distortion x_c and a free betatron oscillation x_β that is calculated with respect to the design orbit by the methods we have been using until now.

Let's look first at the effect of the field errors. Suppose we begin by considering the effect of a field error which exists only in a small azimuthal interval Δs, which we may as well place at $s = 0$. In passing through Δs the displacement x is unchanged, but the slope x' change by the amount

$$\Delta x' = - \frac{ec\,\delta B}{E_0}\,\Delta s\ ,$$

where δB is the deviation of the magnetic field from its nominal value. For the vertical motion we would have the same form if δB were identified as the total radial field at the design orbit (with a suitably chosen sign). In keeping with the definition of Eq. (2.3) we set $ec\,\delta B/E_0 = \delta G$ with a suitable subscript x or z implied when we are considering the radial or vertical motion. We may, as before, consider only a generic x-motion with the understanding that all the results apply equally to x-motion or to z-motion when all identifying subscripts are restored. We write, then the effect of the field error in Δs as

$$\Delta x' = \delta G\,\Delta s$$ (2.84)

The field error at s adds to $x'' = \Delta x'/\Delta s$ a term δG; and is, therefore, equivalent to adding a driving force $\delta G(s)$ to the equation of motion. We get the complete equation of motion for x_c by adding this new force term to the usual equation for x, Eq. (2.31):

$$x'' = K(x)\, x + \delta G(s) \qquad (2.85)$$

The displacement x_c of the disturbed closed orbit is the solution of this equation that is single valued at each physical azimuth.

We may make an estimate of the effect of a localized field error at $s = 0$ by using the approximate harmonic form of the betatron motion described in Section 2.9. Think for the moment of an electron that is traveling along the design orbit — so that its slope x' is zero. When it arrives at $s = 0$ its slope is suddenly changed to $\Delta x'$. See Fig. 21. After $s = 0$ there is no field error (for one full revolution) so the electron begins to oscillate about the ideal orbit with the amplitude

$$b = \overline{\lambda}\Delta x' = \beta_n \Delta x' = \beta_n\, \delta G\, \Delta s. \qquad (2.86)$$

We may expect the closed orbit displacement x_c to be of the same order of magnitude.
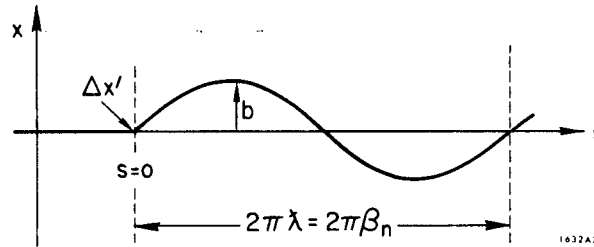
FIG. 21--Effect of a localized field error.

To make a proper calculation of x_c, we should use the correct pseudo-harmonic free oscillation, and remember also that the displaced closed orbit is defined as that particular trajectory which closes on itself after one revolution. In other words x_c must be single valued at each physical azimuth s, namely $s_c(s+L) = x_c(s)$. In particular,

$$x_c(L) = x_c(0); \qquad (2.87)$$

and by Eq. (2.84)

$$x_c'(L) + \delta G\, \Delta s = x_c'(0) \qquad (2.88)$$

But between s = 0 and s = L there are no field errors, so x_c is just a free oscil-lation about the ideal orbit. See Fig. 22. That is, x_c must be given by Eq. (2.46):

$$x_c(s) = a \sqrt{\beta(s)} \cos(\phi - \vartheta), \quad s \neq 0, \tag{2.89}$$

with the arbitrary constants $\underline{a}$ and ϑ to be chosen so that Eqs. (2.87) and (2.88) are satisfied.
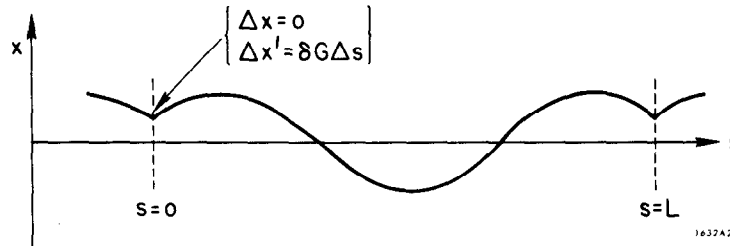
FIG. 22--The disturbed closed orbit for a field error at s = 0.

Using Eq. (2.52) for $x_c'(s)$ — everywhere but at $s = 0$ — you may verify that the appropriate values of $\underline{a}$ and ϑ are[†]

$$a = \frac{\delta G \, \Delta s \, \sqrt{\beta(0)}}{2 \sin \pi \nu} \tag{2.90}$$

$$\vartheta = \pi \nu \tag{2.91}$$

The displacement of the disturbed closed orbit is then

$$x_c(s) = \frac{\delta G \, \Delta s \, \sqrt{\beta(0)}}{2 \sin \pi \nu} \sqrt{\beta(s)} \cos \left\{ \phi(s) - \pi \nu \right\} \tag{2.92}$$

The form of the amplitude invariant $\underline{a}$ displays the two most interesting features of the disturbed closed orbit. Notice first, that the displacement of the closed orbit is everywhere proportional to the "strength" $\delta G \, \Delta s$ of the field error, and to the root of $\beta(0)$, the magnitude of the betatron function at the location of the per-turbation. You see why one may consider that $\beta(s)$ — or more precisely $\zeta(s) = \sqrt{\beta(s)}$ — is a measure of the "sensitivity" to disturbances.

Second, notice that the denominator of $\underline{a}$ goes toward zero, and x_c becomes, therefore, very large whenever the betatron number ν approaches an integer. It is this behavior which was referred to earlier as an integral resonance which must be avoided in choosing the operating point (ν_x, ν_z).

[†]The phase constant θ is, of course, only defined within an integral multiple of 2π.

Notice that the displacement of the closed orbit at the location of the error has a particularly simple form. You just set s = 0 in Eq. (2.92), or generalizing to an error δG located in Δs at an arbitrary azimuth, say s_1, you get

$$x_c(s_1) = \delta G \, \Delta s \, \frac{\beta(s_1)}{2 \tan \pi \nu} \qquad (2.93)$$

The displacement is now proportional to the first power of β, but the same resonance dependence on ν is evident in the tangent factor. Notice also that except for the resonant denominator, this result agrees with the estimate given in Eq. (2.86).

We may also generalize Eq. (2.92) to give the closed orbit distortion for an arbitrary distribution $\delta G(s)$ of field errors around the ring. At each azimuth s the closed orbit displacements caused by the errors at all other azimuths will add. For an error at $\bar{s}$ we should replace s = 0 by $\bar{s}$ in Eq. (2.92) — and at the same time replace $\phi(s)$ by $\phi(s) - \phi(\bar{s})$. We may then sum over all $\Delta\bar{s}$, to get

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2 \sin \pi \nu} \oint \delta G(\bar{s}) \sqrt{\beta(\bar{s})} \, \cos\{\phi(s) - \phi(\bar{s}) - \pi \nu\} \, d\bar{s} \qquad (2.94)$$

If we have a known field deviation $\delta G(s)$, this equation will (with $\beta(s)$ and ν taken as their undisturbed values) give us the form of the displaced closed orbit.

If the field deviations are true "errors" with an unknown statistical distribution, a more complex statistical analysis must be made to arrive at a statistical estimate of x_c. I shall not go into that subject here.

As mentioned earlier the total displacement from the ideal orbit is the sum of x_c and a free betatron oscillation. In the following sections I shall ignore x_c with the understanding that it must always be added in when one wishes to find the total displacement of a trajectory from the design orbit.

2.11. Gradient Errors

Let's turn now to the effects of gradient errors on the betatron oscillations about the ideal closed orbit. These "errors" refer to the deviations of the focussing function K(s) from its initially prescribed — or nominal — value at each azimuth s. Let's write

$$K(s)_{actual} = K(s)_{nominal} + k(s) \qquad (2.95)$$

where we assume k(s) to be a small quantity. The effect of the deviation k(s) will be to change the betatron function from its nominal value $\beta(s)$ to some new value

$\beta(s) + \Delta\beta(s)$. And the betatron number will be changed from its nominal value ν to some new value $\nu + \Delta\nu$. Generally the <u>tune</u> <u>shift</u> $\Delta\nu$ is of more particular concern, because of the need to keep the operating point away from resonances.

Since the evaluation of $\Delta\beta$ is a bit tedious, I shall not give a rigorous deviation here. I shall rather show how a simple calculation of $\Delta\nu$ can be made and then just write down the exact results for $\Delta\beta$ whose derivation can be found elsewhere.[†]

Suppose that there is a gradient error k in only a small azimuthal interval Δs at $s = 0$. Then as an electron passes $s = 0$ it will receive an extra angular kick $\Delta x'$ which is proportional to its displacement x. In fact, by Eq. (2.19)

$$\Delta x' = k \, \Delta s \, x. \qquad (2.96)$$

Let's again approximate the betatron motion by a simple harmonic oscillation; and ask what will happen when an electron arrives at $s = 0$ at the maximum of an oscillation. The motion will be as shown in Fig. 23. Before arriving at $s = 0$ the
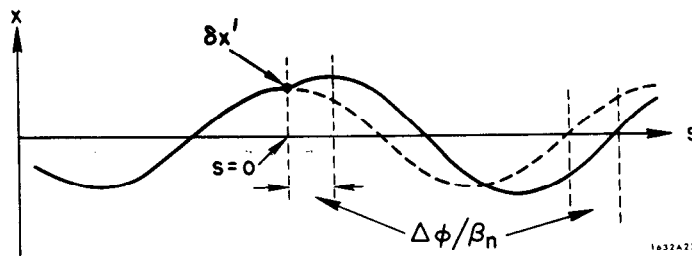
FIG. 23-Effect of a gradient error at $s = 0$.

displacement was given by

$$x = b \cos s/\beta_n; \qquad (2.97)$$

and after $s = 0$ it will follow

$$x = (b + \Delta b) \cos(s/\beta_n + \Delta\phi) \qquad (2.98)$$

where

$$\frac{b + \Delta b}{\beta_n} \sin \Delta\phi = \Delta x' \qquad (2.99)$$

[†] In any book on accelerators; see for example Ref. 5 or 7.

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For small $\Delta x'$, $\Delta \phi$ is small and Δb is much less than b, so we can write that

$$\Delta \phi \approx \frac{\beta_n \Delta x'}{b} \qquad (2.100)$$

Using Eq. (2.96) for $\Delta x'$ — and remembering that at s = 0 the displacement is b — we have that

$$\Delta \phi = \beta_n \, k \, \Delta s \qquad (2.101)$$

The effect of the gradient error is mainly to shift the phase of the oscillation by this $\Delta \phi$. Now recall that the $2\pi\nu$ is just the total phase shift in one revolution; so roughly speaking the gradient error has produced

$$\Delta \nu \approx - \frac{\Delta \phi}{2\pi} = - \frac{\beta_n \, k \, \Delta s}{2\pi} \qquad (2.102)$$

The negative sign comes in because the total phase <u>advance</u> has been reduced.

This result is actually too large by a factor of two. The reason is that we have calculated $\Delta \phi$ for the special case of the electron arriving at s = 0 at the maximum of its oscillation. If the electron arrives at s = 0 with the phase ϕ_0, the phase shift $\Delta \phi$ gets reduced by the factor $\cos^2 \phi_0$ — as you can easily check. Since on successive turns ϕ_0 walks through many values we should expect the average $\Delta \phi$ to be reduced by the average of $\cos^2 \phi_0$, which is just 1/2. With this correction we estimate a $\Delta \nu$ <u>which is precisely what is obtained by a more direct calculation,</u>[†] namely

$$\Delta \nu = - \frac{1}{4\pi} \beta \, k \, \Delta s \qquad (2.103)$$

Notice that the tune shift is just proportional to the gradient error at any point and to the value of β there. We see again that the betatron function is an indicator of the local "sensitivity" to imperfections of the guide field.

If there is a gradient error k(s) distributed around the ring, the total tune shift is

$$\Delta \nu = - \frac{1}{4\pi} \oint \beta(s) \, k(s) \, ds \qquad (2.104)$$

[†]As can be carried out with only a little effort by making use of Eq. (2.79) for ν.

I have said earlier that we might expect $\Delta\nu$ to scale as ν, so that large ν — values are to be avoided. To see that it is so, recall (from Section 2.10) that β is expected to scale roughly as $|K|^{-1/2}$. Then ν should scale as $|K|^{1/2}$. From Eq. (2.104), $\Delta\nu$ would scale as $k\beta$ so $\Delta\nu/\nu$ would scale as k/K. For a given relative size of the gradient errors the tune shift $\Delta\nu$ is proportional to ν. But the spacing between resonances is independent of ν so large ν values imply a more delicate machine.

A change of ν implies that there must have been a change of β which is not evident in the simple calculation above. I shall now just write down an expression for $\Delta\beta$, and make some comments on it. It can be shown that

$$\Delta\beta(s) = \frac{\beta(s)}{2 \sin 2\pi\nu} \oint k(\bar{s})\, \beta(\bar{s})\, \cos 2\left\{\phi(s) - \phi(\bar{s}) - \pi\nu\right\} d\bar{s}, \tag{2.105}$$

where, as usual,

$$\phi(s) = \int_0^S \frac{d\bar{s}}{\beta(\bar{s})} \ .$$

Compare this result with the one obtained in Eq. (2.94) for the closed orbit distortions. The form is similar, but with two important differences. First, while $\beta^{1/2}$ appears in the integral for the closed orbit displacements, the _first_ _power_ of β appears in the integral for $\Delta\beta$. Second, notice that the argument of the sine factor in the denominator is now $2\pi\nu$ instead of $\pi\nu$. The resonant "blow-up" of $\Delta\beta$ occurs at _both_ _integral_ _and_ _half_ _integral_ _values_ _of_ _ν_. The gradient errors introduce a new set of resonances in the operating diagram of ν_x versus ν_z, which must be avoided in a working storage ring.

The tune shift $\Delta\nu$ comes, of course, from the change in the betatron function. From the definition of ν, Eq. (2.60), we can write that

$$2\pi \Delta\nu = -\oint \frac{\Delta\beta(s)}{\beta^2}\, ds \tag{2.106}$$

A straightforward integration over s of $\Delta\beta/\beta^2$ using Eq. (2.105) gives the $\Delta\nu$ of Eq. (2.104).

You are perhaps by now wondering about the following strange point. Although $\Delta\beta$ has a resonance blow-up at half-integral values of ν, the tune shift $\Delta\nu$ does not. How can that be? The reason is that the expression we have derived for $\Delta\nu$ applies only for small changes in β, and is therefore not valid too close to a
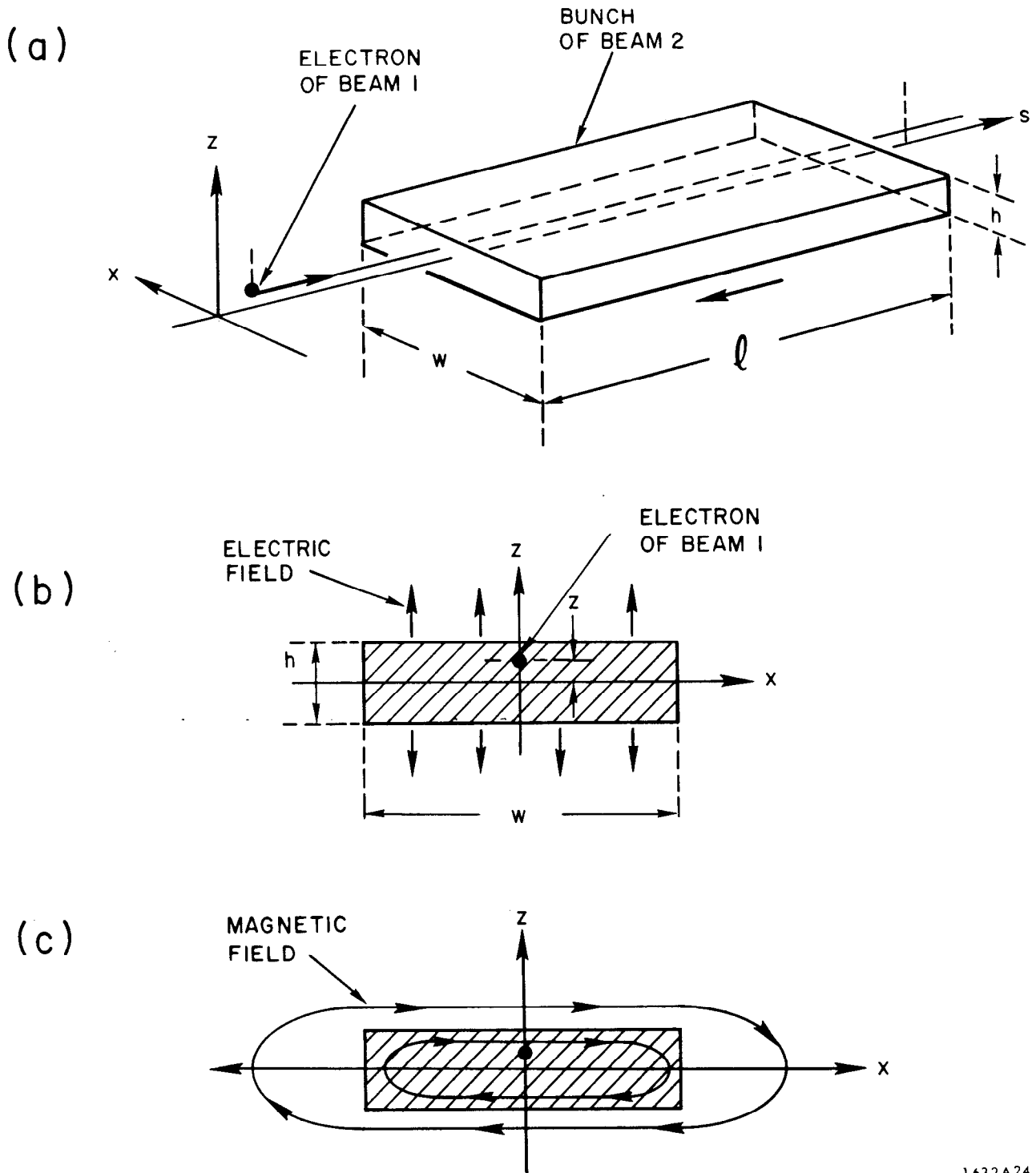
FIG. 24--Electric and magnetic fields seen by an electron of Beam 1 as it passes through a bunch of Beam 2.

resonance, where $\Delta\beta$ diverges. A more precise calculation,which retains second order effects in the perturbation k, must be made to find $\Delta\nu$ — and, in fact, $\Delta\beta$ itself — near a resonance of ν.

2.12. Beam-Beam Interaction; Tune Shift

I have mentioned in Section 1.6 a limit on the intensity of colliding beams. The ground is now prepared for some understanding of this limit. Each time two bunches collide, each electron of one bunch receives a lateral impulse from the macroscopic ("space-charge") electromagnetic field generated by the other bunch. A complete analysis of the effect of these impulses is quite complex. We can, however, take a simple semiquantitative approach to the effects in terms of the tune shift $\Delta\nu$ which arises from the "gradient error" set up by the space-charge forces. In many (perhaps most) storage rings the dominant effect is on the vertical oscillations. To simplify the treatment, I shall make the assumption that that is so and defer until later in this section a consideration of the effects on the horizontal motion. In cases where the horizontal effects dominate the appropriate translation of the discussion and conclusions can easily be made.

Think of the beam bunches as flat rectangular slabs and consider the force on an electron of Beam 1 as it passes through the bunch of Beam 2. See Fig. 24(a). On the axis of the bunch the force is zero. Above the center there is an electric field $\mathscr{E}$ — Fig. 24(b) — which increases linearly with z until the surface of the bunch, and then falls off as sketched in Fig. 25. For all z within the bunch
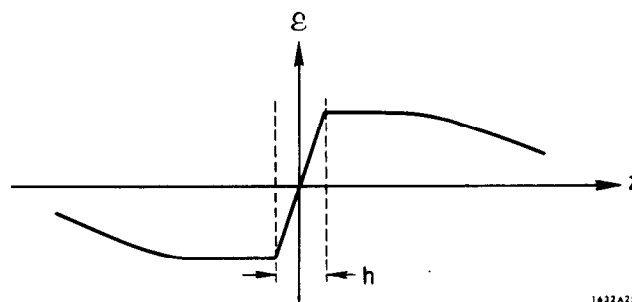
FIG. 25--Electric field strength $\mathscr{E}$ above and below the center of an idealized rectangular bunch.

the electric field is obtained easily from Gauss's Law. The electric field $\mathscr{E}$ at z is given by the superificial charge density between 0 and z. If the bunch has the dimensions w, h, and ℓ and contains N_B electrons, the electric field is

$$\mathscr{E} = \frac{1}{\mathscr{E}_0} \frac{N_B e z}{\ell w h}$$

(2.107)

When an electron passes through a bunch it feels the electric force $e\mathscr{E}$ for a time[†] $\ell/2c$, and receives from it a vertical impulse which changes the vertical component of its momentum p by

$$\Delta p_z(\text{electric}) = \frac{e\mathscr{E}\ell}{2c} \tag{2.108}$$

The moving charges of a bunch also generate a magnetic field — Fig. 24(c) — which is always just $\mathscr{E}/c$. The magnetic force is equal, in both magnitude and direction, to the electric one, so the total impulse received by the electron is just doubled or

$$\Delta p_z = \frac{e^2}{\epsilon_0 c} \frac{N_z}{wh} z. \tag{2.109}$$

The direction of the impulse is repulsive for collisions of like beams (e.g., electron-electron) and attractive for unlike beams (electron-positron). For the present purposes we shall not be concerned with the algebraic sign of the impulse.

Now an electron which receives a vertical impulse Δp_z, suffers a change in slope $\Delta z'$ given by

$$\Delta z' = \frac{\Delta p_z}{p} \tag{2.110}$$

where $p = \gamma mc$. Since this vertical impulse is proportional to z, it has the same effect — see Eq. (2.96) — as an impulsive gradient error of strength $k_z \Delta s$ with

$$k_z \Delta s = \frac{\Delta z'}{z'} \tag{2.111}$$

Such a gradient error produces change in the betatron number of the vertical oscillations whose magnitude is, by Eq. (2.103),

$$\Delta \nu_z = \frac{\beta_z^*}{4\pi} k\Delta s \tag{2.112}$$

where β_z^* is the value of the vertical betatron function at the azimuth of the intersection point. Pulling all the pieces above together we find that

$$\Delta \nu_z = \frac{r_e \beta_z^* N_z}{\gamma \, w \, h} \text{ (rectangular beam)} \tag{2.113}$$

[†] Don't forget that both the bunch and the electron are moving at the speed c.

when r_e is the usual definition of the classical electron radius:

$$r_e = \frac{e^2}{4\pi \, \epsilon_0 mc^2}$$ (2.114)

A real beam does not however, have the ideal rectangular shape assumed. Rather it has a Gaussian distribution of particle density in all three dimensions. The main difference is that variation of the impulse with z is not strictly linear — the linear part in Fig. 25 gets rounded off at its upper and lower edges. To the extent that the impulse depends nonlinearly on z we should no longer speak of a tune shift $\Delta\nu$. But for small betatron amplitudes, the impulse does depend nearly linearly on z and we may <u>define</u> the <u>linear vertical tune shift</u> as the $\Delta\nu$ for these small amplitudes. The calculation proceeds as before; if we now let w/2 and h/2 now stand for the rms half-widths of the distributions, and if the beam is reasonably as flat ($h \ll w$) we get the same result except for a factor of $2/\pi$:

$$\Delta\nu_z = \frac{2r_e}{\pi} \frac{\beta_z^* N_B}{\gamma \, w \, h} \quad \text{(flat Gaussian beam)}$$ (2.115)

If we also extend the calculation to include beam crossing at an angle — see Section 1.5 — we find that the impulse depends only on the projected transverse dimensions w_{eff} and h_{eff}, as did the luminosity. Recalling the definition of A_{int} in Eq. (1.10) we may write the linear vertical tune shift as

$$\Delta\nu_z = \frac{r_e}{2} \frac{\beta_z^* N}{\gamma \, A_{int}} \quad \text{(flat Gaussian beam)}$$ (2.116)

where $N = N_B B$ is the total number of electrons in the B equal bunches of the stored beam. If the aspect of two beams at the intersection is not flat (that is if h_{eff} became comparable to, or greater than w_{eff}) Eqs. (2.115) and (2.116) need correction. The correction would, for example, amount to a factor of 2 for round beams. To simplify the discussions of Part I this small correction has been ignored although it should be kept in mind for more precise calculations. (A complete expression for $\Delta\nu_z$, correct for any ratio of h to w is given in Eq. (2.122).

What is the effect of this vertical tune shift? Suppose we have a stored beam whose vertical betatron number ν_z is placed nicely away from any disturbing resonance — say by 0.1 of an integer. As we put this beam in collision with another beam ν_z is charged by $\Delta\nu_z$, and for a high enough beam intensity the tune shift

may push ν_z to a resonance. We might expect to be in difficulty when $\Delta\nu_z$ reaches some number like 0.1. This is roughly the origin of the beam density limit described in Section 1.6. In any storage ring the tune shift $\Delta\nu_z$ can be no larger than some critical limit, which by Eq. (2.116) sets an upper limit to the transverse charge density N/A_{int}.

The nature of the limiting tune shift is however, somewhat more subtle than just described. If there were just a tune shift $\Delta\nu_z$ due to the beam-beam inter-action, one might, in principle, compensate for it by making an opposite pertur-bation somewhere in the guide field. Remember however, that the $\Delta\nu_z$ evaluated above is an approximation for small oscillation amplitudes. Due to the nonlineari-ties in the forces, the actual effective tune shifts will be different for different amplitudes — being lower for the larger amplitudes. Also, we must remember that an electron oscillates simultaneously in both z and x and that when it is at its maximum excursion in x, it will find smaller forces in the z direction — see Fig. 26. So $\Delta\nu_z$ should more properly be thought of a measure of the spread of tune shifts within the beam. And such a spread cannot be compensated for by any change in a linear guide field.
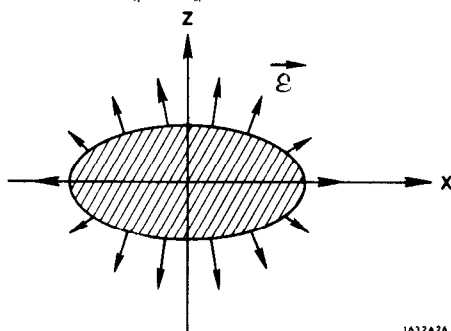
FIG. 26--Electric field $\mathscr{E}$ from a flattened elliptical bunch.

A complete analytical theory of the beam-beam interaction in an electron storage ring is not available. The treatment is complicated by the nonlinearity of the forces and by the quasistocastic variations of the perturbation on successive revolutions — due to the combined oscillations in x and z. Various people[8] have investigated the beam-beam interaction by approximate computer simulations and the results appear to be in reasonable accord with observations on the Princeton-Stanford, Orsay, and Novosibirsk rings. The results may be described as

follows: The expression for $\Delta\nu_z$ should be taken merely as a measure of the strength of space-charge forces. There is a negligible perturbation on the stored beams provided that $\Delta\nu_z$ is less than a certain small number $\Delta\nu_0$. But if $\Delta\nu_z$ exceeds $\Delta\nu_0$ then the electron oscillations of the perturbated beam grow rapidly to amplitudes much larger than normal. There appears to be a "threshold" at $\Delta\nu_0$ for a so-called incoherent instability. The increased beam size from the instability may cause loss of electrons; or, at best, so decrease the beam density that the luminosity is drastically reduced.

The effect we are considering describes the disturbances on the electrons in one beam by the macroscopic fields of the other beam. There is of course, the corresponding effect of the first beam on the individual electrons of the second beam. Notice, however, that if one beam is stronger (more intense) than the other, the instability should occur first in the weaker beam. For this reason, the effect has often been described as the "weak-beam instability." Notice also, that once the weak beam has "blown up" due to the instability, its current density is sharply reduced, and the $\Delta\nu_z$ for the strong beam becomes quite small. Any further increases in the intensity of the weak beam will not lead to an instability of the strong beam. You can see that for two beams of comparable intensity there will be a kind of "flip-flop" effect. Near the instability threshold each beam is trying to make the other blow up. Once one of the beam wins — by making the other go unstable — it can relax, since the $\Delta\nu_z$ it sees then drops sharply. A stable equilibrium will always be reached with one beam blown up and the other undisturbed.

We can now see the nature of the beam intensity limit described in Section 1.6. The currents in the colliding beams must be such that for each beam $\Delta\nu_z \le \Delta\nu_0$; which it is generally agreed, is a number very near 0.025. Equation (2.116) then sets a limit on the current density permitted in each beam. This limit is just the one adopted in Eq. (1.14) if β_V is identified as β_z^*, the vertical betatron function at the interaction point.

Recently, the Orsay group has proposed a way of avoiding the effect of beam-beam interaction by using neutral colliding beams[9] — each beam consisting of equal numbers of positrons and electrons. For such beams $\Delta\nu$ would be obtained by replacing N by $N_+ - N_-$ and clearly, $\Delta\nu$ goes to zero for neutral beams. The potential advantages and disadvantages of this interesting new idea have not yet been fully analyzed (at least not by me), and it will not be considered further in this report.

The vertical tune shift $\Delta\nu_z$ has been emphasized because it tends to play the dominant role in limiting the current of high energy storage rings, but this will be so only so long as the horizontal tune shift $\Delta\nu_x$ is less than $\Delta\nu_z$. Let's look now at $\Delta\nu_x$ so we can tell when the assumed domination of $\Delta\nu_z$ is justified.

Returning to the flat-ribbon model of a beam, consider the force on an electron that passes through a bunch at a small radial distance $x \ll w$ from the axis. See Fig. 27. We can find the electric force on the electron by noticing that — by
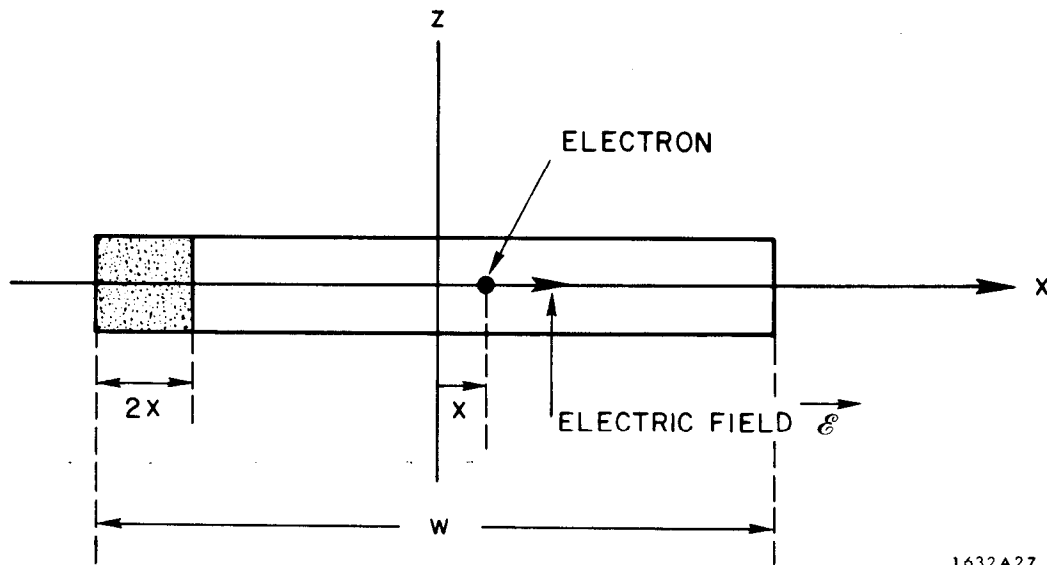
FIG. 27--Electric field at an electron that passes through a bunch at a small radial distance from the axis.

symmetry — all of the unshaded areas of the bunch will contribute no net force and that the resultant force is due just to the shaded area of width 2x. Considering the shaded area as a line charge with the linear charge density λ at the distance $w/2$, the electric field strength $\mathscr{E}$ at the electron at x is

$$\mathscr{E} = \frac{\lambda}{\pi\epsilon_0 w} \ . \tag{2.117}$$

The fraction of the total charge in the shaded area is $2x/w$, so

$$\lambda = \frac{N_B e}{\ell}\ \frac{2x}{w}\ , \tag{2.118}$$

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and

$$\mathcal{E} = \frac{1}{\pi \epsilon_0} \frac{2N_B ex}{\ell w^2} \tag{2.119}$$

Notice that the field strength per unit displacement differs from that in Eq. (2.107) by the factor $2w/\ell$. The rest of the analysis proceeds as before except that when one goes to Gaussian beams the factor of 2 drops out of the ratio and for flat Gaussian beams one finds — corresponding to Eq. (2.115) — that

$$\Delta \nu_x = \frac{r_e \beta_x^* N_B}{2\pi \gamma w^2} \quad \text{(flat Gaussian beams)} \tag{2.120}$$

If the beams cross at an angle, w and h in Eq. (2.115) and (2.120) get replaced by w_{eff} or h_{eff}. See Section 1.5. So in the general case the ratio of the horizontal to the vertical tune shift can be written as

$$\frac{\Delta \nu_x}{\Delta \nu_z} = \frac{\beta_x^* h_{eff}}{\beta_z^* w_{eff}} . \tag{2.121}$$

This result is, it turns out, correct for <u>any aspect ratio</u> of the beam cross section (any ratio of w to h).

We might expect that β_x^* and β_z^* would generally be comparable. And since the natural width of a beam is much larger than its natural height, the ratio above would be expected to be less than 1 — as has been assumed in the discussion of the section. With beam crossings at an angle, however, and by the use of special techniques to modify, almost at will, the beta functions at the intersection, it is possible to have the ratio β_x^*/β_z^* take on almost any value. If the ratio of Eq. (2.121) is greater than 1 the horizontal tune shift dominates the discussion of this section should be modified accordingly. It is, of course, necessary always to insure that both tune shifts are less than $\Delta \nu_0$.

Now that the physics of the tune shift should be clear, let me write down here the complete expressions one obtains for the vertical and horizontal linear tune shifts with Gaussian beams of any aspect ratio. To avoid confusion I write now $\sigma_x, \sigma_z,$ and σ_s for the rms half-widths of the distributions (σ_s refers to the

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longitudinal distribution). For head-on collisions:

$$\Delta \nu_z = \frac{r_e}{2\pi} \frac{N_B \beta_z^*}{\gamma \sigma_z (\sigma_x + \sigma_z)} \tag{2.122}$$

$$\Delta \nu_x = \frac{r_e}{2\pi} \frac{N_B \beta_x^*}{\gamma \sigma_x (\sigma_z + \sigma_x)} \tag{2.123}$$

(The expressions are symmetric in z and x as they should be. Notice also that they go over to the expression derived for flat beams when $\sigma_x \gg \sigma_z$.) If the beams cross at a vertical angle of 2δ, σ_z gets replaced by

$$(\sigma_z)_{\text{eff}} = \left(\sigma_z^2 + \sigma_s^2 \delta^2 \right)^{1/2} \tag{2.124}$$

or for a horizontal crossing angle of 2δ, σ_x gets replaced by

$$(\sigma_x)_{\text{eff}} = \left(\sigma_x^2 + \sigma_s^2 \delta^2 \right)^{1/2} \tag{2.125}$$

I should perhaps emphasize that the beam-beam interaction considered in this sections does not take into account the effects coherent lateral oscillations of the electrons in a bunch — that is, of lateral motions of the bunch as a whole. Such oscillations can occur and can lead to coherent instabilities involving both beams. A discussion of such effects is outside the scope of this report, but may be found elsewhere.[10] For successful operation of a storage ring such oscillations must be inhibited in some way or another. And then the conclusions of this section are applicable.

2.13. Low-Beta Insert

In a high energy storage ring the maximum beam current will, at high energy, be determined by the available radio frequency power — as discussed in Section 1.7. We saw there that if the beam dimensions can be suitably adjusted, the maximum achievable luminosity is proportional to the maximum permitted current density — see Eq. (1.22). If we take $\Delta \nu_z$ from Eq. (2.116) this maximum current density is

$$D_c = \frac{N}{A_{\text{int}}} = \frac{2 \Delta \nu_0 \gamma}{r_e \beta_z^*} \tag{2.126}$$

and the maximum luminosity is

$$\mathscr{L}_2 = \frac{f N_{\text{MAX}} \Delta \nu_0 \gamma}{2 r_e \beta_z^*} \tag{2.127}$$

Given the energy (γ), the beam current (N_{MAX}), and the radius of the storage ring (f = c/2πR), the only "free" parameter is β_z^*. The form of this result lead Robinson and Voss to propose an idea for increasing the luminosity of high energy rings.

A uniformly small betatron function β_z(s) would have the disadvantage (among others) of producing large values of ν_z with the attendant difficulties of avoiding resonances. Robinson and Voss realized, however, that it was possible to obtain an abnormally low value of β at the location of the beam crossing point while leaving the rest of the ring with a normal size β. The idea is to introduce a special section in the guide field which produces only a localized region of very low β_z — the so-called low-beta insert.

To see how the idea works we must return to the discussion of the betatron trajectories in Section 2.6. Imagine that a ring has been designed with generally satisfactory properties for storing the desired beams. Now imagine that this ring is "broken open" at some azimuth and a special section of focussing elements is inserted with the following property: Its transfer matrix $\underline{M}$ is the unit matrix, so that an electron which enters the insert with the displacement and slope (x, x') leaves the insert with the same displacement and slope; and the same is true for the z motion. An insert with these properties will not affect the trajectories in the remainder of the ring; it follows that the betatron functions there will also be un-changed. Thus if we place the constraint on an insert that it shall have unit transfer matrices in both x and z we are otherwise free to choose its structure in any way we wish. And if we are clever enough, we can make the betatron functions take on almost any values we wish at some place inside of the insert.

Actually, the requirement that the insert have a unit transfer matrix is stronger than necessary. For example, you can easily show that a transfer matrix of -1 would also not change the betatron function in the rest of the ring. And, indeed, there are still other possibilities, as you can see by referring to the results of Section 2.9. There we saw that ζ(s), the square root of the betatron function β(s), is that function which satisfies the differential equation (2.74) and is single-valued around the ring. Suppose we wish to "break open" a ring at some point where ζ and ζ' take on the values ζ_0 and ζ_0' and insert a special focussing section, which we may define by its focussing function K(s). Say that the insert will go from s_1 to s_2 — with s now suitably redefined to include the insert. Then the function ζ(s)

will be unchanged outside of the insert provided only that the K(s) of the insert is such that $\zeta(s)$ satisfies Eq. (2.74) <u>with the boundary conditions</u> that $\zeta(s_1) = \zeta(s_2) = \zeta_0$ and $\zeta'(s_1) = \zeta'(s_2) = \zeta_0'$. Or we may say the same thing in another way. Suppose we start at one end of the insert, say at s_1, with the initial conditions that $\zeta(s_1) = \zeta_0$ and $\zeta'(s_1) = \zeta_0'$ and integrate Eq. (2.74) through the insert to s_2. If we arrive at s_2 with $\zeta(s_2) = \zeta_0$ and $\zeta'(s_2) = \zeta_0'$ then the function $\zeta(s)$ through the rest of the ring will be left unchanged. This is the most general definition of an insert that will not disturb the trajectories in the main part of the ring.

A special insert will, necessarily, change the betatron number ν of the whole ring, since there must be some advance of the betatron phase $\phi(s)$ through the insert. For example, an insert with a unit transfer matrix must increase the phase by 2π — or some integral multiple of it. So the ν of the modified ring would be increased by 1 or some integer. Practical inserts which satisfy the more general requirement of the preceding paragraph will generally change ν by something near an integer.

The design of low-beta inserts which satisfy the required conditions, which give the desired low value of β at some chosen interior point, and which can be constructed with real magnetic components is an art the details of which are outside the scope of this report — and which, in any case, I am not competent to discuss. Let me just show one specific example of such an insert and then make some observations that will apply generally to such inserts.

I show in Fig. 28(a) the focussing function K_x for a low-beta insert which was designed for a storage ring proposed at SLAC. For this insert K_z is just $-K_x$. The insert was intended to be inserted between two of the "normal" cells of the magnet whose amplitude functions $\zeta(s) = \sqrt{\beta(s)}$ were shown in Fig. 18. The functions ζ_x and ζ_z of the insert are also shown in Fig. 28. Both functions have zero slopes at the boundary of the insert (as does the normal cell) and their values match the normal cell at the boundary. So all required conditions are satisfied.

The insert shown in Fig. 28 was designed to have a very low β_z at its center — namely 5 cm in comparison with a typical β_z of ≈ 1000 cm in the normal cell — but to have only a normal β_x there. The center is of course, to be the point of intersection of the stored beams. For this reason there are no magnet elements for 2.5 meters on either side of the center — leaving 5 meters free for the detection apparatus.
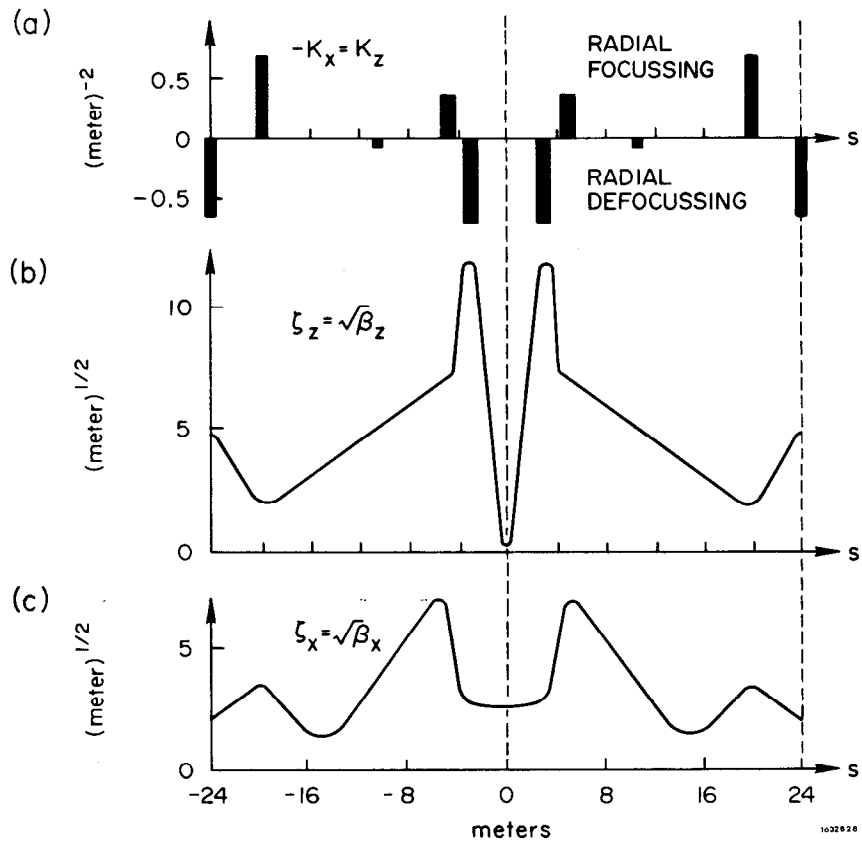
FIG. 28--Focussing function and envelope functions for the SLAC low-beta insert.

A striking feature of the curve for $\zeta_z(s)$ in Fig. 28 is the very large value reached at the quadrupoles nearest to the center of the insert. These large values have several unfortunate consequences. First, since the betatron oscillation amplitudes (for a given amplitude in the normal cell) go in proportion to $\zeta(s)$ they will be very large at these nearest quadrupoles — requiring that they have a large aperture. Second, as we have seen, the closed orbit distortions and the perturbations to $\beta(s)$ and to ν are proportional to $\beta = \zeta^2$ at the location of the perturbation. The effect of any field of gradient errors at the nearest quadrupole is magnified by a large factor. Clearly, the large values of β are undesirable and place stringent technical requirements on the construction of the insert.

Why must β be so large? We worked out in Section 2.10 a simple expression for the form of $\beta(s)$ in a focus–free region — Eq. (2.81). If $\beta(s)$ has the minimum value β_0, then for distances ℓ large compared with β_0 — in a region where $K = 0$ — we found that

$$\beta = \frac{\ell^2}{\beta_0} \tag{2.128}$$

If we ask for 2.5 meters free space to the nearest quadrupole and also for $\beta_0 = 5$ cm, we must get that β becomes as large as 125 meters (!) at the position of the first quadrupole. The large β_z at the quadrupole is the price paid for having a clear experimental region together with the high luminosity which comes from the small β_z at the intersection.

III. ENERGY OSCILLATIONS

3.1. Off-Energy Orbits

In the preceding sections I have been discussing the trajectories in a storage ring of electrons with the nominal energy E_0 — which is the design energy for a given setting of the magnet currents. Stored electrons do not, however, all have this ideal energy. In general, the energy E of a stored electron will deviate from the nominal energy, and, as described in Section 1.2, will oscillate about it. These energy oscillations — often called "synchrotron oscillations" — are the subject of the part.

We must, first, understand the motion of electrons whose energy differs by a small amount ϵ from the nominal energy. Keeping the assumptions of Section 2.3 that the design orbit lies in a horizontal plane, energy deviations will, to first order in small quantities, affect only the radial motion. The vertical displacement will still be described by the betatron oscillations analyzed in Part II, and will not be considered further here. From Section 2.6 onward it was convenient to let the symbol x stand generally for either x_β or z_β, the lateral displacements associated with the betatron oscillations. I now return to the notation in which x represents the <u>total horizontal</u> displacement of a trajectory from the design orbit.

It was shown in Section 2.5 that in an ideal guide field the radial motion for an electron with the energy deviation ϵ can be written as the sum of two parts

$$x = x_\beta + x_\epsilon \tag{3.1}$$

where x_β is the betatron displacement and x_ϵ is a displacement which depends only on the energy of the electron. If we wish to include also the results of Section 2.11, we should include in addition, the distortion of the closed orbit due to magnet imperfections and write

$$x = x_\beta + x_\epsilon + x_c \tag{3.2}$$

Since the various contributions add linearily — under our assumptions of a linear guide field, of small energy deviations, and of small displacements — we have been able to consider separately the several contributions to x. We now ignore the other contributions to x and focus on x_ϵ.

According to Eq. (2.28) the energy displacement x_ϵ can be written as

$$x_\epsilon = \eta(s) \frac{\epsilon}{E_0} \tag{3.3}$$

where $\eta(s)$ is a function of the azimuthal coordinate s which is single valued at each physical azimuth. An off-energy electron with no betatron oscillations runs around a new <u>closed orbit</u> whose displacement from the ideal orbit is everywhere proportional to ε/E_0, with a proportionality factor which depends on the azimuth according to a given function $\eta(s)$, characteristic of the total guide field configuration. I shall call $\eta(s)$ the <u>off-energy function</u> — it is just the closed orbit displacement per unit energy deviation.

Let's look now at the nature of $\eta(s)$. It was defined — see Eq. (2.29) — as that solution of the differential equation [†]

$$\eta'' = K_x(s)\eta + G(s) \tag{3.4}$$

which is periodic in s with period L and is, therefore, single valued for all physical azimuths. The functions G(s) and $K_x(s)$ were defined in Eqs. (2.3) and (2.21).

Let's consider the qualitative behavior implied by this equation for the $\eta(s)$ of a separated function guide field (which was defined in Section 2.2). We may take as an example the guide field of the SLAC proposal which was used for illustration in Sections 2.6 and 2.10. In Fig. 29(a),(b) I show K_x and G for this guide field and in (c), the off-energy function $\eta(s)$.

In a field free section both G and K_x are zero so $\eta(s)$ has a segment of constant slope. In a pure quadrupole G is zero and K_x is just the quadrupole strength. In a focussing quadrupole K_x is negative and $\eta(s)$ follows a segment of a sinusoidal oscillation about zero with the form

$$\eta = a \cos\left(\sqrt{-K_x}\, s + \theta\right)$$

In a defocussing quadrupole K_x is positive and $\eta(s)$ follows a segment of a positive exponential like

$$\eta = a \exp\left(\sqrt{K_x}\, s + \theta\right)$$

The curve of $\eta(s)$ is "attracted" toward the s-axis in a focussing quad and repelled from the axis in a defocussing quad.

Although K_1 is zero in a flat bending magnet, K_x is not. In fact, $K_x = G^2$ and the equation for η is

$$\eta'' = -G^2\eta + G = -G^2\left(\eta - \frac{1}{G}\right) \tag{3.5}$$

[†] $\eta' = d\eta/ds$.
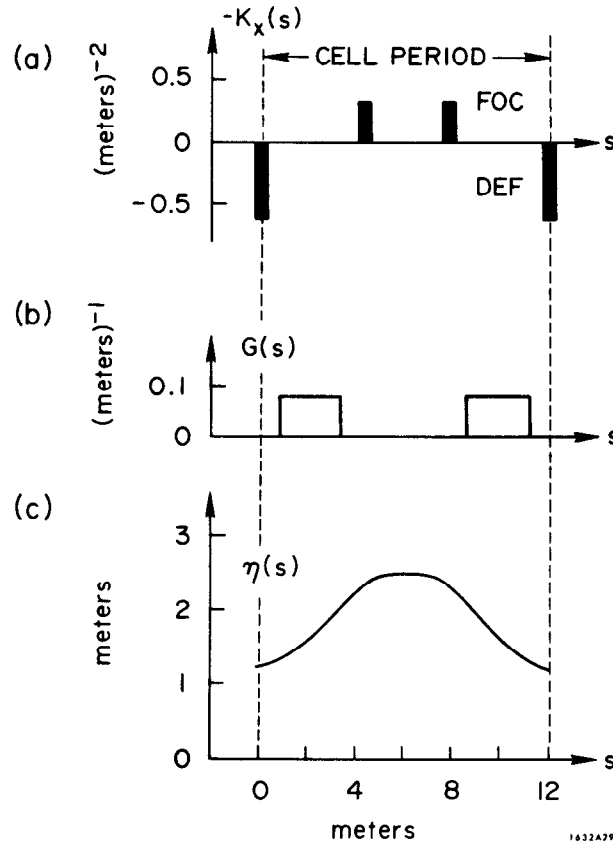
FIG. 29--Guide-field functions and the off-momentum function for
the SLAC guide field.

The curve of η is a segment of a sinusoid which is "attracted" toward the level $\eta_0 = 1/G$ with a "restoring force" proportional to G^2. (The level η_0 is just equal to the radius of curvature ρ of the design orbit.)

From the above discussion you can understand the qualitative features of the variations of $\eta(s)$ that appear in Fig. 29. For all "normal" storage rings it turns out that the off-energy function is everywhere positive.

A storage ring user is not generally faced with the need to make a detailed calculation of $\eta(s)$. Its graph should be provided by the ring designers. I will therefore, only indicate briefly how it may be calculated. For a separated function guide field the preceding discussion can be expanded to give a method for calculating $\eta(s)$. Suppose you begin at s = 0 with some assumed values of $\eta(0)$ and $\eta'(0)$ and evaluate $\eta(s)$ as a succession of segments of the kind described above until you make your way around one complete revolution — until you get to s = L. You will get the true $\eta(s)$ if you then choose $\eta(0)$ and $\eta'(0)$ so that $\eta(L)$ and $\eta'(L)$

are respectively, equal to $\eta(0)$ and $\eta'(0)$. The computation can be carried out most straightforwardly by using a matrix technique. (See Ref. 11).

The off-energy function can also be obtained (for any kind of guide field) by making use of the results we obtained in Section 2.11 for disturbed closed orbits. We may imagine that an off-energy orbit is just a "disturbed" closed orbit since an energy deviation gives rise to a change in curvature just as does a field error. That is to say, that a field error δG in a segment Δs of the orbit produces a change of curvature in the path of an electron of energy E_0 which is the same as the change of curvature that results when an electron with an energy deviation ϵ goes through the nominal field provided that $\delta G/G = \epsilon/E_0$. Since $\eta(s)$ is the ratio of the closed orbit displacement to ϵ/E_0, we may compute $\eta(s)$ by replacing δG in Eq. (2.92) of Section 2.11 by G. This argument can also be justified by noticing that Eq. (3.4) for η has the same form as Eq. (2.85) for x_c in Section 2.11; the latter going into the former if we make the substitutions $x_c \rightarrow \eta$ and $\delta G \rightarrow G$. Making the same substitutions in Eq. (2.94) we get

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2 \sin \pi \nu} \oint G(\bar{s}) \sqrt{\beta(\bar{s})} \cos \left\{ \phi(s) - \phi(\bar{s}) - \pi \nu \right\} d\bar{s}. \tag{3.6}$$

So if $\beta(s)$ is already known we can get $\eta(s)$ by an integration. Notice that $\eta(s)$ too will have a resonance behavior when ν approaches an integer.

If the design orbit does not lie in a plane — as, for example, in the recent DESY or Orsay designs — then the discussion of this Section must be repeated for the vertical displacements. There will in such cases be two curvature functions G_x and G_y as well as the two focussing functions K_x and K_z. The vertical displacements will also have an off-energy contribution which will be proportional to an off-energy function $\eta_z(s)$. And this vertical off-energy function can be evaluated in terms of the vertical focussing and curvature functions. There will be generally one important qualitative difference from the horizontal case in that η_z will have both positive and negative values and its average around the ring will be zero.

3.2   Orbit Length; Dilation Factor

An important consequence of an energy deviation is the associated change in the circumference of the closed orbit. I wish now to take a look at this effect. An electron of the nominal energy E_0 which circulates on the design orbit will, in one revolution, travel the distance L, the circumference of the design orbit. On any

other trajectory, the path length traveled in one revolution will depend on the deviations from the ideal orbit and may be expected to differ from L. We have already noticed in Section 2.4 that an electron which moves from s to s + ds with a displacement x from the design orbit has a path length $d\ell$ different from ds by an amount that depends on the local radius of curvature. See Fig. 9. We found there — Eq. (2.15) — that

$$d\ell = \{1 + G(s) x\} ds, \qquad (3.7)$$

so long as only terms to first order in x are retained.

A betatron oscillation will produce <u>on the average</u> no first order change in the path length. The path is lengthened on a positive swing (x > 0) of the oscillation and shortened on a negative swing. Since the betatron displacements are on the average, symmetric about x = 0, the path length change is zero when averaged over one or more complete betatron cycles. If the betatron number ν is much greater than 1 so that there are several betatron cycles in one revolution, the net change in the path length in one revolution is small. If $\nu \approx 1$ however, there will be changes in the path length from one revolution to the next. We shall however, be interested here only on the <u>average</u> path length (averaged over several revolutions) and the betatron oscillations will not, to first order, affect this average.

There is a second order effect — which gives a time change proportional to the square of the betatron amplitude. It can introduce a very small coupling between betatron oscillations and energy oscillations. I am ignoring here all such second-order processes.

The lateral displacement x_ϵ of an off-energy orbit does give rise to a change in the orbit length — because, for a given energy deviation, x_ϵ has generally the same sign all around the ring. Putting x_ϵ for x in Eq. (3.7) and integrating once around the ring, we get for the circumference ℓ_ϵ of an off-energy closed orbit

$$\ell_\epsilon = \oint d\ell = \oint \{1 + G(s) x_\epsilon(s)\} ds \qquad (3.8)$$

The first term of the integral gives the complete integral of ds which is just L, the length of the design orbit. The second term gives the elongation due to the energy deviation; let's call it $\delta\ell_\epsilon$. Recalling Eq. (3.3) for x_ϵ, we get that

$$\delta\ell_\epsilon = \frac{\epsilon}{E_0} \oint G(s) \, \eta(s) \, ds \qquad (3.9)$$

The change in the orbit length is proportional to the energy deviation ϵ, with a constant of proportionality — the definite integral — which can be obtained from the known properties of the guide field.

It is convenient to define a dimensionless parameter α, which we may call the dilation factor by

$$\frac{\delta \ell_\epsilon}{L} = \alpha \frac{\epsilon}{E_0} \tag{3.10}$$

It follows from Eq. (3.9) that

$$\alpha = \frac{1}{L} \oint G(s) \, \eta(s) \, ds. \tag{3.11}$$

The dilation factor α is a number which like the betatron number ν is a characteristic of the total guide field. It is a crucial parameter of the energy oscillations.

It is with some reluctance that I have introduced the name "dilation factor" for α, since it is quite generally known as the "momentum-compaction." This other name was adopted in the early analyses of alternating gradient guide fields in relation to the lateral spread of orbits of different momenta — which spread is proportioned to our $\eta(s)$. It is true that α is a general indicator of the magnitude of η. However, a greater "compaction" of different momenta generally corresponds to a smaller α. One might therefore think that the "compaction" would be measured by the inverse of α. Since α relates anyway only indirectly to η and is specifically a measure of the variations in orbit length, I would hope that some more appropriate — and less misleading — name might be brought into general use. My candidate is "dilation factor."

We can get a little better understanding of the nature of α by looking at it for the most common kind of guide field, the isomagnetic guide field defined earlier. In an isomagnetic field, G has the value G_0 in all magnets and zero elsewhere (see Eq. (2.9)) so Eq. (3.12) can be expressed by

$$\alpha = \frac{G_0}{L} \int_{Mag} \eta(s) \, ds \quad (\text{isomag}). \tag{3.12}$$

where the integral is to be taken over only those parts of the design orbit which are in the bending magnets.

This result can be written in a more illuminating way. Suppose we define the magnetic average of η as

$$\langle \eta \rangle_{\text{Mag}} = \frac{1}{\ell_{\text{Mag}}} \int_{\text{Mag}} \eta(s) \, ds \qquad (3.13)$$

where ℓ_{Mag} is the total length of the orbit segments in the bending magnets. (This would be the usual definition of the mean value of η in all the magnets.)

But all of the bending magnets must add up to a complete circle so ℓ_{Mag} is just 2π times the constant orbit radius ρ_0 in the magnets which is just $1/G_0$; so

$$\alpha = \frac{2\pi}{L} \langle \eta \rangle_{\text{Mag}} = \frac{\langle \eta \rangle_{\text{Mag}}}{R} \qquad (3.14)$$

Where $R = L/2\pi$ is the gross orbit radius defined earlier. The dilation factor α is just the ratio of the magnetic average of η to the gross radius of the orbit.

The high energy electrons we are concerned with here travel always at a speed which is not noticeably different from the speed of light; and the time required for each revolution of the storage ring is just proportional to the length of the trajectory. On an off-energy orbit corresponding to the energy deviation ε, the change δT in the revolution time is in the same proportion to the revolution time T_0 on the design orbit as the change in length of the closed orbit is to the length of the design orbit:

$$\frac{\delta T}{T_0} = \frac{\delta \ell_\epsilon}{\ell} = \alpha \frac{\varepsilon}{E_0} \qquad (3.15)$$

3.3. Approximations for the Off-Energy Function and the Dilation Factor

For most practical guide fields there is a close relation between the off-energy function $\eta(s)$ and the radial betatron function $\beta_x(s)$ which was our central concern in the preceding part. Since the demonstration of this connection is a bit long, I shall simply offer it to you without proof. (A demonstration can be dug out of the material in Ref. 4.) For an isomagnetic guide field that has a well behaved betatron function (with no wild variations) a rather good approximation to $\eta(s)$ is

$$\eta(s) \approx a_0 \beta_x^{1/2}(s) = a_0 \zeta(s) \quad \text{(isomag)} \qquad (3.16)$$

where a_0 is a constant. Except for the scaling factor a_0, the function $\eta(s)$ has very nearly the same form as our function $\zeta(s)$. You can confirm this similarity for at least one case by comparing Figs. 18 and 29 which show $\zeta_x(s)$ and $\eta(s)$ for the same illustrative guide field. For this example Eq. (3.16) is good to a few per cent.

For most purposes it will be sufficient to take a_0 from the ratio of the known functions. A formula for it comes out of the mathematical derivation of Eq. (3.16) which gives

$$a_0 = \frac{\langle \beta_x^{1/2} \rangle_{\text{Mag}}}{\nu_x} \quad \text{(isomag)}, \qquad (3.17)$$

where ν_x is the radial betatron number and the magnetic average of $\beta_x^{1/2}$ is defined in the same way as we did for η in the preceding section — see Eq. (3.13).

The scaling factor a_0 can also be expressed in terms of already defined ring parameters. If we average both sides of Eq. (3.16) over all the magnets we have that

$$\langle \eta \rangle_{\text{Mag}} \approx a_0 \langle \beta_x^{1/2} \rangle_{\text{Mag}} \quad \text{(isomag)}. \qquad (3.18)$$

By Eq. (3.15), the left-hand side is just αR and by Eq. (3.17) the right-hand side is $a_0^2 \nu_x$ so we get that

$$a_0^2 \approx \frac{\alpha R}{\nu_x} \qquad (3.19)$$

and $\eta(s)$ can be written as

$$\eta(s) \approx \left(\frac{\alpha R}{\nu_x} \right)^{1/2} \beta_x^{1/2}(s) \qquad (3.20)$$

This approximation will generally give a reasonably adequate representation of $\eta(s)$.

A somewhat rougher approximation to a_0 can be obtained by noticing that, generally speaking, the magnetic average of $\beta_x^{1/2}$ should be approximately the same as the square root of the typical value of β_x — which was defined earlier as $\beta_{xn} = R/\nu_x$. (See Eq. (2.72)). Then, using Eq. (3.17), we expect that

$$a_0^2 \approx \frac{R}{\nu_x^3} \qquad (3.21)$$

The last two results also display a useful approximate connection between the betatron number ν_x and the dilation factor α; namely that

$$\alpha \approx \frac{1}{\nu_x^2} \quad \text{(rough)} \qquad (3.22)$$

This simple connection between α and ν_x is useful for an understanding of the general characteristics of high energy storage rings. If we consider ν_x to be a measure of the focussing "strength" of a guide field, the dilation factor is seen to decrease as the inverse square of the strength.

3.4 Energy Loss and Gain

Until now we have ignored those effects which change the energy of a stored electron; it is now necessary to consider the processes by which an electron loses or gains energy. The lateral acceleration along the curved parts of a trajectory causes an electron to radiate away some of its energy. The characteristics of this radiation loss will be discussed in some detail in Section 4.1. If the electron is to remain captured in the storage ring this radiation loss must be compensated for, on the average, by an equal energy gain from the radio frequency accelerating system of the ring — one or more electrode structures which produce, along parts of the orbit, an electric field that can feed energy to the moving electron. It is the interplay of the radiation loss and the acceleration gain — together with the properties of the guide field — that gathers injected electrons into stable circulating bunches and is responsible for the residual small energy oscillations of the electrons in a bunch.

An electron of the nominal energy E_0, moving on the design orbit will radiate away a certain amount of energy, say U_0, each revolution. This radiation loss is always a very small fraction (typically 10^{-4} or less) of the electron's energy. And the energy gain from the acceleration system is of course, of the same order. The small magnitude of the loss in one revolution allows us, fortunately, to make a number of simplifying assumptions without which a study of the energy oscillations would hardly be tractable. We may, to begin with, make the approximation that an electron which starts a revolution with the energy E_0 will also loose the energy U_0 during the revolution. Although the energy will not strictly remain at E_0, nor the trajectory remain on the design orbit, the deviations during one revolution can be neglected. The effects which accumulate over several revolutions must, however, be taken into account.

If an electron with the energy E_0 is given a betatron oscillation its instantaneous rate of radiation loss may change — because of a different lateral acceleration along the trajectory. But the average energy loss over a complete betatron oscillation will not change to first order in the betatron amplitude. (Changes in the lateral oscillation

will be proportional to x and will, to first order, average to zero over a complete cycle.) Since we shall be satisfied to consider only the effects which occur over many betatron oscillations, we need to look only at the average energy loss. So long as we are keeping to our first order view of a storage ring we may ignore any dependence of the radiation loss on the betatron displacements.

The radiation loss will however, change with a change in the energy of an electron. Both its different trajectory and its different energy can contribute to a modified energy loss. Because all energy changes occur slowly, we may consider that an electron is at any instant moving on the off-energy closed orbit which corresponds to its instantaneous energy — or is performing free betatron oscillations about that orbit. Since we know the form of the off-energy orbit, we can compute the energy lost in each revolution. I shall consider this problem later (in Section 4.1); for now we may take it as a given function $U_{rad}(\epsilon)$ of the energy deviation ϵ.

Since we shall generally be interested only in small energy deviations, we need keep only the linear term in the variation of U_{rad} and write that

$$U_{rad} = U_0 + D\epsilon, \tag{3.23}$$

where

$$D = \left(\frac{d\,U_{rad}}{d\epsilon}\right)_0 , \tag{3.24}$$

and the derivative is evaluated at the nominal energy E_0. For the present, then, the radiation loss may be described by the two constants U_0 and D — which will be evaluated in terms of the properties of the guide field in part IV.

Let's now turn to the radio frequency accelerating system — "rf system" for short — which supplies energy to the electrons to compensate for the radiation loss. The rf system consists typically of one or more cavity resonators such as the one shown schematically in Fig. 30, disposed at various places around the storage ring and supplied with rf power from some synchronized radio power sources. These cavities produce oscillating electric fields along the electron trajectories; and it is the component of these fields along the electron's path which feeds energy to the electrons. An electron which goes around once on the design orbit will be given by the rf system an amount of energy U_{rf} equal to the integral of the instantaneous electric force along its trajectory.
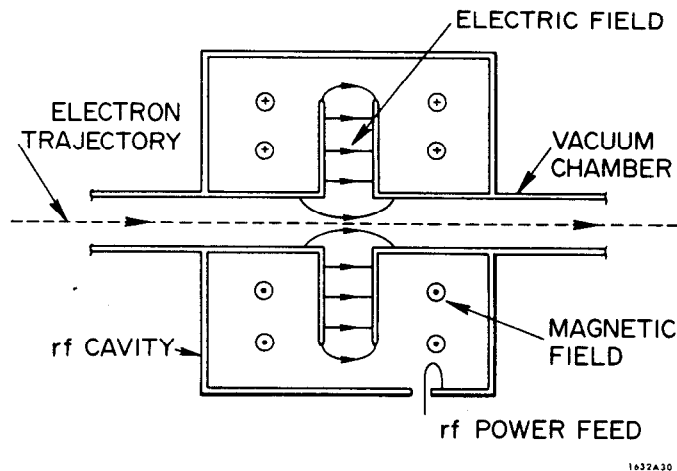
FIG. 30--Schematic diagram of an rf accelerating cavity.

Since the rf fields are time varying,[†] the energy gained by an electron in making one circuit around the ring will depend on the time at which that circuit begins in relation to the oscillations of the accelerating fields. Let's say that the time dependence of the fields is given. Then the energy U_{rf} gained by the electron in one revolution will depend on the time t that it starts its revolution. (We may take that the revolution starts at some reference azimuth, say $s = 0$.)

If electrons are to be stored on (or near) the design orbit, the variation of $U_{rf}(t)$ must have certain characteristics. I shall assume that $U_{rf}(t)$ is a periodic function with a period that is some integral submultiple of the period T_0, the period of revolution of an electron that circulates on the design orbit. That is,

$$U_{rf}(t + T_0/k) = U_{rf}(t) \qquad (3.25)$$

where k is some integer that will be called the harmonic number of the rf system. The variation of $U_{rf}(t)$ might be, for example, like the function shown in Fig. 31. (Although the assumed time variation of U_{rf} is somewhat more restrictive than necessary, the rf fields must have at least similar characteristics if a storage ring is to work. And most storage rings will have generally the characteristics assumed.)

Now consider what can happen with an electron of the nominal energy E_0 that is circulating on the design orbit. Suppose that it is started on its journey at just

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[†]As they must be if there is to be a net integral of the electric field around a closed path!
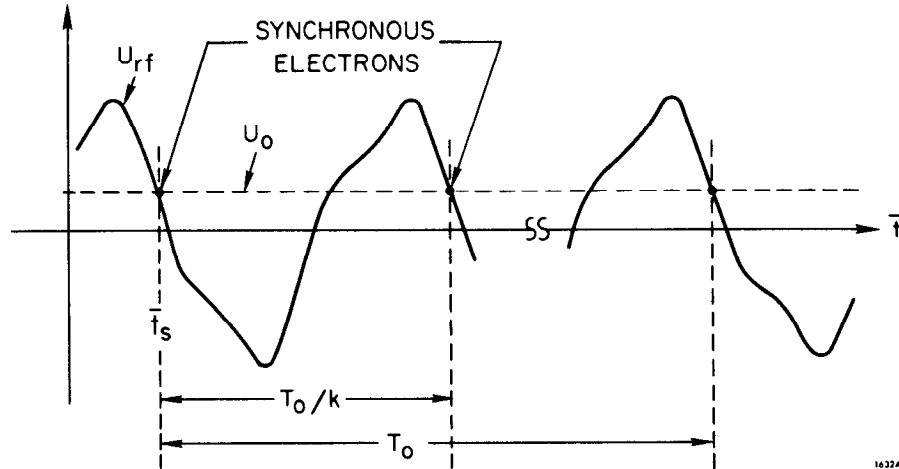
FIG. 31--Energy gain from the rf system as a function of the
starting time t of a revolution.

the right time $\bar{t}_s$ for which the rf gain $U_{rf}(\bar{t}_s)$ is just equal to the radiation loss U_0.
See Fig. 31. In the next revolution the energy lost and gained will compensate
and the electron will return to its starting point again with the energy E_0. The
time taken for the revolution is T_0; so the electron will start the next revolution
at the time $\bar{t}_s + T_0$ and by Eq. (3.25) the rf gain will again be equal to U_0. The
electron will continue to circulate indefinitely on the design orbit. Such an electron
which passes the reference azimuth at the times $\bar{t}_s + j T_0$ (where $j = 1, 2, 3, \ldots$)
is called a synchronous electron — because its rotation is synchronous with the
oscillating rf fields. And $\bar{t}_s$ is generally called the synchronous phase of the rf
system. (Of course with a periodic rf there are equivalent synchronous starting
times once each rf period.)

I have clearly assumed that the peak value of U_{rf} is greater than the radiation
loss U_0 of the synchronous electron. It follows that there will, in actuality, be
two possible choices (at least) of $\bar{t}_s$ in each cycle of U_{rf} — one where U_{rf} has a
positive slope and one where it has a negative slope. Only one of the two — the
one where the slope is negative — corresponds to a phase of stable equilibrium,
as you will presently see. So only that one will be designated as the synchronous
phase $\bar{t}_s$. You can also see from Fig. 31 that with a rf harmonic number k there
will be k different synchronous starting times — and therefore, k distinguishable
synchronous electrons. These k synchronous phases correspond to k possible
stored bunches of electrons.

An electron which is moving with a lateral displacement from the design orbit will see somewhat different electric fields than one moving on the design orbit. It is generally true, however, that its energy gain in one complete revolution — the path integral of the electric force — depends very little on the lateral displacements. I shall therefore ignore any dependence of the energy gain on such lateral displacements — whether they are due to energy deviations or to betatron oscillations — and consider only the important variation of the energy gain with the starting time $\bar{t}$.

The circulating position of a synchronous electron provides a convenient reference point for the study of the longitudinal oscillations of the electrons in a bunch. We may indeed refer to the moving position of the synchronous electron as the "center" of the bunch and describe the instantaneous azimuthal position of any other electron of the bunch by giving its <u>longitudinal displacement</u> y from the bunch center. That is, we define

$$y(t) = s(t) - s_c(t) \tag{3.26}$$

where s is the azimuthal position of any particular electron and s_c refers to the position of the bunch center. See Fig. 32.
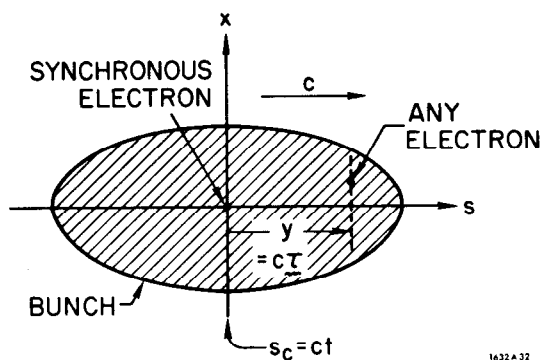
FIG. 32--The longitudinal coordinates y and τ of an electron in a bunch.

For the present discussion I find it somewhat more convenient to describe the longitudinal motion by an equivalent variable τ defined simply by

$$\tau(t) = y(t)/c \tag{3.27}$$

which I shall call the <u>time displacement</u> from the center of the bunch. (The time displacement is very nearly equal to the time interval Δt between the arrival of an electron at any particular azimuth and the arrival of the synchronous electron.

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The difference is equal to the change of τ in the time $\Delta t = \tau$ which because of the slow rate of change of τ can be ignored.) Notice that the time displacement τ is positive when an electron arrives at each azimuth <u>ahead</u> <u>of</u> the synchronous electron.

Because of the time variations of the rf accelerating fields only a synchronous electron will receive the energy U_0 each revolution. Any other electron will gain in one revolution an energy U_{rf} which depends on its time displacement τ. We may follow the conventional notation and write

$$U_{rf} = eV(\tau) \tag{3.28}$$

where e is the electronic charge and $V(\tau)$ is called the "rf voltage" — by analogy with a dc accelerating system. The form of $V(\tau)$ is of course related to $U_{rf}(\tau)$; specifically,

$$eV(\tau) = U_{rf}(\bar{t}_s - \tau) \tag{3.29}$$

The variation with τ is reversed from the variation with $\bar{t}$ so the energy gain function of Fig. 31 would give the $V(\tau)$ shown in Fig. 33 — where now $\tau = 0$
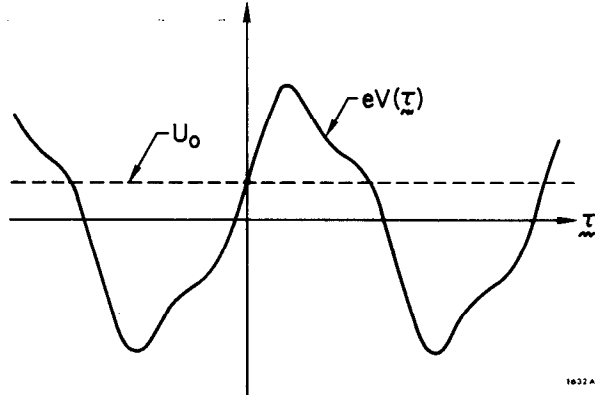
FIG. 33--The rf voltage function $V(\tau)$.

corresponds to the time displacement of a synchronous electron. Notice that the slope of $V(\tau)$ is positive at $\tau = 0$.

It should perhaps be emphasized that the effective "voltage" of a multiple cavity system typical of high energy rings is not simply related to any observable electric "voltage" but depends on the relative positions and oscillation phases of the various rf cavities in the system. The voltage $V(\tau)$ may in fact, depend on the <u>sense</u> of circulation around the ring and may therefore, be quite different for electrons circulating one way around the ring and positrons circulating in the opposite direction.

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We are now ready to consider the energy oscillations of an electron in a circulating bunch in a storage ring. Let's first see qualitatively what will happen. Suppose an electron has initially the nominal energy E_0 but a positive time displacement τ — so that it is ahead of the synchronous position. The radiation loss depends only on the energy so it will be U_0 each revolution. But the energy gain will be greater than U_0. The electron will gain a little bit of energy each revolution. But an increase in energy will, by Eq. (3.15), cause its revolution time to get longer; and its time advance with respect to the bunch center will, accordingly, begin to decrease. After some revolutions the time displacement will decrease to zero. But, by then, the electron's energy will be higher than the nominal energy E_0 — since the electron has continually been gaining energy — so the time displacement will continue to decrease now toward negative values of τ. At negative values of τ however, the energy gain will be too small to compensate for the energy loss by radiation and the electron's energy will begin to decrease toward the nominal energy. When the nominal energy is reached, the time displacement will stop decreasing; but, since it is then negative the energy gain per revolution is below U_0 and the energy will begin dropping below E_0. Now the time displacement will begin returning toward zero. The process will continue until τ returns to its starting value, at which point the energy will again be E_0.

Let's put this description into quantitative terms. First, take the variation of the time displacement τ. It is convenient to keep track of what is happening by observing a bunch once each revolution when the bunch center is at some arbitrarily chosen reference point. The discussion will be easiest if we take the reference point in some field free region (away from any magnets or rf cavities). In Fig. 34 I show two "pictures" of the same bunch on two successive passages of the reference azimuth. In each picture the bunch center is at the reference azimuth so the time between the two pictures is just T_0 the revolution time on the design orbit.

The pictures show also the position of some particular electron of the bunch: "Electron A." In the first picture Electron A is ahead of the bunch center by the distance y_1. In the second picture the longitudinal displacement has decreased to y_2. Between the two pictures the bunch center has traveled once around the design orbit, a distance $L = cT_0$. And since Electron A travels also at the speed c, it also has covered a path length equal to L. But if it has an energy deviation ϵ from the nominal energy, the path length for one complete revolution (back to y_1) —
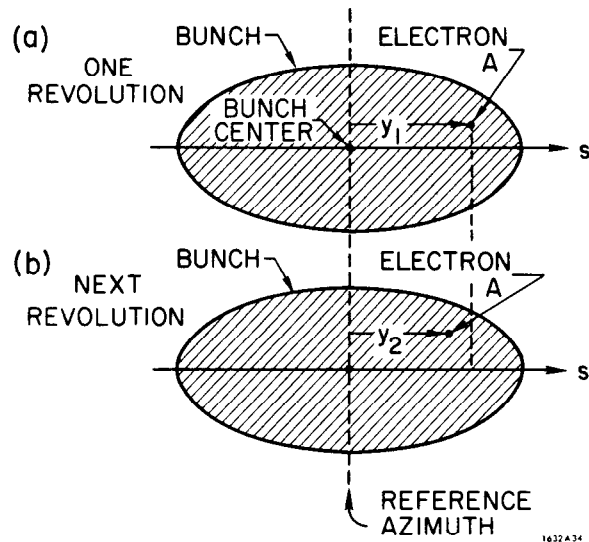
FIG. 34--Longitudinal motion of an electron within a bunch.

would be, as shown in Section 3.2, greater than L by the amount $\delta\ell$ with

$$\frac{\delta\ell}{L} = \alpha \frac{\epsilon}{E_0} .$$

Electron A fails to reach its previous azimuth by the small distance $\delta y = -\delta\ell$, so

$$y_2 - y_1 = \delta y = -\alpha \frac{\epsilon}{E_0} L . \tag{3.30}$$

The change in τ during the revolution is

$$\delta\tau = \frac{\delta y}{c} = -\alpha \frac{\epsilon}{E_0} \frac{L}{c} = \alpha \frac{\epsilon}{E_0} T_0 . \tag{3.31}$$

Since the time between the two pictures is T_0 the time rate-of-change of τ is $\delta\tau/T_0$ or

$$\frac{d\tau}{dt} = -\alpha \frac{\epsilon}{E_0} \tag{3.32}$$

A nice simple result.

Next, the energy variation. During its revolution Electron A has lost by radiation the energy $U_{rad}(\epsilon)$ and gained from the rf system the energy $eV(\tau_1)$. The net change in energy during the revolution is then,

$$\delta U = eV(\tau_1) - U_{rad}(\epsilon). \tag{3.33}$$

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The rate-of-change of the energy deviation ϵ — when averaged over a complete revolution — is $\delta U/T_0$ so we have that

$$\frac{d\epsilon}{dt} = \frac{eV(\tau) - U_{rad}(\epsilon)}{T_0} \tag{3.34}$$

(We may drop the subscripts on τ because we may now take it as a continuous variable, obtained by a smooth interpolation from τ_1 to τ_2 to τ_3, etc.)

The two coupled equations, (3.32) and (3.34) describe the energy oscillations — and the associated oscillations of the time displacement — of a stored electron. They must be solved together to give the time variation of ϵ and of τ.

It will turn out — unfortunately — that the time displacements which are associated with small energy deviations need not themselves be "small," in the sense that they may span a significant fraction of a complete cycle of the variation of $V(\tau)$. This will be the one instance in which we may not look only at linear terms. We shall need at times to take into account the full nonlinear variations of $V(\tau)$. At other times, however, we shall wish to focus our attention on the small energy oscillations which correspond also to small time displacements. For such oscillations we shall need to retain only the linear part of the variation of $V(\tau)$. Since the acceleration energy gain at $\tau = 0$ is by definition U_0, we may then write

$$U_{rf} = eV(\tau) = U_0 + e\dot{V}_0\tau \tag{3.35}$$

where $\dot{V}_0$ stands for $dV/d\tau$ evaluated at $\tau = 0$.

It is quite common for the rf voltage of a storage ring to have a sinusoidal variation with time. In such cases we would have that

$$V(\tau) = \hat{V} \sin \omega_{rf}(\tau - \tau_0) \tag{3.36}$$

where $\hat{V}$ is called the "peak rf voltage" and $\omega_{rf}\tau_0$ is called the "synchronous rf phase angle." With our assumptions

$$\omega_{rf} = 2\pi \frac{k}{T_0} = k\omega_r. \tag{3.37}$$

It also follows that

$$\omega_{rf}\tau_0 = \sin^{-1}(U_0/e\hat{V}) \tag{3.38}$$

and that

$$\dot{V}_0 = \omega_{rf}\hat{V} \cos \omega_{rf}\tau_0 = \omega_{rf}\hat{V}\left\{1 - \left(\frac{U_0}{e\hat{V}}\right)^2\right\}^{1/2} \tag{3.39}$$

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3.5   Small Oscillations

We are now ready to analyse in detail the energy oscillations of the electrons in a bunch. I shall take up first the special case of the small (linearized) oscillations which occur so long as the variations of τ are limited to a small interval that corresponds to an approximately linear segment of $V(\tau)$. And then look later (in the following section) at the nonlinear oscillations which occur when the excursions of τ are large.

For small τ and ϵ, we may replace $V(\tau)$ and $U_{rad}(\epsilon)$ by the linear approximations of Eqs. (3.35) and (3.23). Then Eq. (3.24) becomes

$$\frac{d\epsilon}{dt} = \frac{1}{T_0} (e\dot{V}_0\tau - D\epsilon) \tag{3.40}$$

This equation can now be combined with Eq. (3.32) to give a differential equation for ϵ or τ. Suppose we choose τ. Taking the time derivative of Eq. (3.32) and eliminating ϵ, you can show that

$$\frac{d^2\tau}{dt^2} + 2\alpha_\epsilon \frac{d\tau}{dt} + \Omega^2\tau = 0, \tag{2.41}$$

with[†]

$$\alpha_\epsilon = \frac{D}{2T_0}, \tag{3.42}$$

$$\Omega^2 = \frac{\alpha e \dot{V}_0}{T_0 E_0} \tag{3.43}$$

You will recognize that Eq. (2.43) describes a damped harmonic oscillation with the oscillation (angular) frequency Ω, and damping coefficient α_ϵ. Since the damping rate in a storage ring is always slow ($\alpha_\epsilon \ll \Omega$) the solution of Eq. (3.41) can be written as

$$\tau(t) = A e^{-\alpha_\epsilon t} \cos(\Omega t - \theta_0) \tag{3.44}$$

[†] Careful! There are not enough different letters. The constant α_ϵ is a new quantity quite distinct from the dilation factor α.

with A and θ_0 arbitrary constants. Or, using the usual complex notation

$$\underset{\sim}{\tau}(t) = \tilde{\tau} \, e^{-(\alpha_\epsilon - i\Omega)t} \qquad (3.45)$$

where τ is a complex constant.

Equations (3.40) and (3.32) can be solved instead for $\underset{\sim}{\epsilon}$, which, you can show, satisfies the same differential equation as $\underset{\sim}{\tau}$, Eq. (3.41). And so the time variations of $\underset{\sim}{\epsilon}$ are

$$\underset{\sim}{\epsilon}(t) = \tilde{\epsilon} e^{-(\alpha_\epsilon - i\Omega)t} \qquad (3.46)$$

From Eq. (3.32) $\tilde{\epsilon}$ and $\tilde{\tau}$ are related by

$$\tilde{\epsilon} = -i \frac{\Omega E_0}{\alpha} \tilde{\tau} \qquad (3.47)$$

(because $\alpha_\epsilon \ll \Omega$) and so the oscillations of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\tau}$ will have a phase difference of $\pi/2$.

Notice that the oscillation frequency of the small energy oscillations depends on the rf system only through $\dot{V}_0$. The frequency is proportional to the square root of the rf slope at the synchronous phase. The other parameters, α, T_0, E_0 are characteristics of the guide field (including the energy at which it is operated). The damping constant of the energy oscillations α_ϵ — which is the inverse of the damping time constant — is proportional to D, which is the rate-of-change of the radiation loss with energy. As we shall see, this rate depends on the electron energy and on the properties of the guide field.

I would like to give now some orders of magnitude for the various quantities which have been appearing. The skeptical among you may then be happier about the approximations which have been made. A storage ring for 1 GeV electrons might have the following typical magnitudes for the various (angular) frequencies:

$$\omega_r = 2\pi/T_0 \approx 10^7 \; \text{sec}^{-1}$$

$$\omega_\beta = \nu \, \omega_r \approx 3 \, \omega_r$$

$$\Omega \approx 10^4 \; \text{sec}^{-1}$$

$$\alpha \approx 10 \; \text{sec}^{-1}$$

The large ratios ω_r/Ω and Ω/α_ϵ justify the approximations we have been making.

In the absence of damping $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\tau}$ are conjugate variables. In a "phase diagram," where $\underset{\sim}{\epsilon}$ is plotted versus $\underset{\sim}{\tau}$, the oscillations are described by a point

which moves cyclicly around an ellipse. See Fig. 35(a). The ratio of the two
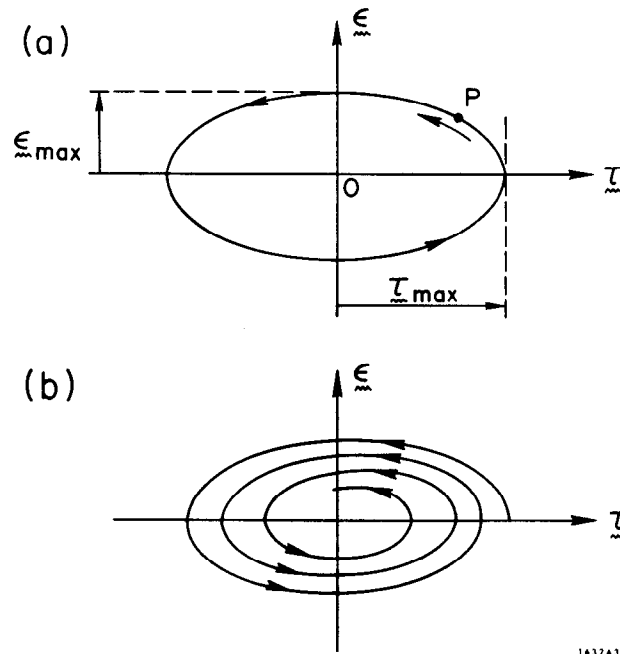
FIG. 35--Phase diagram for energy oscillations. (a) Without damping.
(b) With damping. (The damping rate is very much exaggerated.)

semimajor axes of the ellipse would be — by Eq. (3.45)

$$\frac{\underset{\sim}{\epsilon}\, \text{max}}{\underset{\sim}{\tau}\, \text{max}} = \frac{|\epsilon|}{|\tilde{\tau}|} = \frac{\Omega E_0}{\alpha} \tag{3.47}$$

If the scales are chosen so that the ellipse becomes a circle, the reference point
rotates at the constant angular frequency Ω. With damping, the size of the ellipse
decreases slowly and the phase trajectory is a slow inward spiral as indicated
crudely in Fig. 35(b). The phase diagram also makes transparent why the damping
depends on dU_{rad}/dE. If this derivative is positive, the electron is losing a little
extra amount of energy while on the upper half of the ellipse, and gaining a little
extra energy while on the lower half. So it is always "drifting" toward the axis of
$\underset{\sim}{\tau}$ and the oscillation amplitude is decreasing — in proportion to dU_{rad}/dE.

According to our solution, the energy oscillations of all electrons should
ultimately be completely damped out and they should all end up on top of the syn-
chronous electron. But we have not yet taken into account the excitation of the
oscillations by the quantum effects which "shake up" the oscillations and prevent
them ever from going completely to zero. (They are considered in the next part.)
Under stationary conditions any stored electron will typically be found with some

residual oscillation amplitude in which there is a balance between the excitation and the damping. Since both of these processes are slow we may think of the energy oscillation during any brief time as being described by a fixed phase ellipse such as the one in Fig. 34(a).

I should also remind you that the energy oscillations relate not only to the longitudinal oscillations (in y or τ) of the electrons in a bunch but have also a lateral component. According to Eq. (3.3) an energy deviation ϵ results in a radical displacement x_ϵ which is proportional to ϵ — and in phase with it. So the component x_ϵ of the total horizontal displacement oscillates in synchronism with the energy oscillations. Generally, this transverse manifestation of the energy oscillations has (under stationary conditions) about the same amplitude as the betatron oscillations.

3.6   Large Oscillations; Energy Aperture

A storage ring guide field can usually accept only a small range of energies — typically only a few percent of the nominal energy — and the magnetic focussing forces are usually reasonably linear over the whole energy range. Even much smaller energy deviations however, may correspond to rather large oscillations of the time displacement τ. I mean by "large" oscillations those for which $V(\tau)$ departs significantly from a linear dependence on τ. Such large amplitudes may typically occur when the peak rf voltage is not very much larger than the radiation loss (as is usually the case at very high energies) or when the rf harmonic number k is very large. We should take at least a brief look at the large amplitude oscillations because they are generally responsible for determining the energy "aperture"— or "acceptance" — of a ring. Please keep in mind however, that although we shall be dealing with "large" time displacements — which may encompass a major fraction of an rf period — the maximum energy deviations will still be "small," a very small fraction of the energy itself.

We may begin with the two basic results of the preceding section, Eqs. (3.32) and (3.34). As before, we replace U_{rad} by $U_0 + D\epsilon$, since the energy deviations remain small. But we must retain $V(\tau)$ without any simplification. We get for Eq. (3.34)

$$\frac{d\epsilon}{dt} = \frac{eV(\tau) - U_0}{T_0} - \frac{D\epsilon}{T_0} \tag{3.48}$$

If we now express both ϵ and its time derivative in terms of τ by using Eq. (3.32) we get the following equation.

$$\frac{d^2\tau}{dt^2} = - \frac{\alpha}{E_0 T_0} \left\{ eV(\tau) - U_0 \right\} - \frac{D}{T_0} \frac{d\tau}{dt} \qquad (3.49)$$

This equation describes the variation of τ for all amplitudes.

I now ask you to look at another equation which is probably familiar to you:

$$m \frac{d^2 x}{dt^2} = F(x) - \mu \frac{dx}{dt} \qquad (3.50)$$

It represents the motion in one dimension, x, of a particle of mass m, which moves in a conservative force field $F(x)$, and suffers a frictional drag force proportional to its speed. We can understand Eq. (3.49) by making a direct comparison between it and Eq. (3.50). The motion in ϵ is <u>exactly</u> like the motion of a particle of unit mass which moves in the conservative force field

$$F(\tau) = - \frac{\alpha}{E_0 T_0} \left\{ eV(\tau) - U_0 \right\}, \qquad (3.51)$$

and which is subject to a frictional drag proportional to the velocity with a drag coefficient D/T_0.

The motion in τ can, in general, only be evaluated by a numerical computation. We can however, get a good heuristic idea of the motion by considering first what happens if the friction term is zero. It is small anyway and can be taken into account later as a perturbation. We wish to study the motion

$$\frac{d^2\tau}{dt^2} = F(\tau) \qquad (3.52)$$

with $F(\tau)$ given by Eq. (3.51). As you know such an equation is often handled by defining a "potential energy" function $\phi(\tau)$ which is the negative of the integral of the force. Let's define

$$\phi(\tau) = \frac{\alpha}{E_0 T_0} \int_0^\tau \left\{ eV(\bar{\tau}) - U_0 \right\} d\bar{\tau}. \qquad (3.53)$$

We can then analyze the motion by evoking the principle of conservation of "energy." At each instant the sum of the "potential energy" $\phi(\tau)$ and the "kinetic energy" — here $\frac{1}{2}(d\tau/dt)^2$ — must be a constant, the "total energy." The total energy is also the maximum ϕ_0 that can be reached by $\phi(\tau)$ — which will occur when $d\tau/dt$

is zero — so we may write that

$$\frac{1}{2}\left(\frac{d\tau}{dt}\right)^2 = \phi_0 - \phi(\tau) \tag{3.54}$$

Suppose that the energy gain function $eV(\tau)$ has the form shown in Fig. 36(a) and that the synchronous energy gain U_0 is as shown there. Then $\phi(\tau)$ will be as drawn in part (b) of the figure. The form shown is quite typical. Notice that there is a general downward trend of $\phi(\tau)$ with an average slope of $-U_c$. This must occur because the rf accelerating fields must integrate to zero over each complete cycle (at least over each complete cycle of the lowest frequency present).

You can now visualize the nature of the time displacement oscillations. The motion is like that of a point particle (an "electron") which slides around "on" the hilly surface represented by $\phi(\tau)$ — where you must of course, think of τ as a horizontal spatial coordinate. First, there is a potential minimum at $\tau = 0$. If you place an electron there it remains stationary; it is a "synchronous electron."[†] If however, you place an electron at τ_1 — so that it is at point A on the hill — it will slide down the hill and coast up the other side to point B. Both A and B are at the same height $\phi_0 = \phi(\tau_A)$. At τ_A and τ_B the "kinetic energy" will be zero. The kinetic energy will reach its maximum value as the electron passes $\tau = 0$. At each τ the kinetic energy is given by Eq. (3.54) and from it we can obtain the "velocity" at each τ:

$$\frac{d\tau}{dt} = \pm \sqrt{2}\left[\phi_0 - \phi(\tau)\right]^{1/2} \tag{3.55}$$

Remember, now, that according to Eq. (3.32) the "velocity" is

$$\frac{d\tau}{dt} = -\alpha \frac{\epsilon}{E_0}$$

so that the energy deviation (of the real electron) at each τ is given by

$$\frac{\epsilon(\tau)}{E_0} = \mp \frac{\sqrt{2}}{\alpha}\left[\phi_0 - \phi(\tau)\right]^{1/2} \tag{3.56}$$

[†] There are of course, stationary points at each potential minimum and these correspond to the synchronous electrons at the centers of other bunches (so long as $\tau < T_0$).
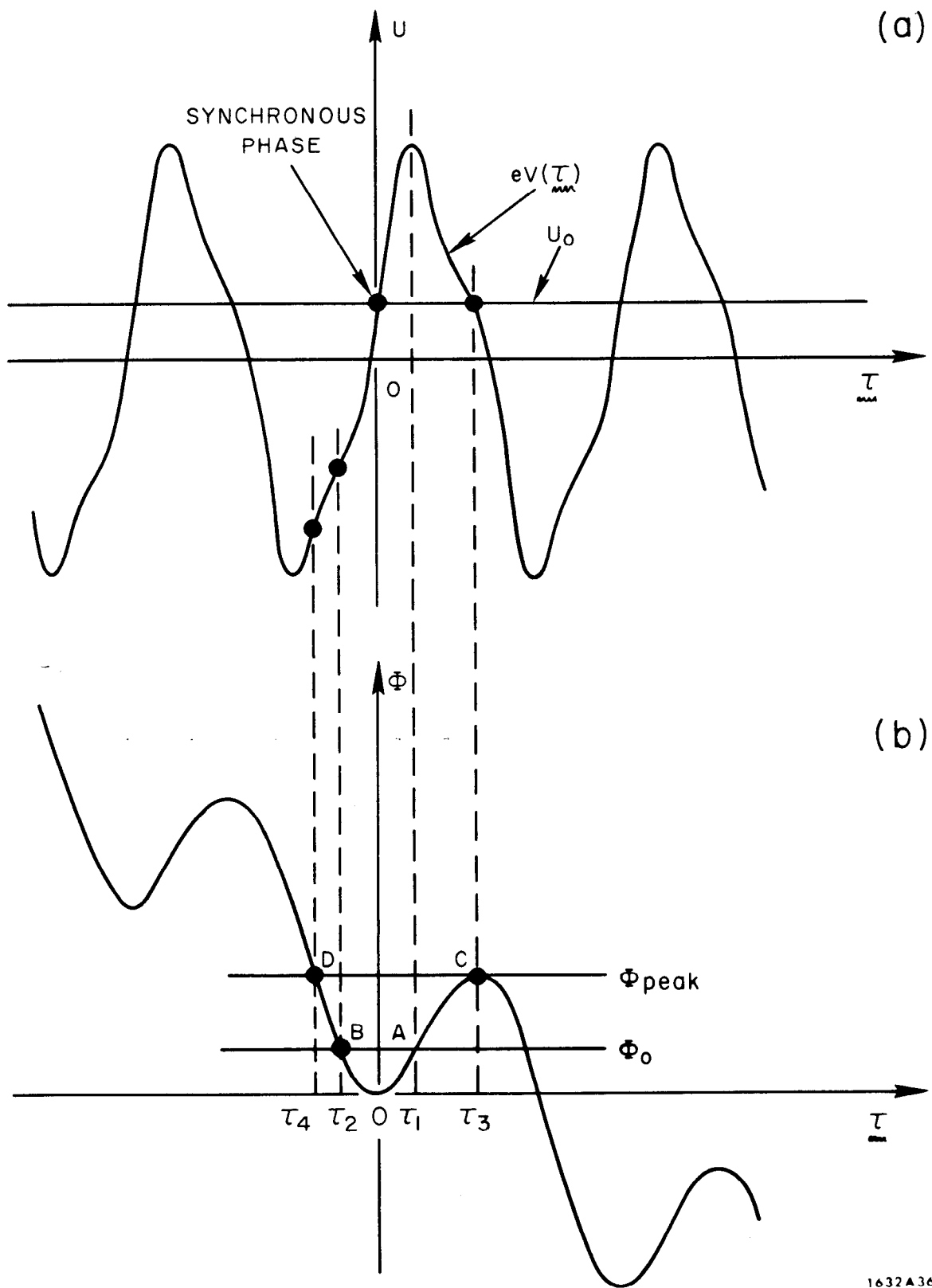
FIG. 36--(a) The rf acceleration function eV(τ), and (b) the effective potential energy function $\phi(\tau)$.

you can easily see that if you plot a phase diagram — ϵ versus τ — you will get a more-or-less elliptical curve much like the curve a drawn in Fig. 37. You
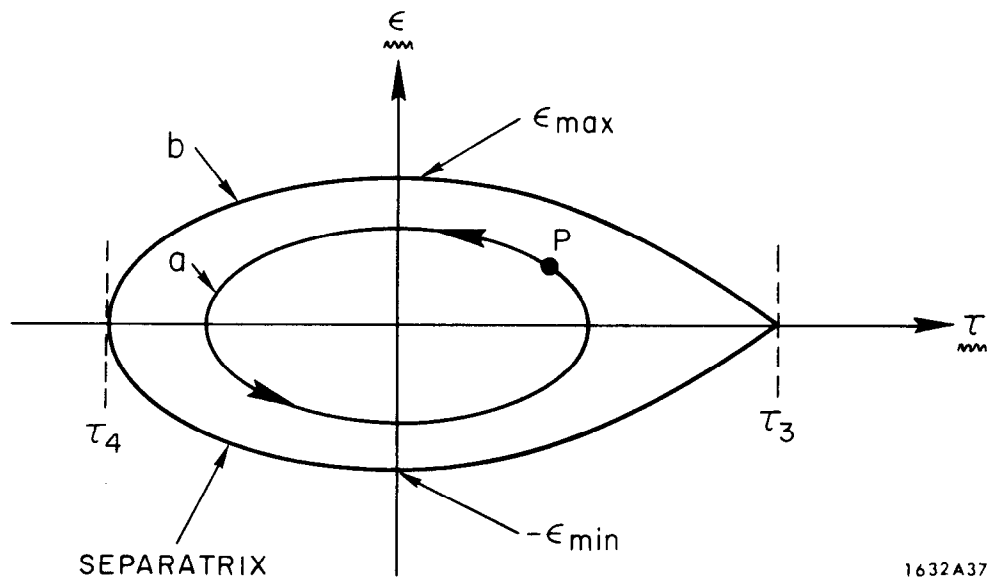
FIG. 37--Phase diagram for large oscillations. Bounded energy
oscillations occur only inside of the separatrix.

must only use your common sense to choose the proper sign for the square root on each half cycle.

Also you can see what would happen if you were now to include the friction term — the radiation damping. During each oscillation cycle a small amount of energy would be lost in a resulting decrease of the total "energy." (You could even estimate this loss by, say, approximating the motion by a sinusoid.)

It should also be apparent that there will be a <u>maximum amplitude</u> of a stable (periodic) oscillation of τ. It occurs when the electron can just reach the peak of the hill at τ_3 — corresponding to the point C in Fig. 35(b) — where $\phi(\tau)$ takes on the value ϕ_{max}. An electron with any larger amplitude will sail on over the peak and on into the next valley where it will have so much "kinetic energy" that it will keep on going forever — until it is lost from the storage ring.

The maximum stable oscillation goes back and forth between the points C and D. Notice that the point C is also where $eV(\tau)$ is again equal to U_0. (To the left of C the real electron always gains energy and may have some hope of returning to the origin of τ). The other extreme of the oscillation at point D has no special quality except that $\phi(\tau)$ is again equal to ϕ_{max}, the value at C. The phase diagram

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of the extreme oscillation is a little peculiar, since both the velocity <u>and</u> the acceleration go to zero at C but not at D. The electron "lingers" at C — in the ideal case for an infinite time! As a result the phase diagram will have a "corner," as shown by the curve b of Fig. 37. This special curve is called the <u>separatrix</u> because it separates the stable oscillations from the unstable trajectories. An electron injected into a storage ring with a certain energy deviation ϵ and time displacement τ corresponding to the point P in Fig 37 will circulate on a more-or-less elliptical closed curve (neglecting damping). If an electron is injected at a point outside the separatrix it is "lost."

You can now see how the rf system can determine the energy aperture of a storage ring. Energy deviations larger than $\pm \epsilon_{max}$ — of Fig. 37 — cannot be held in the storage ring. Electrons may be lost at smaller energy deviations if the lateral displacements x_ϵ associated with ϵ cause the electron to collide with some physical obstruction that limits the radial aperture. Normally, however, the rf limitation sets in first and the energy aperture is $\pm \epsilon_{peak}$. From Eq. (3.56)

$$\frac{\epsilon_{max}}{E_0} = \frac{1}{\alpha} (2 \phi_{max})^{1/2} \tag{3.57}$$

If you work out $\phi(\tau)$ for the special case of an rf voltage function that is a pure sinusoid — as described by Eq. (3.36) — you will find that

$$\phi_{max} = \frac{\alpha U_0}{2\pi k E_0} F(q) \tag{3.58}$$

in which

$$q = e\hat{V}/U_0 \tag{3.60}$$

is the <u>overvoltage</u> — namely the ratio of the peak rf voltage to the minimum voltage required to store a synchronous electron — and

$$F(q) = 2 \left\{ \sqrt{q^2 - 1} - \cos^{-1}(1/q) \right\} \tag{3.60}$$

The energy aperture ϵ_{max} for this case is then given by

$$\left(\frac{\epsilon_{max}}{E_0} \right)^2 = \frac{U_0}{\pi \alpha k E_0} F(q) \tag{3.61}$$

The aperture function F(q) is plotted in Fig. 38. Notice that for large q

$$F(q) \rightarrow 2q - \pi. \qquad (3.62)$$
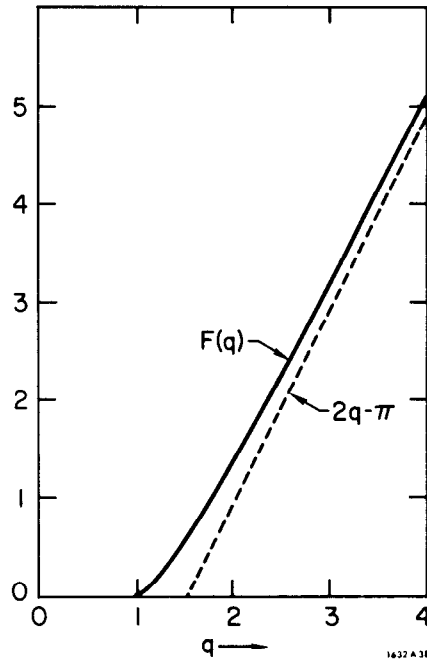
FIG. 38--The energy aperture function F(q).

Finally if you think about what happens if you start an electron outside of the energy aperture — say at points above the point D on the curve of $\phi(\tau)$ in Fig. 36(b) and figure out what their phase trajectories will be you will see that they become curves like the ones drawn in Fig. 39. Three successive separatrices are shown
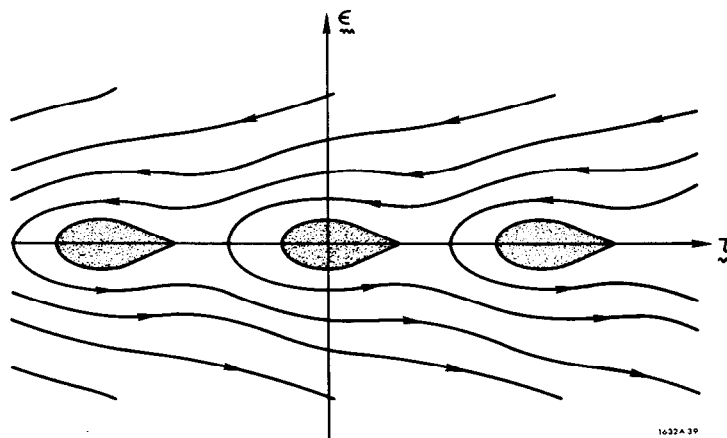
FIG. 39--Phase trajectories for electrons not captured in a bunch. (A qualitative sketch.)

and several examples of unstable trajectories. Again you see that an electron once outside a stable region will — barring a fortunate accident — stay outside forever.

IV. RADIATION DAMPING[†]

4.1. <u>Energy Loss</u>

A relativistic electron which is accelerated in a macroscopic force field will radiate electromagnetic energy at a rate which is proportional to the square of the accelerating force. The rate depends on the angle between the force and the electrons velocity and is larger by the factor $\gamma^2 = (E/mc^2)^2$ when the force is perpendicular to the velocity than when the force is parallel to the velocity. In a storage ring the typical longitudinal forces (from the accelerating system) are much smaller than the typical transverse magnetic forces and γ^2 is a large number indeed, so we need consider only the radiation effects that accompany the magnetic forces.

Let P_γ stand for the rate of loss of energy by radiation; it may be written

$$P_\gamma = \frac{2}{3} \frac{r_e c}{(mc^2)^3} E^2 F_\perp^2 \qquad (4.1)$$

where m is the rest mass of the electron, r_e is the classical electron radius, and $F_\perp$ is the magnetic force on the electron. It will be convenient to define the constant

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(mc^2)^3} = 8.85 \times 10^{-5} \text{ meter-GeV}^{-3} \qquad (4.2)$$

Then, since $F_\perp = ecB$, the radiated power is

$$P_\gamma = \frac{e^2 c^3}{2\pi} C_\gamma E^2 B^2. \qquad (4.3)$$

This instantaneous power is proportional to the square of both the energy and the local magnetic field strength. It is sometimes useful to express the magnetic force in terms of the local radius of curvature ρ of the trajectory; then

$$P_\gamma = \frac{cC_\gamma}{2\pi} \frac{E^4}{\rho^2} \qquad (4.4)$$

[†]I shall assume that you are familiar with the classical theory of electromagnetic radiation by relativistic electrons (see e.g., Ref. 10) and will only review briefly the results needed for our purposes.

An electron circulating on the design orbit has the nominal energy E_0 and moves on the radius $\rho_s = 1/G$ — see Section 2.2. To find the energy U_0 radiated in one revolution we must integrate P_γ with respect to time once around the ring. Since $dt = ds/c$,

$$U_0 = \frac{C_\gamma E_0^4}{2\pi} \oint G^2(s)\, ds. \qquad (4.5)$$

We may write the integral as the mean of G^2 multiplied by $L = 2\pi R$, the distance around the ring:

$$U_0 = C_\gamma E_0^4 R \langle G^2 \rangle. \qquad (4.6)$$

For an isomagnetic guide field[†] $G = G_0 = 1/\rho_0$ along the curved parts of length $2\pi\rho_0$ and zero elsewhere, so

$$\langle G^2 \rangle = \frac{G_0}{R} = \frac{1}{R\rho_0} \quad \text{(isomag)} \qquad (4.7)$$

and

$$U_0 = \frac{C_\gamma E_0^4}{\rho_0} \qquad (4.8)$$

For a <u>fixed</u> radius ρ_0, the energy radiated per turn varies as the <u>fourth</u> power of the electron energy. A 1 GeV electron moving on a 5 meter radius looses 17 keV each revolution.

The average <u>power</u> radiated is U_0/T_0 where $T_0 = c/2\pi R$ is the time elapsed during one revolution. For the general guide field

$$\langle P_\gamma \rangle = \frac{cC_\gamma}{2\pi} E_0^4 \langle G^2 \rangle \qquad (4.9)$$

And for an isomagnetic ring,

$$\langle P_\gamma \rangle = \frac{cC_\gamma}{2\pi} \frac{E_0^4 G_0}{R} = \frac{cC_\gamma E_0^4}{L \rho_0} \quad \text{(isomag)} \qquad (4.10)$$

An electron that is not on the ideal orbit radiates at a different rate. Consider first an electron that has the nominal energy E_0 but is circulating with a betatron

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[†]See Section 2.2.

oscillation. It's rate of radiation will be different from an electron moving on the design orbit only because it moves through a slightly different magnetic field — due to its betatron displacement. But at each azimuth its displacement is equally often positive or negative. And we have assumed that the fields have only a linear variation with displacement. So to first order in the betatron amplitude the radiated power averaged over a betatron cycle is the same as that of an electron on the design orbit.

The same is not true of an electron with an energy different from E_0. That case will be analyzed in the next section.

For ultra-relativistic electrons the radiation is emitted primarily along the direction of motion. Most of the radiation is emitted within the angle $1/\gamma$. The radiation reaction force — and therefore, the accompanying momentum change — is exactly opposite to the direction of motion.[†] The only effect of the radiation is then to decrease the energy of the electron without changing its direction of motion.

4.2. Damping of the Energy Oscillations

In Section 3.5 we saw that small energy oscillations were damped at a rate proportional to the change of the radiation loss with energy. From Eqs. (2.43) and (3.24) the damping coefficient α_ϵ is

$$\alpha_\epsilon = \frac{D}{2T_0} = \frac{1}{2T_0}\left(\frac{dU_{rad}}{dE}\right)_0 \tag{4.11}$$

where U_{rad} is the energy loss per revolution. When the energy of an electron deviates from the nominal energy E_0, the energy radiated in one revolution changes in part because of the energy change, in part because the electron travels in a different magnetic field and in part because its path length is different. Let's look at how dU_{rad}/dE may be evaluated.

We have already seen that a betatron oscillation does not, to first order, change the average power radiated. So to get U_{rad} at any energy we must merely integrate the P_γ of Eq. (4.3) with respect to time around one complete off-energy closed orbit. It will, however, be convenient to change the variable of integration

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[†]Neglecting quantum effects; see Section 5.1.

to s. Then

$$U_{rad} = \oint P_\gamma dt = \oint P_\gamma \frac{dt}{ds}\, ds. \qquad (4.12)$$

We have earlier evaluated dt/ds, see Eq. (2.15):

$$\frac{dt}{ds} = \frac{1}{c}\left(1 + \frac{x}{\rho_s}\right),$$

where x is the displacement from the design orbit and $\rho_s(s)$ is the radius of cur-vature of the design orbit. Since we are now interested in the energy loss on an off-energy closed orbit we should take $x = \eta \epsilon / E_0$, where $\epsilon = E - E_0$ and $\eta(s)$ is the off-energy function. See Eq. (2.28). Then

$$U_{rad} = \frac{1}{c}\oint\left(1 + \frac{\eta}{\rho}\frac{\epsilon}{E_0}\right) P_\gamma\, ds. \qquad (4.13)$$

We have already looked at this integral for $\epsilon = 0$; it is just U_0. So let's differentiate now, evaluating the derivative at $\epsilon = 0$.

$$\frac{dU_{rad}}{dE} = \frac{1}{c}\oint\left\{\frac{dP_\gamma}{dE} + \frac{\eta}{\rho}\frac{P_\gamma}{E}\right\}_0 ds, \qquad (4.14)$$

where the subscript "0" on the curly brackets means that all quantities in the integrand are to be evaluated on the design orbit, and at the energy E_0. From Eq. (4.3) P_γ is proportional to the product $E^2 B^2$ — and remember that when E changes, the orbit moves to a different location so that B also changes. We may then write that

$$\frac{dP_\gamma}{dE} = 2\frac{P_\gamma}{E_0} + 2\frac{P_\gamma}{B_0}\frac{dB}{dE}.$$

But

$$\frac{dB}{dE} = \frac{dx}{dE}\frac{dB}{dx} = \frac{\eta}{E_0}\frac{dB}{dx},$$

where dB/dx is a property of the guide field. Putting these last two together and into Eq. (4.14)

$$\frac{dU_{rad}}{dE} = \frac{1}{c}\oint\left\{2\frac{P_\gamma}{E} + 2\frac{P_\gamma}{B}\frac{\eta}{E_0}\frac{dB}{dx} + \frac{P_\gamma}{E}\frac{\eta}{\rho}\right\}_0 ds$$

The integral of the first term yields just $2U_0/E_0$ so our result for the variation of the radiated energy is

$$\frac{dU_{rad}}{dE} = \frac{U_0}{E_0} \left[2 + \frac{1}{cU_0} \oint \left\{ \eta P_\gamma \left(\frac{1}{\rho} + \frac{2}{B} \frac{dB}{dx} \right) \right\}_0 ds \right] \qquad (4.15)$$

We may now write for the damping constant:

$$\alpha_\epsilon = \frac{1}{2T_0} \frac{dU_{rad}}{dE} = \frac{U_0}{2T_0 E_0} (2 + \mathscr{D}) \qquad (4.16)$$

with

$$\mathscr{D} = \frac{1}{cU_0} \oint \left\{ \eta P_\gamma \left(\frac{1}{\rho} + \frac{2}{B} \frac{dB}{dx} \right) \right\}_0 ds \ . \qquad (4.17)$$

Taking P_γ and U_0 from Eqs. (4.3) and (4.8) and expressing B and dB/dx in terms of G(s) and K_1(s) as defined in Section 2.2, we may rewrite $\mathscr{D}$ as

$$\mathscr{D} = \frac{\oint \eta G \left(G^2 + 2K_1 \right) ds}{\oint G^2 ds} \qquad (4.18)$$

This form makes clearer the fact that $\mathscr{D}$ is just a number which is a property of the total guide field configuration — obtained from integrations around the ring of expressions involving only the guide field functions G, K_1, and η. The number $\mathscr{D}$ is typically a positive number quite a bit smaller than 1.

Equation (4.16) has a nice physical interpretation. Since $\mathscr{D}$ is usually small we have the approximate relation:

$$\alpha_\epsilon \approx \frac{U_0}{E_0 T_0} = \frac{\langle P_\gamma \rangle}{E_0} \qquad (4.19)$$

where $\langle P_\gamma \rangle$ is the average rate of energy loss. The damping <u>time constant</u> for energy oscillations — which is the inverse of α_ϵ — is just <u>the time</u> it <u>takes an electron to radiate away its total energy</u>!

The expression above for $\mathscr{D}$ becomes simpler if the guide field is isomagnetic. Then G(s) is either zero or equal to some constant in the magnets and the integrals extend only over the magnets. Equation (4.18) becomes

$$\mathscr{D} = \frac{1}{2\pi} \int_{Mag} \eta(s) \left\{ G_0^2 + 2K_1(s) \right\} ds \quad (isomag) \qquad (4.20)$$

If the guide field is also "separated function," the magnets have no gradients and

$$\mathscr{D} = \frac{G_0^2}{2\pi} \int_{\text{Mag}} \eta(s) \, ds \qquad \binom{\text{isomag.}}{\text{sep. func.}} \qquad (4.21)$$

The integral is familiar; it appeared earlier when we calculated the dilation factor α for an isomagnetic guide field. Using Eqs. (3.13) and (3.14)

$$\mathscr{D} = G_0 \langle \eta \rangle_{\text{Mag}} = G_0 \alpha R = \frac{\alpha R}{\rho_0} \qquad \binom{\text{isomag.}}{\text{sep. func.}} \qquad (4.22)$$

For this type of ring, the number $\mathscr{D}$ is just the dilation factor α increased by the ratio of the gross orbit radius R to the magnetic radius ρ_0. Typical values for these parameters of a ring might be:

$$\alpha \approx 0.05; \qquad R/\rho_0 \approx 3; \qquad \mathscr{D} \approx 0.15.$$

Recapitulating, for energy oscillations in an isomagnetic, separated function guide field, the damping coefficient for energy oscillations is

$$\alpha_\epsilon = \frac{\langle P_\gamma \rangle}{2E_0} \left(2 + \frac{\alpha R}{\rho_0} \right) \qquad \binom{\text{isomag.}}{\text{sep. func.}} \qquad (4.23)$$

4.3. Damping of Betatron Oscillations

It is now time to take a look at the so-called radiation damping of the betatron oscillations. I shall give here only an approximate treatment, but using a method which can — with only a bit of tedious algebra — be extended to an exact calculation. The exact result is, in any case, obtained more easily by a general theorem that will be discussed in the next section.

Let's look first at the vertical betatron oscillations. (The notation will be the one used in Part II.) I shall approximate the motion by ignoring the variation of β with s, then I may write (see Section 2.8)

$$z = A \cos \phi, \qquad z' = \frac{A}{\beta} \sin \phi, \qquad (4.24)$$

where ϕ is s/β. The amplitude A of the oscillations can be obtained from z and z' at any instant by

$$A^2 = z^2 + (\beta z')^2 \qquad (4.25)$$

Suppose we are looking at an electron of energy E_0 — which is then oscillating vertically about the design orbit. In any element of azimuth δs the electron will lose by radiation the small amount of energy δE. Its momentum vector $\underset{\sim}{p}$ will be changed by $\delta\underset{\sim}{p}$ and, as was remarked earlier, $\delta\underset{\sim}{p}$ is parallel (and opposite) to $\underset{\sim}{p}$, so $|\delta\underset{\sim}{p}| = c\delta E$. See Fig. 40(a). The radiation loss does not change either the
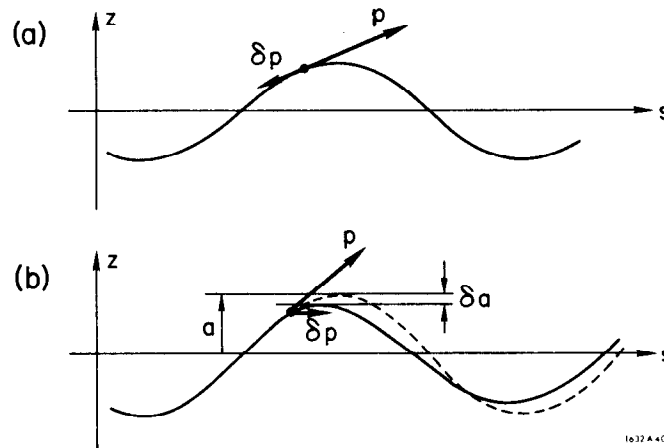
FIG. 40--Effect of an energy change on the vertical betatron oscillations:
(a) for radiation loss, (b) for rf acceleration.

displacement or the slope of the trajectory; and so the amplitude A is <u>unchanged</u> <u>by the radiation</u>. (There is a small effect due to the fact that the effective focussing forces and, therefore, also β are changed with a change of energy but this so-called "adiabatic" damping effect is of second order and can, anyway, be neglected since the energy is not changing on the average when the rf acceleration is also taken into account.)

Notice now, that the effect of the rf accelerating force is quite different. This force is, on the average, parallel to the design orbit. Then the momentum increment $\delta\underset{\sim}{p}$ received in the azimuthal element δs is no longer exactly parallel to $\underset{\sim}{p}$. See Fig. 40(b). Let's write $p_\perp$ for the component of $\underset{\sim}{p}$ perpendicular to the design orbit; then, since the angles are small we may write

$$z' = \frac{p_\perp}{p}$$

Again, the accelerating force doesn't change z. But now it does change z' which goes over to

$$z' - \frac{p_\perp}{p+\delta p} = \frac{p_\perp}{p}\left(1 - \frac{\delta p}{p}\right) = z'\left(1 - \frac{\delta p}{p}\right). \tag{4.26}$$

The change in z' is

$$\delta z' = - z' \frac{\delta p}{p} = - z' \frac{\delta E}{E} . \tag{4.27}$$

There is a corresponding change in the amplitude A;

$$A \delta A = \beta^2 z' \, \delta z' = -(\beta z')^2 \frac{\delta E}{E} \tag{4.28}$$

Now the phase of the oscillation at the arrival of the electron at the point s is arbitrary (and all values between 0 and 2π are equally probable) so we should inquire only about the average change in A. The average of $(z')^2$ is $A^2/2\beta^2$, so

$$A \langle \delta A \rangle = - \frac{A^2}{2} \frac{\delta E}{E_0} \tag{4.29}$$

Suppose we now sum over all the elements of acceleration gain in one revolution. Since all of the δE must add up to the radiation loss U_0, we find for the change ΔA that occurs in A during one revolution (due to the rf acceleration):

$$\frac{\Delta A}{A} = - \frac{U_0}{2E_0} \tag{4.30}$$

Since ΔA in each revolution time T_0 is proportional to A, the motion is exponentially damped — as $e^{-\alpha_z t}$. That is,

$$\frac{1}{A} \frac{dA}{dt} = \frac{\Delta A}{A T_0} = - \frac{U_0}{2E_0 T_0} \tag{4.31}$$

so the damping coefficient is

$$\alpha_z = \frac{U_0}{2E_0 T_0} = \frac{\langle P_\gamma \rangle}{2E_0} . \tag{4.32}$$

You can show that an exact calculation — using the full-blown form for the vertical betatron oscillation — yields the same result. Notice that the damping rate for the vertical oscillations is just 1/2 the typical rate for the energy oscillations (when $\mathscr{D}$ is small); see Eq. (4.20).

It is amusing to notice that the "radiation" damping does not occur in the radiation process, but rather in the process of energy gain from the rf system. One might question the appropriateness of the name "radiation damping." But on second thought, there would be no opportunity for damping by the rf fields if there were not the necessity to compensate for the energy loss by radiation. So the name "radiation damping" is not so bad.

Now let's turn to the radiation effects on the radial betatron oscillations. You might at first, think that the radial betatron oscillations would be radiation damped in the same way as the vertical ones. But there are additional complications so we shall have to treat them as a new problem. One new element arises from the change in the betatron <u>displacement</u> that occurs when there is an energy change. Remember that the total radial displacement x is the sum of two parts: the displacement x_ϵ of the off-energy closed orbit, plus the betatron displacement x_β with respect to the closed orbit,

$$x = x_\epsilon + x_\beta \ .$$

(4.33)

When the energy of an electron changes by δE, there is a change of x_ϵ by the amount, see Eq. (2.28),

$$\delta x_\epsilon = \eta \frac{\delta E}{E_0}$$

(4.34)

But since the <u>position</u> <u>in</u> <u>space</u> of the electron is not changed by a finite momentum impulse, the total x does <u>not</u> change, so there must be a compensatory change in $\bar{x}_\beta$. That is, from Eq. (4.33)

$$\delta x = \delta x_\epsilon + \delta x_\beta = 0,$$

from which

$$\delta x_\beta = - \delta x_\epsilon = - \eta \frac{\delta E}{E_0}$$

(4.35)

When there is an energy change, the electron doesn't instantaneously move, but the reference axis of its oscillations does and the displacement <u>with</u> <u>respect</u> <u>to</u> <u>that</u> <u>axis</u> is therefore changed — as is illustrated in Fig. 41.

Something similar occurs for the betatron slope. Corresponding to Eq. (4.33) we must have

$$x' = x'_\epsilon + x'_\beta \ .$$

(4.36)

Only now, an elementary impulse may change the total x' by some $\delta x'$ so we should have for the change in the <u>betatron</u> slope

$$\delta x'_\beta = \delta x' - \delta x'_\epsilon$$

(4.37)
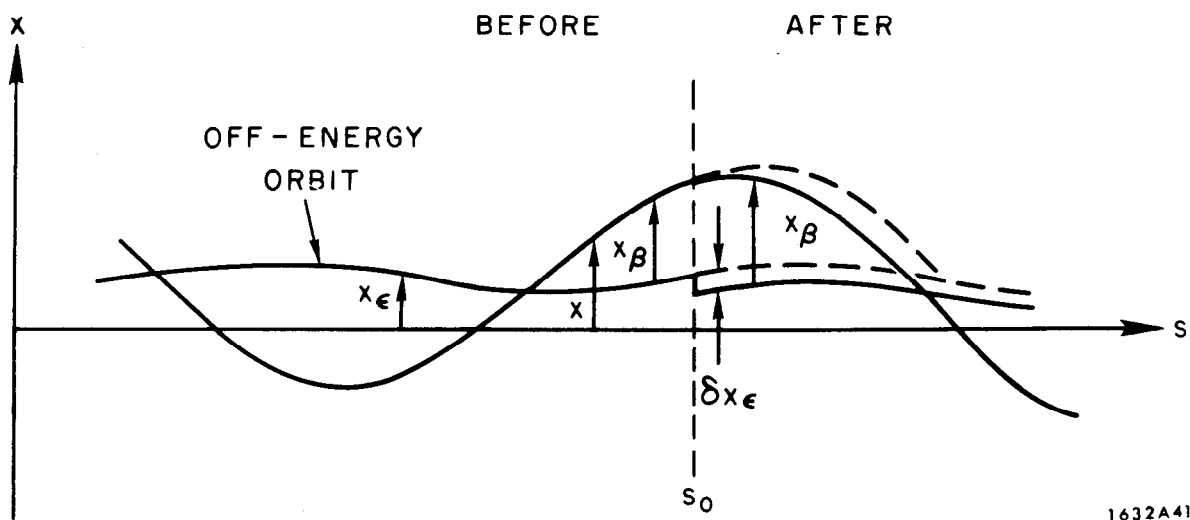
FIG. 41--Effect of a sudden energy change at s_0 on the betatron displacement.

Taking the derivative x'_ϵ from Eq. (2.28),

$$\delta x'_\beta = \delta x' - \eta' \frac{\delta E}{E_0} \tag{4.38}$$

where η' is, of course, $d\eta/ds$. Even if $\delta x'$ were zero, a change in the slope of x_ϵ — the baseline of the oscillations — would produce a change in the slope of the betatron oscillation.

Still an additional complication arises from the curvature of the reference orbit. The positive and negative halves of a betatron oscillation occur in equal intervals of s, but the electron travels a greater path length on the positive swing than on the negative swing — see Eq. (3.7). Although the net effect on the path length is zero, over a complete oscillation, there is, in general, a different amount of energy lost by radiation during the two halves of an oscillation. And the amplitude of the oscillation is thereby affected.

Now let's apply these ideas to the radiation loss δE in an azimuthal element δs. A precise calculation would proceed from the changes in x and x' found in Eqs. (4.35) and (4.38). In keeping with the approximations made earlier in this section, however, I am going to make the simplifying assumption that η is a constant, so that $\eta' = 0$; and write the variation of x_β with s in the same form that

I took for z ; namely,

$$x_\beta = A \cos \phi; \quad x'_\beta = \frac{A}{\beta} \sin \phi. \tag{4.39}$$

This time we have that

$$A\delta A = x_\beta \delta x_\beta + x'_\beta \delta x'_\beta / \beta^2 , \tag{4.40}$$

and since only δx_β is different from zero,

$$A\delta A = -x_\beta \eta \frac{\delta E}{E_0} \tag{4.41}$$

Again let's take the energy change δE as the radiation loss in an azimuthal element δs. For the z-motion we assumed that the electron was always moving with zero <u>radial</u> displacement so the rate of radiation loss was the same (to first order in z) as the rate of energy loss on the design orbit. Things are different for the x-motion if the magnets have a field gradient. To simplify the discussion here I will restrict consideration to an isomagnetic and separated function guide field (see Section 2.2). In a separated function machine the rate of radiation loss is independent of x — to first order.[†] I may then take that (for an electron of the nominal energy) the rate of radiation loss $P_\gamma(s)$ does not depend on x, but only on s. The energy change in a path element $\Delta\ell$ is then

$$\delta E = - \frac{P_\gamma}{c} \Delta\ell \tag{4.42}$$

Taking for $\Delta\ell$ the expression in (2.15),

$$\delta E = - \left(1 + \frac{x_\beta}{\rho_s}\right) \frac{P_\gamma}{c} \Delta s \tag{4.43}$$

Combining this result with Eq. (4.41) we have for the amplitude change,

$$A\delta A = x_\beta \eta \left(1 + \frac{x_\beta}{\rho_s}\right) \frac{P_\gamma}{c} \Delta s \tag{4.44}$$

Again we are interested only in the expectation value of δA — the average over all phase angles ϕ. The expectation value of x_β is zero and of x_β^2 is $A^2/2$; we get that

$$\frac{\langle\delta A\rangle}{A} = \frac{\eta}{2\rho_s} \frac{P_\gamma}{cE_0} \Delta s. \tag{4.45}$$

Since I am assuming an isomagnetic guide field wherever P_γ is different from zero $\rho = \rho_s = 1/G_0$ and we can easily sum up the effect at each Δs to get the change ΔA

[†] There is only a field gradient in the quadrupoles; where B is proportional to x. Since the rate of radiation varies is B^2 there is no first order effect.

in one complete revolution. The sum of all P_γ s/c is just the energy loss U_0 in one complete turn. So we have for the effect of the radiation

$$\left(\frac{\Delta A}{A}\right)_{\text{rad}} = \frac{\eta}{2\rho_0} \frac{U_0}{E_0} \tag{4.46}$$

Observe that the sign on the right hand side is <u>positive</u>. There is an <u>increase</u> of the amplitude due to the radiation!

Fortunately, this is only part of the story. We must also take into account the effect of the rf acceleration. For it however, there is no corresponding "path length" effect. Generally the rf cavities are located in places where $\rho = \infty$; but in any case, it is a property of such cavities that the energy gain is (to first order at least) independent of the betatron displacement. The calculation of the contribution from the rf acceleration goes exactly the same as for the vertical oscillations with the result shown in Eq. (4.31). To get the total effect in one revolution we must add the contributions from the radiation loss and from the acceleration to get

$$\frac{\Delta A}{A} = -\left(1 - \frac{\eta}{\rho_0}\right) \frac{U_0}{2E_0} , \tag{4.47}$$

which gives for the damping coefficient α_x of the radial oscillations

$$\alpha_x = -\left(1 - \frac{\eta}{\rho_0}\right) \frac{U_0}{2E_0 T_0} \tag{4.48}$$

A precise calculation for a separated function isomagnetic guide field gives exactly the same result, if we replace η by $\langle\eta\rangle_{\text{Mag}}$, the mean value of $\eta(s)$ in the magnets. But recalling Eq. (3.14), $\langle\eta\rangle_{\text{Mag}} = \alpha R$ so

$$\alpha_x = \left(1 - \frac{\alpha R}{\rho_0}\right) \frac{U_0}{2E_0 T_0} \qquad \left(\begin{array}{c}\text{isomag.}\\ \text{sep. func.}\end{array}\right) \tag{4.49}$$

Provided $\alpha R/\rho_0$ is less than 1— as it usually is — the damping coefficient is positive and the radial oscillations are damped. But there is an "antidamping" effect of the radiation — the term $\alpha R/\rho_0$ — which counteracts somewhat the positive damping from the rf system. So long as the antidamping term is small no harm is done.

If you compare Eq. (4.49) with the results of the preceding section you will see that we may also write our result in terms of the parameter $\mathscr{D}$ defined there:

$$\alpha_x = (1 - \mathscr{D}) \frac{U_0}{2E_0 T_0} \qquad \text{(general)} \tag{4.50}$$

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Although we have demonstrated this result only for a special kind of guide field (and with some approximations) Eq. (4.50), it turns out, is <u>exactly</u> true <u>for</u> <u>any</u> <u>guide</u> <u>field</u>. That is, if we had in our treatment kept account of the effect of the variation of η with s we would have found that in place of η/ρ_0 in Eq. (4.48) we would have the complete expression for $\mathscr{D}$ in Eq. (4.17). More will be said about this interesting "coincidence" in the next section.

4.4. Radiation Damping Rates

Radiation damping effects have now been considered for all three degrees of freedom of an electron in a bunch: the two transverse betatron displacements x_β and z_β and the energy oscillations — which show up also in associated oscillations of τ and x_ϵ. Each of the three oscillation modes has a natural exponential decay with damping coefficients α_i (with i = x, z, or ϵ) that can be conveniently expressed as

$$\alpha_i = J_i \alpha_0 = J_i \frac{\langle P_\gamma \rangle}{2E_0} \tag{4.51}$$

with

$$J_x = 1 - \mathscr{D}; \quad J_z = 1, \quad J_\epsilon = 2 + \mathscr{D}. \tag{4.52}$$

The damping time constants are just $1/\alpha_i$ so

$$\tau_i = \frac{2E_0}{J_i \langle P_\gamma \rangle} \tag{4.53}$$

For an isomagnetic storage ring $\langle P_\gamma \rangle$ may be taken from Eq. (4.10) then

$$\tau_i = \frac{4\pi}{C_\gamma} \frac{R\rho_0}{J_i E_0^3} \quad \text{(isomag.)} \tag{4.54}$$

where C_γ is the constant defined in Eq. (4.2). In a given storage ring the damping time constants vary as the inverse cube of the energy.

The number $\mathscr{D}$ is a property of the guide field and may be evaluated from one of the equations (2.10), (2.12), or (2.13). The numbers J_i are known as the <u>damping</u> <u>partition</u> <u>numbers</u> since their sum is a constant:

$$\sum_i J_i = J_x + J_z + J_\epsilon = 4. \tag{4.55}$$

Although I have not actually proved this last result, it does indeed follow from detailed calculations for a general guide field. (See e.g., Ref. 5.) Such calculations are, however, not really necessary because Robinson[12] has proved on very general grounds a theorem that yields Eq. (4.55) directly. The theorem required only that all of the fields acting on the particle are determined a priori and are not in any way influenced by the motion of the electron. These conditions apply if we consider only the prescribed magnetic and rf fields of a storage ring.

The damping rates for an individual electron — and more importantly, for the coherent motion of a clump of them — can be modified from the above numbers if additional forces are introduced that depend on the details of the electron motion. Such forces may, for example, come from image currents in the wall of the vacuum chamber or from currents induced by the beam in rf cavities, or from forces from auxiliary electrode systems powered via amplifiers from detectors that sense the displacement of the electrons. In actual rings, the first effect has led to unstable transverse coherent oscillations and the last one has been used to tame them. The second effect has been both the cause and the cure of unstable longitudinal oscillations of a bunch. Since such effects require the coherent cooperation of many electrons they are beyond the scope of the report and will not be considered further.

From Eq. (4.55) one would also obtain the more particular result that $J_x + J_\epsilon = 3$. This result depends, however, on one restrictive assumption — that the design orbit lies in a plane and that the magnetic fields are symmetric with respect to that plane. We have already referred briefly (at the end of Section 3.1) to one of the consequences of dropping this assumption. Off-energy orbits may generally have "vertical" displacements z_ϵ as well as the "radial" displacements x_ϵ. Most of the developments made in this report become more complicated and, in particular, the partition numbers will not be given by (4.54). The "conservation" theorem Eq. (4.55) will, however, remain valid.

Two other remarks about the consequences of this theorem are perhaps in order. First, for "alternating gradient" guide fields — such as those used universally in electron synchrotrons and in most proton synchrotrons — the number $\mathscr{D}$ is greater than 1. As a consequence the radial betatron oscillations are antidamped — and grow exponentially with time at a fixed energy. This effect has not been grave for the synchrotrons because the amplitude growth due to the antidamping is quite small during the acceleration time. It has however, posed a special problem in

the adaptation of the CEA synchrotron for use as a storage ring. For this adaptation it has been necessary to install special magnetic devices designed to modify $\mathscr{D}$ without affecting significantly the other characteristics of the ring.

Finally, you will appreciate that no real guide ever satisfies exactly the postulated symmetry of the fields with respect to the plane of the design orbit. The accidental asymmetries are generally small but they will, in general, lead to some coupling of the horizontal and vertical betatron oscillations. When such coupling is taken into account, x and z are no longer the coordinates of the normal modes. And the new normal modes will have damping coefficients which are somewhat different from α_x and α_z.

V. RADIATION EXCITATION

5.1. Quantum Radiation

Until now we have considered only the total energy loss due to synchrotron radiation — assuming implicitly that the energy loss is a continuous process. Such a view is all right for a first approximation since the energy loss is indeed fairly smooth on the average. But we know that all electromagnetic radiation occurs in quanta of discrete energy. And this quantization of the energy loss has significant effects on the behavior of electrons in a storage ring.

Each time a quantum is emitted the energy of the electron makes a small discontinuous jump. As we shall see later, the most significant quanta have energies which range from that of visible light out into soft x-rays. Although one is on shaky ground in trying to speak too quantitatively about quantum effects in a classical way, the following quasi-classical statements can be rigorously justified. First, the "time" during which a typical quantum is emitted is certainly no greater than $\rho/\gamma c$, where ρ is the radius of curvature of the trajectory and γ is the electron energy in units of its rest energy. Since this time is much less than any other relevant time — such as the period of a betatron or synchrotron oscillation — we may consider it to be instantaneous. Second, the emission times of the individual quanta are statistically independent. Since the energy change in any emission event is a very small fraction of the electron energy we may consider that the emission of successive quanta is a purely random (that is, Poisson) process.

The discontinuous energy change from the emission of a quantum disturbs the trajectory of the electron. The cumulative effect of many such disturbances introduces a kind of "noise" into the various oscillation modes causing their amplitudes to grow until the quantum excitation is, on the average, balanced by the damping of the oscillations. This process will be considered in detail for both betatron and energy oscillations in later sections.

A remark is perhaps required here about damping. We have, in the preceding sections, related the damping effects to radiation. You should notice that the damping depends only on the <u>average</u> rate of emission of energy and not on any of its other statistical properties. So when considering quantum effects we may take the same damping we have already found — understanding that it is due to the average rate of energy loss in all quantum energies. The excitation effects will be due to the fluctuations in the radiation about its average rate. (One could, of

course, treat both the average and fluctuation effects together, but to do so would only add unwarranted complications.)

In considering the effects of radiation fluctuations on the oscillations of an electron in a storage ring we shall need to know certain properties of the quantized radiation. I wish now to look at these properties.

From the classical view, the synchrotron radiation is emitted with a continuous spectrum of frequencies. (The frequency spectrum was first calculated by Schwinger,[13] a derivation is also given in Jackson.[10]) Consider the radiation emitted by an electron in some finite time interval Δt. Suppose we examine the radiation field which corresponds — by a suitable time retardation — to the emission in Δt, and for each direction in space, make a Fourier analysis of the radiation field. The frequency spectrum will, in general, be different for each direction. But we may average the spectrum over all directions to define a <u>radiated power spectrum</u> $\mathscr{P}(\omega)$ such that $\mathscr{P}(\omega)d\omega \, \Delta t$ is the energy radiated in Δt with angular frequencies between ω and $\omega + d\omega$. Clearly, the definition makes sense only if Δt is sufficiently large that most of the energy is found in frequencies greater than $1/\Delta t$. Recall now, that the radiation is typically emitted within the angle $1/\gamma$ of the electron's velocity vector. Such an angle is swept out in the time $\rho/\gamma c$, where ρ is the local radius of curvature of the trajectory. So a time interval $\approx \rho/\gamma c$ should contain most of the impulse of radiation; and it should, therefore, represent a suitable magnitude for Δt. We shall see later that it is indeed so.

With the definition given for $\mathscr{P}(\omega)$ we may permit it to be a slowly varying function of time and we shall not be in any difficulty provided only that $\mathscr{P}(\omega)$ (and therefore, ρ or γ on which it depends) does not change appreciably in Δt. This condition is generally satisfied[†] for storage rings, so we may consider that $\mathscr{P}(\omega)$ is an "instantaneous" power spectrum whose integral over ω is the instantaneous radiated power defined earlier,

$$P_\gamma = \int_0^\infty \mathscr{P}(\omega) \; d\omega. \tag{5.1}$$

[†]The important results of this part actually require only that ρ and γ do not change appreciably in a time $\rho/\gamma^3 c$ which is much smaller than Δt.

The power spectrum can be written in the convenient form:

$$\mathscr{P}(\omega) = \frac{P_\gamma}{\omega_c} \, S\!\left(\frac{\omega}{\omega_c}\right), \tag{5.2}$$

with ω_c a constant defined by

$$\omega_c = \frac{3}{2} \frac{c\gamma^3}{\rho} \tag{5.3}$$

The number ω_c is called the critical frequency,[†] notice that it is approximately equal to γ^3 times the angular revolution frequency of the electron. The spectral function $S(\omega/\omega_c)$ is a pure algebraic function of its argument which can be expressed by

$$S(\xi) = \frac{9\sqrt{3}}{8\pi} \, \xi \int_\xi^\infty K_{5/3}(\bar{\xi}) \, d\bar{\xi} \tag{5.4}$$

where $K_{5/3}$ is a modified Bessel function. It follows from the definition of Eq. (5.2) that S is normalized so that

$$\int_0^\infty S(\xi) d\xi = 1 \tag{5.5}$$

The form of the spectral function is shown in Fig. 42. Its behavior for large and for small arguments — which can easily be obtained from the asymptotic
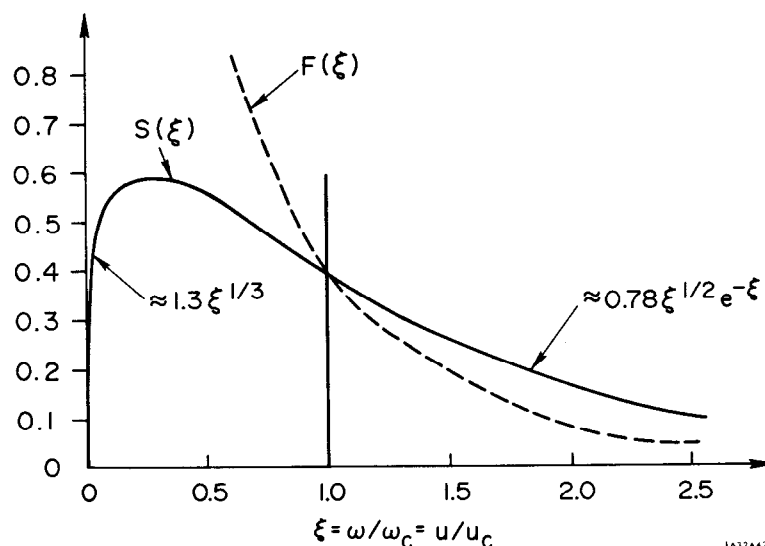
FIG. 42--Normalized power spectrum S and photon number spectrum F of synchrotron radiation.

[†] Caution! Some writers, for example Jackson, define the critical frequency with a different numerical factor.

behavior of the Bessel function — is sometimes useful.

$$\text{For} \quad \xi \ll 1; \qquad S(\xi) \approx 1.34 \, \xi^{1/3}$$

$$\text{For} \quad \xi \gg 1; \qquad S(\xi) \approx \frac{9\sqrt{3}}{8\sqrt{2\pi}} \, \xi^{1/2} e^{-\xi} \tag{5.6}$$

The power spectrum $\mathscr{P}(\omega)$ is obtained from $S(\xi)$ by Eq. (5.2). Don't forget that both P_γ — see Eq. (4.4) — and ω_c depend on both γ and ρ. It is clear from Fig. 42 that most of the power is found in frequencies near ω_c. (Since ω_c is γ^2 larger than the inverse of the Δt defined earlier, the assumptions made there are now justified.)

You know that electromagnetic radiation at the angular frequency ω is emitted in quanta of energy $u = \hbar\omega$, where $\hbar$ is Plank's constant reduced by 2π ($\hbar = h/2\pi = 6.85 \times 10^{-16}$ eV-sec). Let $n(u)du$ be the number of quanta emitted per unit time with energies between u and $u + du$. The power emitted in these quanta is $un(u)du$, which must be the same as the power emitted in the frequency interval $d\omega = du/\hbar$ at the frequency $\omega = u/\hbar$; namely,

$$un(u) \, du = \mathscr{P}(u/\hbar) \, du/\hbar \tag{5.7}$$

Taking $\mathscr{P}(\omega)$ from Eq. (5.2) the quantum distribution function can be written as

$$n(u) = \frac{P_\gamma}{u_c^2} \, F\!\left(\frac{u}{u_c}\right) \tag{5.8}$$

with

$$u_c = \hbar\omega_c = \frac{3}{2} \frac{\hbar c \gamma^3}{\rho} \, , \tag{5.9}$$

$$F(\xi) = \frac{1}{\xi} S(\xi) \tag{5.10}$$

Like the frequency spectrum, the quantum spectrum is, apart from the scale factor P_γ/u_c^2, a universal function of the ratio u/u_c.

The function $F(\xi)$ is also shown in Fig. 42. The rate of emission of quanta per unit energy interval diverges at low energies.[†] But only as $u^{-2/3}$, so the

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[†]The spectrum is, anyway, questionable for $u < u_c/\gamma^3$, according to the conditions mentioned earlier.

rate of emission of the quanta in any finite interval of quantum energies — an integral over u — is finite.

Let's let $\mathcal{N}$ stand for the total rate of emission of quanta (of all energies):

$$\mathcal{N} = \int_0^\infty n(u)\,du \ . \tag{5.11}$$

From the asymptotic expressions for $S(\xi)$ in (5.6), its complete integral is clearly ≈ 1. It is actually $15\sqrt{3}/8$ so

$$\mathcal{N} = \frac{15\sqrt{3}}{8} \ \frac{P_\gamma}{u_c} \ . \tag{5.12}$$

The mean quantum energy would be defined:

$$\langle u \rangle = \frac{1}{\mathcal{N}} \int_0^\infty u\,n(u)\,du \ . \tag{5.13}$$

The integral is just P_γ so the mean quantum energy is

$$\langle u \rangle = \frac{8}{15\sqrt{3}} \ u_c = (0.32\ldots)\,u_c \tag{5.14}$$

Speaking roughly, we may say that the radiation is emitted in quanta of a typical energy about u_c, and at a mean rate of about P_γ/u_c. For a 1 GeV electron moving on a 5 meter radius trajectory,

$$P_\gamma = 1.7 \times 10^{11}\ \text{eV sec}^{-1},$$

$$u_c = 437\ \text{eV},$$

$$\mathcal{N} = 3.2\ P_\gamma/u_c = 1.3 \times 10^9\ \text{sec}^{-1} \ .$$

It is amusing to notice that the mean number of quanta emitted per radian of trajectory depends only on the electron energy. It is, in fact, very nearly equal to simply the product of γ and the fine structure constant:

$$\text{(Mean number of quanta per radian)} = \frac{5}{2\sqrt{3}} \ \frac{\gamma}{137} \tag{5.15}$$

For a 1 GeV electron, the number is about 20. The actual number in any time interval fluctuates as the Poisson distribution corresponding to the mean number. It is then understandable that with such small numbers the fluctuations may be significant.

We shall see later that the quantum excitation of electron oscillations in a storage ring depends not only on the mean rate of quantum emission, but also on the mean-square quantum energy. We would expect $\langle u^2 \rangle$ to be about equal to u_c^2;

as indeed it is. If you work it out in detail you will find that

$$\langle u^2 \rangle = \frac{11}{27} u_c^2 \, .$$ (5.16)

The quantity that will enter in the quantum excitation of oscillations is in fact, the product of the mean square quantum energy with the mean rate $\mathcal{N}$; namely.

$$\mathcal{N}\langle u^2 \rangle = \int_0^\infty u^2 n(u) du \, .$$ (5.17)

It will be convenient to write, using Eq. (5.12)

$$\mathcal{N}\langle u^2 \rangle = C_u u_c P_\gamma$$ (5.18)

with

$$C_u = \frac{55}{24\sqrt{3}} = 1.32 \ldots$$ (5.19)

It is important to keep in mind that both u_c and P_γ are functions of the electron energy and of the local radius of curvature ρ of the trajectory. Taking u_c from Eq. (5.9) and P_γ from Eq. (4.4)

$$\mathcal{N}\langle u^2 \rangle = \frac{3 C_u C_\gamma}{4\pi} \frac{\hbar c^2}{(mc^2)^3} \frac{E^7}{\rho^3} = \frac{55}{24\sqrt{3}} r_e \hbar mc^4 \frac{\gamma^7}{\rho^3}$$ (5.20)

At a fixed radius the quantum excitation varies as the seventh power of the energy!

5.2. Energy Fluctuations

I turn now to an examination of the effect of quantum emission on the energy oscillations of a stored electron. When a quantum of energy u is emitted, the energy of the electron is suddenly decreased by the amount u. This impulsive disturbance sets up a small energy oscillation. The cumulative effect of many such disturbances — occurring at random times — causes the energy oscillation to grow (as in a random walk). The growth is limited — on the average — by the damping; and under stationary conditions the energy oscillations of any particular electron will fluctuate about some mean amplitude. I want now to look at these fluctuating energy oscillations.

At first, I shall be concerned only with one measure of the typical energy oscillation — namely the root-mean-square deviation from the mean energy — without considering in detail the probability distribution of the energy deviation. The nature of the distribution will be considered later on.

In Section 3.5 we looked at the small oscillations of the energy deviation of a stored electron. In the absence of any disturbances, and ignoring for the moment any damping, the energy deviation $\underset{\sim}{\epsilon}$ is described by

$$\underset{\sim}{\epsilon} = A_0 e^{i\Omega(t - t_0)} \tag{5.21}$$

where Ω is the (real) synchrotron frequency and the amplitude A_0 is a real number. Now suppose that at some instant t_i the energy is suddenly decreased the amount u — by a quantum emission. After t_i the energy will go as

$$\underset{\sim}{\epsilon} = A_0 e^{i\Omega(t - t_0)} - ue^{i\Omega(t - t_i)} \tag{5.22}$$

See Fig. 43. This new oscillation can be written as
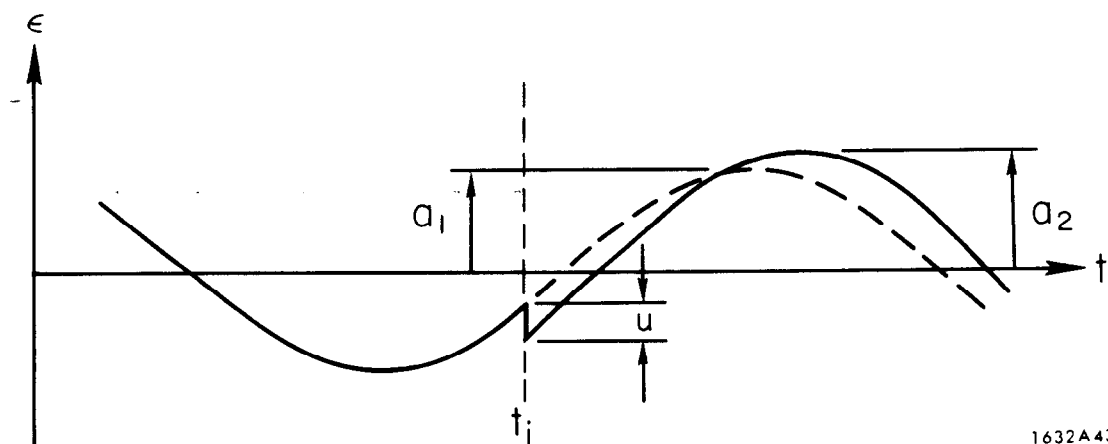
$$\underset{\sim}{\epsilon} = A_1 e^{i\Omega(t - t_1)} \tag{5.23}$$

FIG. 43--Effect on the energy oscillations of the emission of a quantum of energy u.

where[†]

$$A_1^2 = A_0^2 + u^2 - 2A_0u \cos \Omega(t_i - t_0) \tag{5.24}$$

[†]Obtained from $A^2 = \underset{\sim}{\epsilon}\underset{\sim}{\epsilon}^*$.

and t_1 is some time displacement of no concern to us now. The quantum emission changes the amplitude of the oscillation to a new value which depends on the initial amplitude and on $(t_i - t_0)$. Since the time t_i at which a quantum emission occurs is completely unpredictable — and since we are interested only in the cumulative affect of many such events — we should ask only statistical questions. Such as: What is the probable amplitude change? In general, the phase $(t_i - t_0)$ is completely random and the expectation value of $\cos\Omega(t_i - t_0)$ is therefore zero. Then the probable amplitude change due to the quantum event is

$$\langle \delta A^2 \rangle = \langle A_1^2 - A_0^2 \rangle = u^2 \tag{5.25}$$

Notice that our result says that the probable change in A^2, which occurs when we add with random phase a new increment of oscillation of amplitude u, is just u^2 — the same result we would have obtained for δA^2 if we had started with $A = 0$.

Suppose now that such quantum events occur in a random time sequence at the mean rate $\mathcal{N}$ (number per unit time). Each event changes A^2 by u^2; and since the mean time between events is $1/\mathcal{N}$, we expect that

$$\langle \frac{dA^2}{dt} \rangle = \mathcal{N}u^2 \tag{5.26}$$

But the probable rate-of-change of A^2 is equal to the rate-of-change of the probable value of A^2 or

$$\frac{d\langle A^2 \rangle}{dt} = \mathcal{N}u^2 \tag{5.27}$$

In addition to exciting energy oscillations, the quantized energy losses contribute to a cumulative energy change. We have however, considered such average effects earlier. Their effect is to produce the energy oscillations as well as to cause the slow exponential damping of the amplitude A with a time constant $\tau_\epsilon = 1/\alpha_\epsilon$. With such damping the amplitude decreases at the rate A/τ_ϵ; or its square at the rate $2A^2/\tau_\epsilon$. The probable amplitude must be similarly decreased by the damping which would contribute to the rate-of-change of $\langle A^2 \rangle$ the amount

$$\frac{d\langle A^2 \rangle}{dt} = -2 \frac{\langle A^2 \rangle}{\tau_\epsilon} . \tag{5.28}$$

When both quantum excitation and damping are at work — and other conditions are stationary — the rates of Eqs. (5.27) and (5.28) must sum to zero. We find that

the probable value of A^2 is given by

$$\langle A^2 \rangle = \frac{1}{2} \tau_\epsilon \mathcal{N} u^2 \qquad (5.29)$$

For the sinusoidal energy oscillations (as they are very nearly) the expectation value of ϵ is zero, and of its square — which we shall call σ_ϵ^2 — is just 1/2 the probable amplitude squared:

$$\sigma_\epsilon^2 = \langle \epsilon^2 \rangle = \frac{\langle A^2 \rangle}{2} = \frac{1}{4} \tau_\epsilon \mathcal{N} u^2 \qquad (5.30)$$

This then, would be the mean-square energy fluctuation in the energy oscillation which would be produced by the random emission of quanta all of the same energy u. It should correspond approximately to the energy fluctuations in a storage ring if we were to put for u, the typical quantum energy u_c and for $\mathcal{N}$ the mean rate P_γ/u_c.

An approximately equivalent result can be obtained from the following simple argument: The typical energy fluctuation comes from the deviation from its mean of the number of quanta emitted in one damping time τ_ϵ. The mean number emitted in τ_ϵ is $\mathcal{N}\tau_\epsilon$, and so the rms deviation from the mean is $\sqrt{\mathcal{N}\tau_\epsilon}$ (Poisson distribution). Since each quantum has about the energy u_c, on the average,

$$\sigma_\epsilon \approx \sqrt{\mathcal{N}\tau_\epsilon} \, u_c \qquad (5.31)$$

The result is roughly the same as Eq. (5.30). It is amusing to notice that, since $\mathcal{N} \approx P_\gamma/u_c$ and $\tau_\epsilon \approx E_0/P_\gamma$, we may also write that

$$\sigma_\epsilon \approx \sqrt{E_0 u_c} \qquad (5.32)$$

The energy fluctuation is roughly the geometric mean between the electron energy and the critical photon energy!

Let's now do a precise calculation which is somewhat more complicated — first, because there is a distribution of quantum sizes and second, because both the distribution and the mean rate may vary around the storage ring. Returning to Eq. (5.28) we should consider separately the contribution to $d\langle A^2 \rangle/dt$ from each interval of quantum sizes. Those quanta with energies between u and $u + \Delta u$ — of which there are $n(u)\Delta u$ — will give the contribution

$$\Delta \left\{ \frac{d\langle A^2 \rangle}{dt} \right\} = u^2 n(u) \, \Delta u \qquad (5.33)$$

But since the emission of quanta at the various different energies is also uncorrelated, each energy will contribute independently to the random-walk growth of $\langle A^2 \rangle$. We need only sum the contributions from each interval Δu:

$$\frac{d\langle A^2 \rangle}{dt} = \int_0^\circ u^2 n(u)\, du \tag{5.34}$$

You will recognize the integral as just the product $\mathcal{N}\langle u^2 \rangle$ considered in the preceding section — Eq. (5.17):

$$\frac{d\langle A^2 \rangle}{dt} = \mathcal{N}\langle u^2 \rangle. \tag{5.35}$$

The rate of growth just obtained depends on the electron energy — which we may take to be the nominal energy E_0 — and on ρ, the local radius of curvature of the trajectory, both of which may vary around the ring. From our derivation we may expect that the time for a "significant" change in the amplitude of the energy oscillation will be of the order of the damping time constant τ_ϵ. Since both the period of the oscillation $\approx 1/\Omega$, and the damping time τ_ϵ are each much longer than a revolution time T_0 we may, without injustice, replace the rapidly varying quantity $\mathcal{N}\langle u^2 \rangle$ by its <u>mean</u> <u>value</u> <u>over</u> <u>one</u> <u>revolution</u> <u>of</u> <u>the</u> <u>ring</u>. We shall also make a negligible error (on the average) if we replace the instantaneous radius of curvature ρ of the trajectory at each azimuth s by the local radius of curvature of the design orbit. Taking the average of $\mathcal{N}\langle u^2 \rangle$ over one revolution by integrating with respect to the azimuthal coordinate s, we may define[†]

$$Q_\epsilon = \mathcal{N}\langle\!\langle u^2 \rangle\!\rangle_s = \frac{1}{2\pi R} \oint \mathcal{N}\langle u^2 \rangle\, ds. \tag{5.36}$$

Following through the rest of the derivation as before we get for the mean-square energy fluctuation:

$$\sigma_\epsilon^2 = \frac{1}{4} \tau_\epsilon Q_\epsilon \tag{5.37}$$

The simple form of our result is misleading; the complexities are hidden in τ_ϵ and Q_ϵ. Let's look first at Q_ϵ. We need to evaluate $\mathcal{N}\langle u^2 \rangle$ on the design orbit.

[†]The index s on the brackets indicates that the average is taken over the coordinate s as distinct from the average of u^2 which is over the distribution in u.

Suppose we begin with the form derived in Eq. (5.18). P_γ on the design orbit is obtained from Eq. (4.4) by setting $E = E_0$ and $(1/\rho) = G$ (see Section 2.2), so

$$(P_\gamma)_{\text{design orbit}} = \frac{cC_\gamma}{2\pi} E_0^4 G^2, \tag{5.38}$$

which may be written — using Eq. (4.9) — as

$$(P_\gamma)_{\text{design orbit}} = \frac{\langle P_\gamma \rangle_s G^2}{G^2{}_s} \tag{5.39}$$

And U_c on the design orbit is from Eq. (5.9),

$$(u_c)_{\text{design orbit}} = \frac{3}{2} \hbar c \gamma_0^3 G. \tag{5.40}$$

We have then that

$$\left\{ \mathcal{N} \langle u^2 \rangle \right\}_{\text{design orbit}} = C_u \cdot \frac{3}{2} \hbar c \gamma_0^3 G \cdot \frac{\langle P_\gamma \rangle_s G^2}{G^2{}_s} . \tag{5.41}$$

The only quantity which varies around the design orbit is G so that Q_ϵ can be written as†

$$Q_\epsilon = \frac{3}{2} C_u \hbar c \gamma_0^3 \frac{P_\gamma G^3}{\langle G^2 \rangle} . \tag{5.42}$$

Taking τ_ϵ from Eq. (4.53),

$$\tau_\epsilon = \frac{E_0}{J_\epsilon \langle P_\gamma \rangle} \tag{5.43}$$

we may finally rewrite Eq. (5.37) as

$$\sigma_\epsilon^2 = \frac{3 C_u \hbar mc^3 \gamma_0^4 \langle G^3 \rangle}{4 J_\epsilon \langle G^2 \rangle} \tag{5.44}$$

The <u>relative</u> energy spread σ_ϵ / E_0 is usually more significant. We may write it as

$$\left(\frac{\sigma_\epsilon}{E_0} \right)^2 = \frac{C_q \langle G^3 \rangle \gamma_0^2}{J_\epsilon \langle G^2 \rangle} \tag{5.45}$$

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† We may now leave off the subscript s on the average since it is clear that all quantities shown are to be averaged over s. I hope it is clear that $\langle G^2 \rangle$, for example, means $\oint G^2(s)\, ds/L$ where L is the orbit length.

with C_q — which we may call the <u>quantum</u> <u>constant</u> — given by

$$C_q = \frac{3 C_u \hbar}{4mc} = \frac{55}{32 \sqrt{3}} \frac{\hbar}{mc} = 3.84 \times 10^{-13} \text{ meter} \qquad (5.46)$$

It is very nearly, just the Compton wavelength of the electron.

The quantity $\langle G^3 \rangle / J_\epsilon \langle G^2 \rangle$ is a geometrical property of the guide field. Specifically,

$$\frac{\langle G^3 \rangle}{J_\epsilon \langle G^2 \rangle} = \frac{1}{J_\epsilon} \frac{\oint G^3(s)ds}{\oint G^2(s)ds} \; . \qquad (5.47)$$

It is roughly equal to the inverse of the "typical" radius-of-curvature of the design orbit. The result of Eq. (5.45) is then roughly γ^2 times the ratio of the Compton wavelength to the orbit radius. For any ring the quantum induced spread in the relative energy deviation — namely σ_ϵ / E_0 varies in direct proportion to the electron energy.

In a storage ring with an isomagnetic guide field (one which has a constant radius ρ_0 in the magnets and is straight elsewhere) the geometrical expression above is just $1/J_\epsilon \rho_0$, and

$$\left(\frac{\sigma_\epsilon}{E_0} \right)^2 = \frac{C_q \gamma_0^2}{J_\epsilon \rho_0} \qquad \text{(isomag)}. \qquad (5.48)$$

In an isomagnetic storage ring with a 5 meter magnetic radius, electrons stored with an energy of 1 GeV will have an energy spread very nearly 0.04% of the energy — or about 40 keV.

5.3.   Distribution of the Fluctuations

The energy deviation ϵ at any instant t is the result of a super position of the contributions from the emission of quanta at all earlier times t_i. We may in fact, write for $\epsilon(t)$

$$\epsilon(t) = \sum_{t_i < t} u_i e^{-(t - t_i)\tau} \cos \Omega(t - t_i) \; , \qquad (5.49)$$

where u_i is the energy of the quantum emitted at t_i. Since the typical value of $\epsilon(t)$ is much larger than the typical quantum energy — see Eq. (5.32) — and since the times t_i are randomly distributed, the sum at any instant t consists of contributions from a large number of small terms which are all statistically independent, and

which are positive and negative with equal probability. It is well known[†] that the result of such a sum is a stochastic quantity whose most probable value is zero and which is otherwise distributed as a <u>normal error function</u> — a so-called Gaussian distribution. That is the probability $w(\epsilon)d\epsilon$ that the energy deviation will be found in an interval $d\epsilon$ at ϵ is distributed according to

$$w(\epsilon)d\epsilon = \frac{1}{\sqrt{2\pi}\ \sigma_\epsilon}\ \exp\left(-\epsilon^2/2\sigma_\epsilon^2\right)d\epsilon\ . \tag{5.50}$$

The parameter σ_ϵ, often called the <u>standard deviation</u>, is equal to the root-mean-square spread of the distribution — that is, the square root of $\langle\epsilon^2\rangle$ — as can easily be shown by a direct integration:

$$\sigma_\epsilon^2 = \int_{-\infty}^{+\infty}\epsilon^2 w(\epsilon)d\epsilon\ . \tag{5.51}$$

(The distribution function $w(\epsilon)$ is properly normalized so that its complete integral is equal to 1.) The standard deviation σ_ϵ is then, the same quantity we have evaluated in the preceding section.

In a stored beam we have, normally, a large number N of stored electrons. So long as any interactions among them can be ignored, the distribution of energies within the bunch will — under stationary conditions — also be described by Eq. (5.40). That is, the <u>number</u> of electrons with energies between ϵ and $\epsilon + d$ will be just $Nw(\epsilon)\,d\epsilon$. And the "half-width" of the spread of energies in the beam is described by σ_ϵ.

The distribution function of Eq. (5.50) and also our calculation of σ_ϵ assume that the energy oscillations are <u>linear</u>. (With nonlinearities, Eq. (5.49) is not correct and the effects of the individual quanta are no longer independent.) We have already seen however, that the energy oscillations are <u>not</u> linear for large energy deviations. If the rf voltage function is significantly nonlinear over the time displacements that correspond to the likely energy deviations, we must expect the probability distribution for ϵ to be distorted from the ideal distribution of Eq. (5.50). If, however, the nonlinearity is not too great over the largest part

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[†]And follows from the Central Limit Theorem of probability theory.

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of the distribution, we may expect that neither σ_ϵ nor the distribution function of the energy deviations will be affected very much.

The distribution of energy deviations just considered implies related distributions in other parameters of the energy oscillations. The relationships are most easily understood by considering the electron's trajectory in a "phase diagram" such as the one discussed in Section 3.5. Suppose we describe the state of the energy oscillation by giving its energy deviation ϵ and its "scaled" time displacement θ, which we define by

$$\theta = \frac{\Omega E_0}{\alpha} \tau .$$ (5.52)

Ω, the angular frequency of the energy oscillation and α, the dilation factor are constants so θ is just a scaled equivalent of τ the time displacement coordinate of the energy oscillations. (See Section 3.5.) So long as the damping rate is small, θ could equally well be defined by

$$\theta = \frac{1}{\Omega} \frac{d\epsilon}{dt} ,$$ (5.53)

so it may also be viewed as a normalized derivative of ϵ. We may now represent the state of motion of an electron by a point on a two-dimensional graph in which ϵ and θ are orthogonal coordinates — see Fig. 44(a) — and in which an oscillation
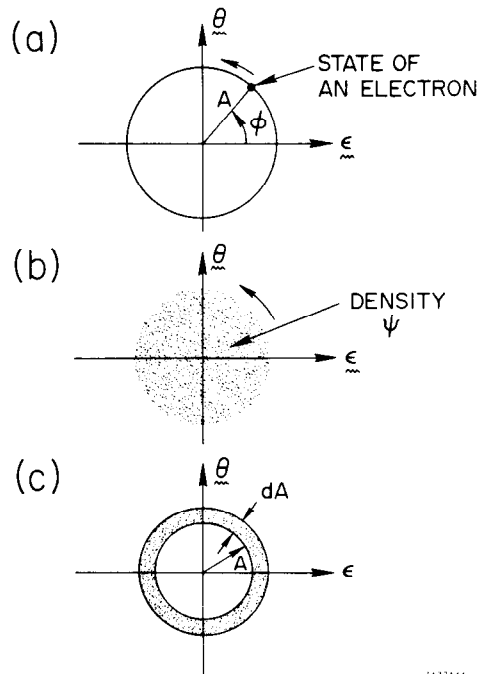

FIG. 44--Scaled phase space of the energy oscillations.

of constant amplitude would describe a circle. Then so long as the damping and the quantum effects are small, we may consider that for any small interval of time, ϵ and θ vary as

$$\epsilon = A \cos \phi, \tag{5.54}$$

$$\theta = A \sin \phi; \tag{5.55}$$

where

$$\phi = \Omega t - \phi_0, \tag{5.56}$$

and A is slowly varying amplitude. The quantities A and ϕ are a polar representation of the representative point and ϕ increases as Ωt.

The distribution of energy oscillations of the electrons in a stored bunch can now be represented by a distribution of points in the phase plot as indicated schematically in Fig. 44b. A complete description of the distribution is given by specifying the density $\psi(\epsilon, \theta)$ in the ϵ, θ plane. That is $\psi(\epsilon, \theta)d\epsilon d\theta$ is to represent the number of electrons found in the element of area $d\epsilon d\theta$ located at (ϵ, θ). We already know the projection of $\psi(\epsilon, \theta)$ on the horizontal axis. If there are N electrons in the bunch it is just $Nw(\epsilon)$. But in one-quarter of an oscillation each electron rotates one-quarter of a revolution about the origin of the figure. And since we are assuming a stationary distribution — that is one with no time variations — the projection on the vertical and on the horizontal axes must be identical. It must be then, that the number of electrons in an element of area $d\epsilon d\theta$ is given by

$$\psi(\epsilon, \theta)d\epsilon d\theta = \frac{N}{2\pi\sigma_\epsilon^2} \exp\left(-\frac{\epsilon^2 + \theta^2}{2\sigma_\epsilon^2}\right) d\epsilon d\theta . \tag{5.57}$$

The projection on the horizontal axis is

$$\int \psi(\epsilon, \theta)d\theta = \frac{N}{\sqrt{2\pi}\,\sigma_\epsilon} \exp\left(-\epsilon^2/2\sigma_\epsilon^2\right)d\epsilon,$$

which agrees with the $w(\epsilon)$ of Eq. (5.50). Similarly, the distribution in θ is

$$\frac{N}{\sqrt{2\pi}\,\sigma_\epsilon} \exp\left(-\theta^2/2\sigma_\epsilon^2\right). \tag{5.58}$$

We may now ask what is the distribution of oscillations <u>amplitudes</u>. Since $A^2 = \epsilon^2 + \theta^2$, the <u>density</u> of electrons in the ϵ, θ plane at the amplitude A is

just

$$\frac{N}{2\pi\sigma_\epsilon^2} \exp\left(-A^2/2\sigma_\epsilon^2\right) . \tag{5.59}$$

If we now let g(A)dA be the number of electrons in an amplitude interval dA at A, that number is just $2\pi A dA$ times the density at A:

$$g(A)dA = \frac{NA}{\sigma_\epsilon^2} \exp\left(-A^2/2\sigma_\epsilon^2\right) . \tag{5.60}$$

(See Fig. 44(c).) The mean-square of A in this distribution is just the $\langle A^2 \rangle$ that was discussed in the preceding section. By direct integration of $A^2 g(A)dA$ you can see that $\langle A^2 \rangle = 2\sigma_\epsilon^2$, as was argued earlier. So the last equation can be written as

$$g(A)dA = N \frac{2A}{\langle A^2 \rangle} \exp\left(-A^2/\langle A^2 \rangle\right)dA. \tag{5.61}$$

Suppose we take the number $W = A^2$ as a measure of the "oscillation energy,"[†] and compute the mean oscillation energy $\langle W \rangle$. Since the energy interval dW corresponds to 2AdA, the number of electrons which are found in the interval dW at W is

$$h(W)dW = \frac{N}{\langle W \rangle} \exp\left(-W/\langle W \rangle\right)dW . \tag{5.62}$$

The distribution in oscillation energies is a pure exponential and corresponds to the Boltzman distribution of energies in an ensemble of mechanical systems in thermal equilibrium — with the characteristic energy $\langle W \rangle$ given by

$$\langle W \rangle = \langle A^2 \rangle = 2\sigma_\epsilon^2 . \tag{5.63}$$

5.4.   Bunch Length

We have just seen that the distribution in the normalized time displacement θ is as a Gaussian with a standard deviation that is equal to the standard deviation σ_ϵ of the energy oscillations — refer to Eq. (5.50). It follows that the fluctuating energy oscillations are accompanied by associated fluctuations in the time

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[†]W is proportional to — but different by a numerical factor from — the "oscillation energy" defined in Section 3.6.

displacement τ, and that the standard deviation σ_τ of these fluctuations is — see Eq. (5.52) —

$$\sigma_\tau = \frac{\alpha}{\Omega E_0} \sigma_\epsilon. \tag{5.64}$$

For an isomagnetic guide field Eq. (5.45) gives

$$\sigma_\tau^2 = \frac{\alpha^2}{\Omega^2} \frac{C_q \gamma_0^2}{J_\epsilon \rho_0^2} \quad \text{(isomag).} \tag{5.65}$$

Taking Ω^2 from Eq. (3.44)

$$\sigma_\tau^2 = \frac{\alpha T_0 E_0}{e\dot{V}_0} \frac{C_q \gamma_0^2}{J_\epsilon \rho_0} = \frac{2\pi C_q}{(mc^2)^2} \frac{\alpha R}{J_\epsilon \rho_0} \frac{E_0^3}{e\dot{V}_0} \quad \text{(isomag).} \tag{5.66}$$

The spread σ_τ in the time displacement gives when multiplied by c, also the spread of longitudinal displacement from the bunch center — or, what we may call the bunch half-length.

If the energy E_0 of the stored beam in a particular storage ring is varied while holding constant the slope of the rf voltage ($\dot{V}_0$), the bunch length will increase with the energy as $E_0^{2/3}$. However, we shall see later (Section 6.5) that it may be advantageous to adjust the rf voltage when changing the energy and, in fact, in such a way that the peak rf voltage is kept proportional to E_0^3. If the rf voltage is sinusoidal, the slope $\dot{V}_0$ will also vary as E_0^3 and, by Eq. (5.66) the bunch-length will then be independent of energy. The constant bunch length, $2c\sigma_\tau$ is in such a case, typically about 10% of the distance between bunch centers.

In several of the storage rings that have been constructed to date the bunch length is observed to be larger than is predicted here by a significant factor which depends on the number of electrons in the stored bunch. The mechanism responsible for this anomalous lengthening is not understood at this time.

5.5. Beam Width

The emission of discrete quanta in the synchrotron radiation will also excite random betatron oscillations and these quantum-induced oscillations are responsible for the lateral extent of a stored electron beam. Let's look first at the quantum effects on the horizontal betatron oscillations. (As in the preceding section, I will consider first only the gross statistical properties of the fluctuations.)

In Section 4.3 we considered the effect of a small radiation loss δE — which was assumed there to occur continuously in a path length $\delta\ell$ — under the assumption that the momentum loss was parallel to the direction of motion. We may take over the results obtained there and adapt them to the case of quantum emission by setting δE to the quantum energy u — keeping for the moment the assumption that quantum emission gives only a change in the magnitude of the momentum and not in its direction. You will recall from Section 4.3 that a change in energy is accompanied by a change in the betatron displacement only because of the sudden displacement of the reference orbit — the energy displaced orbit — about which the betatron oscillations occur. Taking the results of Eqs. (4.35) and (4.38), the emission of a quantum of energy u will result in a change δx_β in the betatron displacement and a change $\delta x'_\beta$ in the betatron slope given by

$$\delta x_\beta = -\eta \frac{u}{E_0} \quad ; \qquad \delta x'_\beta = -\eta' \frac{u}{E_0} \tag{5.67}$$

The effect that such a sudden disturbance will have on the betatron oscillations will depend on where in the storage ring the quantum emission occurs — and on where we observe the oscillation. From Section 2.6 we know how to relate the oscillations observed at one azimuth to those that will be found at another azimuth; so we can for convenience, evaluate the quantum effects by what they do to the oscillations at some fixed azimuth — say at s_1 — and later transfer the result to any other azimuth. Our program can then be the following: (1) We ask what is the effect at s_1 of a quantum emission that occurs at some other azimuth s_2. (2) We average over all quanta which might be emitted at s_2. (3) We sum the contributions from all possible values of s_2.

In Section 2.6 we considered the motion which resulted at s_1 from the "initial conditions" x_2 and x'_2 at s_2, the result can be written in the form[†]

$$x_\beta(s_1, t_j) = a \sqrt{\beta_1} \cos \phi_j \tag{5.68}$$

where the ϕ_j are the oscillation phases at the times t_j of the successive passages of the electron by the azimuth s_1, β_1 is the betatron function at s_1 and $\underline{a}$ is an

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[†] It will be understood that here β means β_x.

invariant amplitude factor given by

$$a^2 = \frac{1}{\beta_2} \left\{ x_2^2 + \left(\beta_2 x_2' - \frac{1}{2}\beta_2' x_2 \right)^2 \right\} . \tag{5.69}$$

If we put for x_2 and x_2' the disturbance of Eq. (5.67) and write δa^2 for the resulting amplitude, we have that the emission of a quantum of energy u at s_2 gives the amplitude

$$\delta a^2 = \frac{u^2}{E_0^2} \frac{1}{\beta_2} \left\{ \eta_2^2 + \left(\beta_2 r'_2 - \frac{1}{2}\beta_2' \eta_2 \right)^2 \right\} . \tag{5.70}$$

All of the s-dependent quantities on the right-hand-side are to be evaluated at s_2, so let's define a new function of s:

$$\mathcal{H}(s) = \frac{1}{\beta} \left\{ \eta^2 + \left(\beta\eta' - \frac{1}{2}\beta'\eta \right) \right\} \tag{5.71}$$

which is specified by the properties of the guide field. Then Eq. (5.70) becomes simply

$$\delta a^2 = \frac{u^2}{E_0} \mathcal{H}(s_2). \tag{5.72}$$

This result gives the amplitude produced when we start with zero amplitude. What happens if we already have some amplitude $\underline{a}$, and then a quantum is emitted? As we saw at the beginning of Section 4.6 for the amplitude A, so long as there is no correlation between the phase of the initial oscillation and the occurrence of the quantum event — that is, if the quantum emission is completely random — then the change in the probable value of a^2 is just the δa^2 we have calculated. We may, therefore, say that the change in the probable invariant amplitude of the betatron oscillation caused by the emission of a quantum of energy u at s_2 will be

$$\delta \langle a^2 \rangle = \frac{u^2}{E_0^2} \mathcal{H}(s_2). \tag{5.73}$$

We now know what will be the result if a quantum is emitted at s_2; we must next ask what is the likelihood that such an event will occur. Consider what happens as the electron travels the distance Δs at s_2 — which will take the time $\Delta t = \Delta s/c$. Taking the definitions of Section 4.5, the probability that a quantum will be emitted is $\mathcal{N}\Delta s/c$, and the probable value of u^2 for the quantum emitted is $\langle u^2 \rangle$. So the change in the probable value of a^2 due to the element Δs of the

trajectory can be written as

$$\delta \langle a^2 \rangle = \frac{\left\{ \mathcal{N} \Delta s \langle u^2 \rangle \mathcal{H}(s) \right\}_2}{c E_0^2} \tag{5.74}$$

The subscript on the curly brackets means that all quantities inside are to be evaluated at s_2. (Both $\mathcal{N}$ and $\langle u^2 \rangle$, you will remember, depend on the local radius-of-curvature of the trajectory.)

Suppose we now add up the contributions to changes in $\langle a^2 \rangle$ during one trip of the electron around the ring. The resulting change, which we may call $\Delta \langle a^2 \rangle$, is obtained by integrating the right-hand-side of Eq. (5.74) once around the ring:

$$\Delta \langle a^2 \rangle = \frac{1}{c E_0^2} \oint \left\{ \mathcal{N} \langle u^2 \rangle \mathcal{H} \right\}_2 ds_2 \tag{5.75}$$

As before,[†] it will be convenient to represent the integral as the product of the length of the orbit $2\pi R$, with the mean value — with respect to s — of the integrand.

$$\Delta \langle a^2 \rangle = \frac{2\pi R}{c E_0^2} \langle \mathcal{N} \langle u^2 \rangle \mathcal{H} \rangle_s \tag{5.76}$$

Although $\mathcal{N}$ and $\langle u^2 \rangle$ depend on the actual electron trajectory — and so may change from one turn to the next — they will differ little from the values on the design orbit. Also the differences will, to first order in the displacements from the design orbit, average to zero. Since we are going to be interested, anyway, only in effects which accumulate over many revolutions, we will make no significant error if we take (as we did for the energy oscillations) the average in Eq. (5.76) by evaluating $\mathcal{N} \langle u^2 \rangle$ on the design orbit. We shall therefore, interpret the average over s in that way.

The change $\Delta \langle a^2 \rangle$ of Eq. (5.76) occurs in the time of one revolution, namely $2\pi R/c$. So we may write that

$$\frac{d \langle a^2 \rangle}{dt} = Q_x = \frac{\langle \mathcal{N} \langle u^2 \rangle \mathcal{H} \rangle_s}{E_0^2} \tag{5.77}$$

[†] For the remainder of the development I shall follow the same line of argument used in the preceding section and will not repeat all of the details. You should refer to that section for any details that are not clear.

This is of course, only the contribution from the quantum noise. As in Section 5.2, we must still add in the average effect of the radiation which contributes a damping term

$$\frac{d\langle a^2 \rangle}{dt} = -\frac{2\langle a^2 \rangle}{\tau_x} \tag{5.78}$$

where τ_x is the damping time constant of the radial betatron oscillations. Under stationary conditions the total time derivative — the sum of Eqs. (5.77) and (5.78) is zero. We get for the stationary expectation value of a^2:

$$\langle a^2 \rangle = \frac{1}{2}\, \tau_x Q_x. \tag{5.79}$$

We may now return to Eq. (5.68) to get the expected spread in the betatron displacements. Squaring and taking the expectation of $x_\beta(s_1)$ we may write for the rms spread in the radial betatron displacement at s_1:

$$\sigma_{x\beta}^2(s_1) = \langle x_\beta^2(s_1) \rangle = \frac{1}{2} \langle a^2 \rangle \beta_1. \tag{5.80}$$

Since the azimuth s_1 may be anywhere, we may now drop the subscript. Combining the last two equations, we may write that

$$\sigma_{x\beta}^2(s) = \frac{1}{4}\, \tau_x Q_x \beta(s). \tag{5.81}$$

The form of the result is similar to that obtained for σ_ϵ. Both τ_x and Q_x are numbers which are determined from the overall properties of the guide field — and do not, therefore, vary with s. The only variation of $\sigma_{x\beta}$ comes from the factor $\beta(s)$. This then is our result for the horizontal spread of a stored electron beam due to quantum induced betatron oscillations.

To see the physical significance of our result we must recover the complexities hidden in τ_x and Q_x. Taking $\mathcal{N}\langle u^2 \rangle$ from Eq. (5.41)

$$Q_x = \frac{3}{2}\, C_u \hbar c \gamma_0^3\, \frac{\langle P_\gamma \rangle_s \langle G^3 \mathcal{H} \rangle_s}{\langle G^2 \rangle} \tag{5.82}$$

were G(s) is the inverse radius of the orbit, and $\mathcal{H}(s)$ is the function of Eq. (5.71). Taking τ_x from Eq. (4.53) we get that

$$\frac{\sigma_{x\beta}^2}{\beta} = \frac{1}{4}\, \tau_x Q_x = \frac{C_q \gamma_0^2 \langle G^3 \mathcal{H} \rangle_s}{J_x \langle G^2 \rangle_s}, \tag{5.83}$$

where C_q is the quantum coefficient defined in Eq. (5.46).

For an isomagnetic guide field $(G = 1/\rho_0,$ or zero) the result simplifies to

$$\frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q \gamma_0^2 \langle \mathcal{H} \rangle_{\text{Mag}}}{J_x \rho_0} \qquad \text{(isomag)}, \qquad (5.84)$$

where $\langle \mathcal{H} \rangle_{\text{Mag}}$ is the average of $\mathcal{H}$ taken only in the magnets. That is,

$$\langle \mathcal{H} \rangle_{\text{Mag}} = \frac{1}{2\pi\rho_0} \int_{\text{Mag}} \frac{1}{\beta} \left\{ \eta^2 + (\beta\eta' - \tfrac{1}{2}\beta'\eta)^2 \right\} ds. \qquad (5.85)$$

Comparing Eq. (5.84) with Eq. (5.48) we see that for an isomagnetic guide field we may write that

$$\frac{\sigma_{x\beta}^2(s)}{\beta(s)} = \frac{J_\epsilon \langle \mathcal{H} \rangle_{\text{Mag}}}{J_x} \left(\frac{\sigma_\epsilon}{E_0} \right)^2 \qquad \text{(isomag)} . \qquad (5.86)$$

For a precise calculation of $\sigma_{x\beta}$ the integral of Eq. (5.85) must be evaluated. We can however, get a simple — but usually quite good — approximation by making use of the approximate relations discussed in Section 3.3. Equation (3.21) gives as a good approximation to (s):

$$\eta(s) \approx \left(\frac{\alpha R}{\nu_x} \right)^{1/2} \beta^{1/2}(s). \qquad (5.87)$$

To the extent that this approximation is valid $\beta\eta'$ and $1/2 \ \beta'\eta$ are everywhere equal and H(s) is just a constant! Namely,

$$\mathcal{H} \approx \frac{\alpha R}{\nu_x} \qquad (5.88)$$

and Eq. (5.86) becomes

$$\frac{\sigma_{x\beta}^2(s)}{\beta(s)} \approx \frac{J_\epsilon}{J_x} \frac{\alpha R}{\nu_x} \left(\frac{\sigma_\epsilon}{E_0} \right)^2 \qquad \text{(isomag)} . \qquad (5.89)$$

Alternatively, Eq. (5.84) can be written as

$$\frac{\sigma_{x\beta}^2(s)}{\beta(s)} \approx \frac{C_q \alpha R \gamma_0^2}{J_x \rho_0 \nu_x} \qquad \text{(isomag)} \qquad (5.90)$$

The radial betatron spread is proportional to the energy of the stored electrons and to the geometric mean of C_q and a length that is a characteristic of the guide field.

To get a qualitative feeling for the order of magnitude of the effect, we may in Eq. (5.89) repace β on the right-hand side by its typical value $\beta_n = R/\nu_x$ and replace α by its approximate equivalent $1/\nu_x^2$ — see Sections 2.8 and 3.3 — to obtain

$$\left(\frac{\sigma_{x\beta}}{\beta_n}\right)^2 \approx \frac{J_\epsilon}{J_x \nu_x^2} \left(\frac{\sigma_\epsilon}{E_0}\right)^2 . \tag{5.91}$$

The ratio of $\sigma_{x\beta}$ to β_n is the same as the ratio of σ_ϵ to E_0 except for the factor $J_\epsilon/J_x\nu_x^2$, which might typically be $\approx 1/3$. As noted earlier, σ_ϵ/E_0 is just γ times the ratio of the Compton wavelength to the magnetic radius. For a typical 1 GeV storage ring we might have $\beta_n \approx 6$ meters and $\sigma_\epsilon/E_0 \approx .4 \times 10^{-4}$ (as found earlier); then $\sigma_{x\beta} \approx 1.4$ millimeters.

As argued in Section 5.3 for the energy deviations, the likelihood of finding any particular betatron displacement will vary as a normal error function. That is, the probability of finding a particular electron with a betatron displacement between x_β and $x_\beta + dx_\beta$ will be

$$w(x_\beta)dx_\beta = \frac{1}{\sqrt{2\pi}\,\sigma_{x\beta}} \exp\left(-x_\beta^2/2\sigma_{x\beta}^2\right)dx_\beta . \tag{5.92}$$

If we think of a particular bunch of electrons which contains, say, N electrons, then as it passes any particular azimuth s, the number of electrons $n(x_\beta)$ which lie in the radial interval dx_β at x_β is

$$n(x_\beta)dx_\beta = Nw(x_\beta)dx_\beta ,$$

and so has also a Gaussian distribution. We may think then, of a stored beam as a fuzzy object with a half-width (which depends on s) given by the standard deviation $\sigma_{x\beta}$ of its distribution in radius.

We should not forget, however, that the total radial spread has contributions from both the betatron and energy oscillations, since the spread of energies of the electrons in a bunch gives rise to an associated radial spread. Recalling that an electron with the energy deviation ϵ moves on an orbit whose radial displacement varies with the azimuthal position $\underline{s}$ according to $x_\epsilon(s) = \eta(s)\,\epsilon/E_0$, it follows that the mean-square radial spread due to the energy spread is

$$\sigma_{x\epsilon}^2(s) = \eta^2(s)\,\frac{\sigma_\epsilon^2}{E_0^2} \tag{5.93}$$

Now the periods of the energy oscillations and of the betatron oscillations are widely different, and certainly not precisely commensurate. We may, therefore, consider that — although they are stimulated by the same stochastic events — they will be statistically independent. We may then add their contribution to the total radial spread as the squares and write that

$$\sigma_x^2 = \sigma_{x\beta}^2 + \sigma_{x\epsilon}^2 . \tag{5.94}$$

Let's consider only an isomagnetic guide field. Taking $\sigma_{x\beta}^2$ from Eq. (5.86) and using Eq. (5.48) for σ_ϵ^2, we may write that

$$\sigma_x^2(s) = \frac{C_q \gamma_0^2}{\rho_0} \left\{ \frac{\langle \mathcal{H} \rangle_{\text{Mag}} \beta(s)}{J_x} + \frac{\eta^2(s)}{J_\epsilon} \right\} \quad \text{(isomag)} . \tag{5.95}$$

Or if we are willing to use the approximate expressions of (5.87) and (5.88) for η and $\mathcal{H}$, the ratio of the two terms in the curly brackets is just J_x/J_ϵ and we may write that

$$\sigma_x^2(s) \approx \sigma_{x\beta}^2 \left(1 + \frac{J_x}{J_\epsilon} \right) \quad \text{(isomag)} . \tag{5.96}$$

The two contributions to the radial spread vary together so their sum is a constant factor. Recalling that J_ϵ is typically about twice J_x we have that

$$\sigma_x \approx \sqrt{1.5}\, \sigma_{x\beta} \tag{5.97}$$

The results of the section do not take into account the effects of coupling between radial and vertical oscillations. If such coupling exists the results must be modified as described in the following section.

5.6. Beam Height

In calculating the beam width we assumed that the emission of a quantum did not change the direction of motion of the electron. This assumption is not strictly correct. Any individual quantum event may give a small transverse impulse to the electron. We may think that the quantum event corresponds to the emission of a photon of momentum u/c at, say the angle θ_γ with respect to the electron's momentum. It will carry off a transverse component of momentum equal to $\theta_\gamma u/c$. Conservation of momentum requires that there be a corresponding change in the transverse momentum $x'E_0/c$ of the electron — see Fig. 45. That is there will be a change in x'
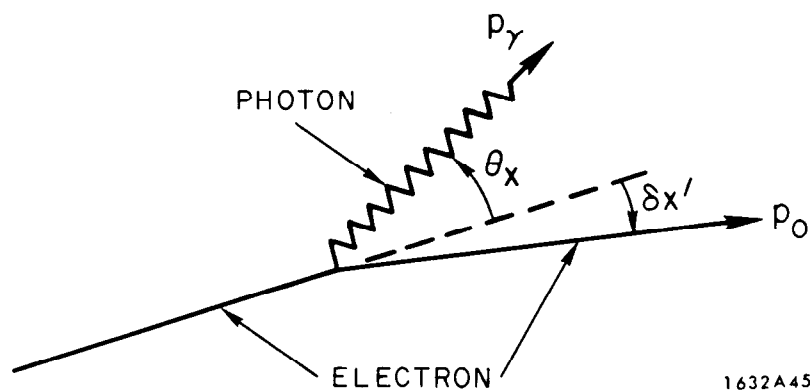
FIG. 45--Change in the direction of an electron due to the emission of a photon.

given by

$$\delta x' = \frac{u}{E_0} \theta_x, \qquad (5.98)$$

where θ_x is the horizontal projection of θ_γ. The synchrotron radiation is emitted generally along the direction of motion of the electron, but is spread out in a cone of half-angle $1/\gamma$. So we may consider that θ_γ is typically of the order of $1/\gamma$. The quantity η' which appears in Eq. (5.67) is of order-of-magnitude 1, so the neglect of the contribution from (5.98) on the radial motion was well justified.

Consider however, what may be the quantum effects on the <u>vertical</u> betatron motion. If the design orbit lies strictly in a plane there are no first-order effects from quantum emission on the vertical motion. (That is, the vertical function which corresponds to   is precisely zero.) The only remaining effect would be from the angular distribution of the radiation. Let's see what the magnitude of effect would be.

We may take over the results of the preceding section by replacing Eq. (5.67) by

$$\delta z = 0; \qquad \delta z' = \frac{u}{E_0} \theta_z, \qquad (5.99)$$

where θ_z is the projected vertical angle of emission of the photon. Equation (5.73) would become — using the subscript z to remind us that we are now dealing with a vertical oscillation —

$$\delta \langle a_z^2 \rangle = \frac{u^2}{E_0^2} \theta_z^2 \beta_z(s_2). \qquad (5.100)$$

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Following through the derivations we would find in place of Eq. (5.81)

$$\sigma_{z\beta}^2(s) = \frac{1}{4} \tau_z Q_z \beta_z(s),$$ (5.101)

with

$$Q_z = \frac{\langle \mathcal{N} \langle u^2 \theta_z^2 \rangle \beta_z \rangle_s}{E_0^2} .$$ (5.102)

To evaluate Q_z we would need to take into account the variation of the frequency spectrum of synchrotron radiation with the angle of emission. Since the effect we are dealing with is in any case small, an approximate calculation will do. Suppose we first make the approximation

$$\langle u^2 \theta_z^2 \rangle \approx \langle u^2 \rangle \langle \theta_z^2 \rangle .$$ (5.103)

For the mean-square projected angle, we may take 1/2 the mean-square polar angle of the radiation

$$\langle \theta_z^2 \rangle \approx \frac{1}{2\gamma_0^2} .$$ (5.104)

Also let's replace $\beta_z(s)$ by a typical value β_n. We then get that

$$Q_z \approx \frac{\langle \mathcal{N} \langle u^2 \rangle \rangle_s \beta_n}{\gamma_0^2 E_0} .$$ (5.105)

Recall that the average of $\mathcal{N} \langle u^2 \rangle$ is just what we called Q in Section 5.2. We may then write that

$$\frac{\sigma_z^2}{\sigma_\epsilon^2} \approx \frac{\tau_z Q_z \beta_n}{\tau_\epsilon Q_\epsilon} \approx \frac{J_\epsilon}{J_z} \frac{\beta_n^2}{\gamma_0^2 E_0^2} .$$ (5.106)

For a flat design orbit $J_z \approx 1$. Considering only the isomagnetic case, we may take σ_ϵ^2/E_0^2 from Eq. (5.48) and get

$$\sigma_z^2 \approx \frac{C_q \beta_n^2}{\rho_0} \qquad \text{(isomag)} .$$ (5.107)

Roughly speaking, β_n is the same order as ρ_0 and

$$\sigma_z^2 \approx C_q \beta_n .$$

(5.108)

The vertical oscillations induced by the quantum emission are energy independent and less than the radial oscillations by roughly the factor $1/\gamma_0^2$. They are very small indeed.

The vertical oscillations given by Eq. (5.107) are so small that they will always be negligible in comparison with the vertical oscillations produced by another much larger effect — a coupling of oscillation energy from the horizontal betatron oscillations into the vertical ones. We did not analyze such effects when we were considering, in Part II, the nature of the betatron oscillations because they would be essentially perturbations of second order. An analysis of the perturbations expected from the construction imperfections in a real ring shows that the coupling between horizontal and vertical oscillations is likely to produce a beam height in the ring which is at least a few percent of the beam width — and is therefore much larger than the minimum intrinsic width calculated above.

Indeed, it is — as we shall see later — sometimes desirable to obtain a beam height larger than is produced by accidental imperfections. And this can be done by introducing an intentional augmentation of the coupling between the horizontal and vertical oscillations — as can be effected by special magnetic elements (skew quadrupoles) or by operating the ring near a resonance between ν_x and ν_z, or by a combination of the two.

A detailed analysis of the coupling of vertical and horizontal oscillations is beyond the scope of this report, but a phenomenological approach will serve our purposes. Suppose we let g_x and g_z represent the invariant mean-square amplitudes of the radial and vertical oscillations. That is,

$$g_x = \frac{\sigma_x^2(s)}{\beta_x(s)} ; \qquad g_z = \frac{\sigma_z^2(s)}{\beta_z(s)} .$$

(5.109)

For the special case in which the damping rates of the vertical and horizontal oscillations are equal, we may now argue as follows. In the absence of coupling

$$g_x = g_0 = \frac{1}{4} \tau_x Q_x$$

(5.110)

— from Eq. (5.81). When coupling is taken into account, the quantum excitation of the radial oscillations can be shared with the vertical oscillations in any proportion up to an equal division. That is, we may have that

$$g_z = \kappa g_x,$$ (5.111)

where κ is the "coefficient of coupling." In principle κ may be any number between 0 and 1, although it is probably difficult in practice to reduce κ below one percent or so. Since the excitation is being shared, the combined excitations must still be equal to g_0.

$$g_x + g_z = g_0.$$ (5.112)

We may equivalently write that

$$g_z = \frac{\kappa}{1+\kappa} g_0 ,$$

$$g_x = \frac{1}{1+\kappa} g_0 .$$ (5.113)

The excitation g_0 is to be taken from any of the expressions for $\sigma_{x\beta}^2/\beta$ derived (without taking coupling into account) in Section 5.6. Given any coupling coefficient κ, g_z and g_x are obtained; and from them the beam half-width and half-height σ_x and σ_z can be found using Eq. (5.109).

The maximum beam height that can be obtained in this way will occur when $\kappa = 1$. Then $(g_z)_{max} = g_0/2$, and

$$\frac{\left(\sigma_z^2\right)_{max}}{\beta_z} = \frac{1}{8} \tau_x Q_x.$$ (5.114)

Using the approximate results of the preceding section for an isomagnetic guide field we may write for the maximum vertical beam spread

$$\frac{\left(\sigma_z^2\right)_{max}}{\beta_z} \approx \frac{C_q \propto R \gamma_0^2}{2 \rho_0 \nu_x} \quad \text{(isomag)}$$ (5.115)

where, since we have assumed that $\tau_x = \tau_z$, I have set $J_x = J_z = 1$.

In principle, either or both of the width and height of a beam can be increased by the artificial stimulation of the transverse oscillations — for example, by the periodic application of impulsive electric or magnetic forces to the stored beam.

In practice, however, such external stimulation will give rise also to coherent motions of large numbers of the electrons in a bunch which has been found to have deleterious effects on the luminosity of colliding beams. It is likely however, that artificial beam enlargement can be used in future rings which will have the possibility of operating with different betatron numbers for the two colliding beams.

5.7. Beam Lifetime from Radial Oscillations

In the preceding section I have argued that the likelihood that a stored electron will pass a given azimuth s with a radial displacement between x and x + dx is distributed as a Gaussian error function — namely as

$$w(x)dx = \frac{1}{\sqrt{2\pi}\,\sigma_x}\ \exp\left(-x^2/2\sigma_x^2\right)dx, \qquad (5.116)$$

with σ_x a function of s. Such a distribution can clearly not be completely correct since it has "tails" which extend to arbitrarily large positive and negative displacements while an actual stored beam must live in a vacuum chamber with a finite aperture! The probability distribution of Eq. (5.116) can be only an approximation which we may expect to be reasonably correct so long as the radial aperture is much larger than σ_x everywhere around the ring.

Even when the aperture is large however, there may still be a significant effect from its finite extent. Sooner or later an electron will suffer a sufficiently large fluctuation in its emission of quanta to produce a radial displacement as large as the aperture limit. Then the electron will be lost by a collision with the edge of the vacuum chamber — or whatever obstruction defines the limit of the aperture. Alternatively if we take into account the nonlinearities of the guide field, large amplitude oscillations may become unstable leading to the loss of the electron from the stored beam. It will be convenient for the present discussion to think in terms of an aperture that is limited by a physical obstruction. An extension of the discussion to a magnetic aperture limit is relatively straight-forward.

So long as the chance of an electron being lost at the aperture limit is small — by which we should mean that it is much less than 1 in a damping time — the proba-bility per unit time of getting lost is the same for all electrons. Then the loss rate from a stored beam will be proportional to the number N of electrons present; and N will therefore, decrease exponentially and with a time constant τ_q related

to the loss rate by

$$\frac{1}{\tau_q} = -\frac{1}{N}\frac{dN}{dt} \qquad (5.117)$$

The number τ_q is usually referred to as the <u>quantum lifetime</u> of the stored beam.

A precise evaluation of the quantum lifetime for all conditions is a bit intricate. I shall therefore, show a way to compute it which is reasonably accurate only when the lifetime is long — which is, after all, the condition of most interest for a storage ring. I shall first look at the lifetime due to lateral oscillations and then look later at the lifetime due to energy oscillations.

Let's think now of a somewhat over-simplified situation in which we imagine that only the radial betatron oscillations are excited — ignoring for the moment the radial spread associated with the energy oscillations. We saw in Section 2.6 that in the absence of radiation effects the betatron oscillations of an electron sweep out a band between the envelope limits $X(s) = a\sqrt{\beta(s)}$ — recall Fig. 12. When we include quantum effects and radiation damping, the "invariant" amplitude factor $\underline{a}$ of any particular electron wanders up and down in a random way. The time scale of the variations of $\underline{a}$ is however, rather slow — that is, much larger than the revolution time — so we may think that the electron continuously sweeps out a radial band whose envelope is slowly varying.

Suppose now that there is at some azimuth, say s_1, an obstruction which defines an aperture limit of the ring. By that I mean that as the invariant amplitude $\underline{a}$ is varied, the envelope $X(s)$ will first encounter an obstruction at s_1. See Fig. 46. All losses will occur at s_1 and we need only consider the radial distribution at this azimuth.
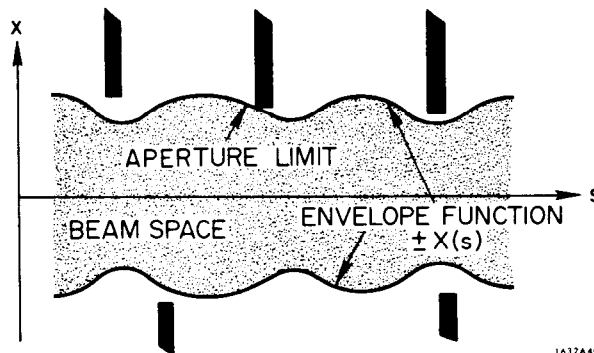

FIG. 46--Radial aperture limit.

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We have seen in Section 2.7 that the radial displacement on successive passages of any chosen azimuth varies with time as

$$x = a\sqrt{\beta_1}\,\cos\omega t \qquad\qquad (5.118)$$

As we did in Section 5.3 for the energy oscillations we may take the square of the amplitude factor as a measure of the "effective energy" of the oscillations. Let's define

$$W = a^2\beta_1. \qquad\qquad (5.119)$$

Quantum effects and radiation damping produce slowly varying fluctuations in W. The same arguments made in Section 5.3 can be used again to show that — in the absence of any aperture limit — the electrons in a beam will have a distribution of W's according to (see Eq. (5.62))

$$h(W) = \frac{N}{\langle W\rangle}\,\exp\,(-W/\langle W\rangle), \qquad\qquad (5.120)$$

where the mean value $\langle W\rangle$ is equal to $2\sigma_x^2$. (The function $h(W)$ is defined such that the number of electrons with "oscillation energy" between W and W = dW is $h(W)dW$.) The function $h(W)$ is shown by the solid curve in Fig. 47.
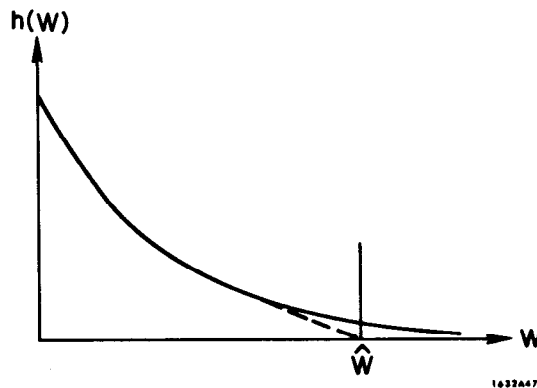
FIG. 47--Distribution of oscillation energies.

Now consider what happens when there is an aperture limit that removes any electron for which W exceeds some limiting value $\hat{W}$ — which we may call "W-peak". There can be no electrons with $W > \hat{W}$, so the actual distribution $h(W)$ must change for large W to correspond to something like the broken line curve in Fig. 47. We may think about what is happening in the following way. The quantum effects are continually trying to fill in the ideal distribution by a "diffusion" of electrons from

the region of small W into the region of large W. But each time an electron reaches $\hat{W}$ it is "wiped off", so there is a continuous loss out of the tail of the distribution. I would like now to make an estimate of this loss rate.

We may make a rough estimate in the following way. We have said that there is a characteristic "relaxation time" for the quantum fluctuations equal to the damping time constant τ_x. We may guess that there is an "attempt" to fill in the tail of the ideal distribution once each damping time. Then the number of electrons lost in each damping time will be equal to the number of electrons in the ideal distribution with $W > \hat{W}$. That number is

$$N(> \hat{W}) = \int_{\hat{W}}^{\infty} h(W)\, dW = N \exp(-\hat{W}/\langle W \rangle) \,. \tag{5.121}$$

The electron loss rate will then be estimated by

$$-\frac{dN}{dt} \approx \frac{N}{\tau_x} \exp(-\hat{W}/\langle W \rangle)\,, \tag{5.122}$$

which would give a quantum lifetime of

$$\tau_q \approx \tau_x \exp(\hat{W}/W)\,. \tag{5.123}$$

We would estimate that the lifetime itself depends exponentially on $\hat{W}/\langle W \rangle$.

An exact calculation of τ_q requires setting up a diffusion equation for $h(W)$ and solving it numerically with the appropriate boundary conditions. I shall not attempt to do this but rather show how a good approximation to the exact result can be obtained.

Consider what would be happening in the neighborhood of some particular W_0 that is much greater than $\langle W \rangle$ if there were no aperture limit. The chance of finding any particular electron with $W > W_0$ in the ideal distribution is very small. We may expect that if an electron once gets into the tail $(W > W_0)$ it is most likely to return rather quickly to the main body of the distribution — being replaced in the tail by some other unfortunate electron. Consider now the flux of electrons passing through a small zone[†] near W_0. The electrons which have been populating the tail will be passing to the left through this zone and an equal flux of electrons

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[†] We should think of passage through a "zone" so that we may ignore the microscopic fluctuations in the amplitude.

will (in the stationary state) be passing toward the right throught the zone due to abnormal quantum fluctuations. (We are neglecting the unlikely events in which an electron leaving the tail would have at that instant an abnormal fluctuation and reenter the tail of the distribution right away.)

Let's estimate the flux of electrons coming $\underline{\text{out}}$ of the tail. When W is large the "normal" energy fluctuations can be neglected in comparison with the rate of decrease of W due to the damping. For any electron the damping gives

$$\frac{dW}{dt} = -\frac{2W}{\tau_x} \quad , \tag{5.124}$$

and the flux of electrons through W_0 due to the damping will be

$$\left\{ Nh(W)\frac{dW}{dt} \right\}_{W_0} = \frac{2N W_0 h(W_0)}{\tau_x} \tag{5.125}$$

In the absence of an aperture limit the net flux through any W — and past W_0 in particular — must be zero so there would also be an outward flux of electrons quite equal to the inward flux of (5.125).

Now put in the aperture limit at $\hat{W}$. If it is sufficiently large, the main body of the distribution is little affected. The flux outward through $\hat{W}$ will be unchanged while the return flux will of course, be zero. We have that the flux of (5.124), evaluated at $W_0 = \hat{W}$, is also an estimate of the outward flux of lost electrons. The loss rate will be

$$-\frac{dN}{dt} = \frac{2N\hat{W}h(\hat{W})}{\tau_x} . \tag{5.126}$$

Using Eq. (5.120) for h(W) we obtain,

$$\tau_q = \frac{\tau_x}{2} \frac{\langle W \rangle}{W} \exp\left(\hat{W}/\langle W \rangle\right). \tag{5.127}$$

Remember that $\hat{W}$ and $\langle W \rangle$ are related to the limiting radial excursion permitted by the aperture (assumed to occur at some azimuth s_1) and the rms radial displacement at that azimuth by

$$\hat{W} = \left[a^2 \beta_x(s_1) \right]_{\text{Max}}; \qquad \langle W \rangle = 2\sigma_x^2(s_1), \tag{5.128}$$

with both numbers evaluated at the azimuthal position of the limiting aperture.

This result differs from the estimate in Eq. (5.123) by the factor $\langle W \rangle/2\hat{W}$ and gives, therefore, a lifetime smaller by a factor which might be typically 5 or

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10. The discrepancy can be explained by arguing that the "relaxation time" is shorter by this factor for the population of the tail of the distribution than for the main body of it — which is understandable since a large fluctuation has a better chance of dominating the radiation damping if it accumulates during a relatively short time span. Although Eq. (5.127) was derived by making some approximations whose quantitative significance we have not tried to estimate, the same result has been obtained by more sophisticated — although still approximate — techniques.[5,15]

In our derivation of the quantum lifetime we have assumed that the radial fluctuations were due solely to betatron oscillations. As we have seen in Section 5.5 however, the radial beam spread has contributions from both the betatron and energy oscillations. And the analysis is complicated by the fact that the two components have different damping time constants. I shall not attempt to refine the calculation but settle for the following comments. The two damping time constants are not very different — usually within a factor of two of each other. It is then clear that Eq. (5.125) will give a reasonable approximation if we use for σ_x^2 the total mean-square beam spread and for τ_x some value between the betatron and synchrotron damping time constants. Or alternatively we may get a "safe" estimate of τ_q — that is a lower limit — by using for τ_x the smaller of the two time constants.

The quantum lifetime increases approximately exponentially with the square of the limiting radial excursion — an exceedingly rapid variation. There is then, a rather precise criterion for the aperture required. If the aperture is just a little bit too small the lifetime will be disastrously short, but if it is a little larger than necessary the lifetime will be astronomically large and will be of no consequence.[†] The "critical" aperture limit occurs at about

$$\frac{|x_{max}|}{\beta_x} = 6\sigma_x \tag{5.129}$$

which gives $\hat{W}/\langle W \rangle \approx 18$ and from Eq. (5.120),

$$\tau_q = \frac{\tau_x}{36} e^{18} \approx 1.5 \times 10^6 \, \tau_x . \tag{5.130}$$

[†] Since other loss mechanisms will then dominate.

Since τ_x is typically about 0.1 sec, the critical aperture gives a quantum lifetime of about one day. Other effects such as gass scattering usually give lifetimes of several hours and the filling time (time to store an operating beam) is generally a fraction of an hour, so a quantum lifetime of one day is quite "safe." We can understand the "rule-of-thumb" that the full aperture must be at least 12 times the standard deviation σ_x of the radial distribution. A similar rule clearly holds for the vertical aperture.

5.8. Beam Lifetime from Energy Oscillations

In the preceding section we have examined the loss of electrons due to abnormal fluctuations in the amplitudes of the radial oscillations. Loss of electrons from a stored beam will also occur when abnormal fluctuations in the energy oscillations result in energy excursions so large that they can no longer be contained within the energy aperture that is determined by the radio frequency accelerating system.

In Section 3.6 we saw that the energy oscillations correspond to the motion of an ideal particle in a potential well, one of whose walls is a potential "hill" of limited height. The situation was described by Fig. 36(b), a part of which is redrawn in Fig. 48(a). The horizontal coordinate is the time displacement τ associated with the energy oscillations and the vertical coordinate is a fictitious "potential energy"
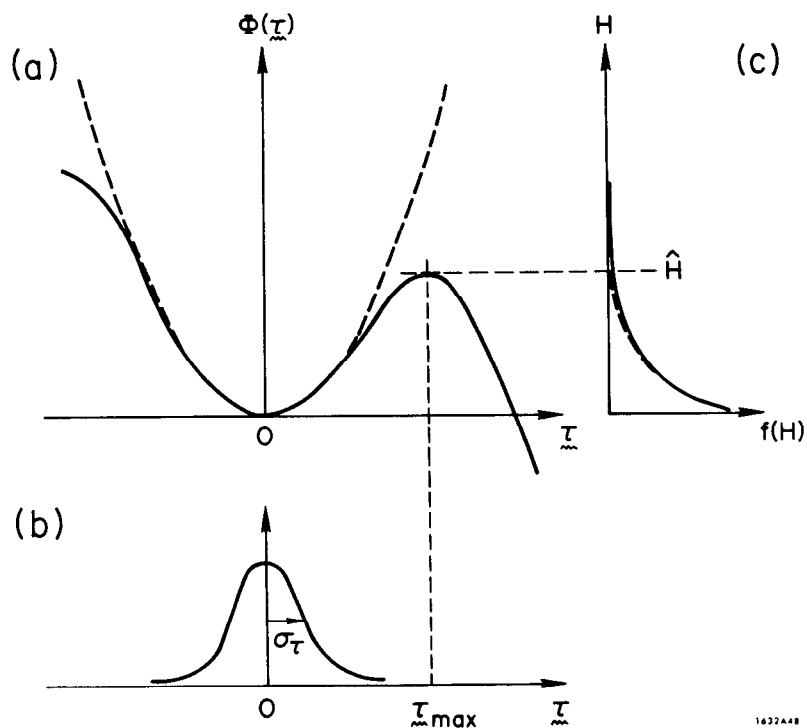

FIG. 48--Quantum spread in the energy oscillations.

of the oscillation. The corresponding "kinetic energy" is

$$\frac{1}{2}\left(\frac{d\tau}{dt}\right)^2 = \frac{\alpha^2}{2}\left(\frac{\varepsilon}{E_0}\right)^2 \qquad (5.131)$$

where ε is the instantaneous energy deviation of the real energy oscillation.

Suppose we let H represent the "total oscillation energy" — that is, the sum of the "potential energy" and the "kinetic energy" of Eq. (5.131) —

$$H = \Phi(\tau) + \frac{\alpha^2}{2}\left(\frac{\varepsilon}{E_0}\right)^2 \qquad (5.132)$$

(Φ is taken to be zero at the bottom of the potential well.) During the oscillation of any particular electron the "potential energy" reached at the maximum of τ is equal to H. And the peak "kinetic energy" — which occurs as the electron passes $\tau = 0$ — is also equal to H, so

$$H = \frac{\alpha^2}{2}\frac{\hat{\varepsilon}^2}{E_0^2} \qquad (5.133)$$

where $\hat{\varepsilon}$ is the peak value of ε during its oscillation. An electron is captured in a stable energy oscillation if H is less than Φ_{max}, the maximum height of the potential well. (See Section 3.6.) Otherwise it will be lost.

In Sections 5.2 and 5.3 we have examined the quantum-induced energy oscillations under the assumption that they were ideally linear — which would correspond to the ideal parabolic potential-well indicated by the broken line in Fig. 48(a). Under these assumptions, the distribution of time displacements in a stored bunch of electrons would be as the Gaussian function drawn in Fig. 48(b) — whose standard deviation σ_τ was evaluated in Section 5.4.

We have also seen that the energy fluctuations yield an exponential distribution in the square of the amplitude of the energy oscillations — as described by Eq. (5.62). The quantity W used there is just the square of the amplitude (of the oscillation in ε) and is therefore, proportional to the "total energy" H. In fact,

$$H = \frac{\alpha^2}{2E_0^2}\,W. \qquad (5.134)$$

It follows that the distribution over H for the electrons stored in a bunch is also exponential. Specifically, if we let f(H)dH represent the number of electrons with "total oscillation energies" between H and H + dH, then a direct translation of

Eq. (5.62) gives

$$f(H)dH = \frac{N}{\langle H \rangle} \exp(-H/\langle H \rangle),$$

(5.135)

where

$$\langle H \rangle = \frac{\alpha^2}{2E_0^2} \langle W \rangle = \frac{\alpha^2}{E_0^2} \sigma_\epsilon^2 .$$

(5.136)

This distribution in oscillation energies is shown in Fig. 48(c).

The real situation must evidently be different. Any electron whose time displacement once exceeds τ_{max}, the value of τ at the top of the actual potential hill — or equivalently, one whose "oscillation energy" H exceeds Φ_{max} — will be lost from the stored bunch. As we saw in the preceding section for the radial oscillations, we must expect that the actual distributions will fall to zero at τ_{max} — and therefore at $H = \hat{H} = \Phi_{max}$. And there will be a continuous loss of electrons due to diffusion out of the tail of the distribution.

The situation here is similar to the one discussed in the preceding section, which would correspond to a parabolic potential well which is suddenly truncated at τ_{max}. The smooth rounding of the potential maximum will have a somewhat different effect on the comportment of the distribution of electrons near the edge of the distribution. One may expect however, that so long as $\tau_{max} \gg \sigma_\tau$, the rate of loss of electrons may be estimated in the same way for both situations.

Without repeating the argument here we may write the result which corresponds to Eq. (5.127), translated to the case of the energy oscillations,

$$\tau_q = \frac{\tau_\epsilon}{2} \frac{e^\xi}{\xi} ,$$

(5.137)

with

$$\xi = \frac{\hat{H}}{\langle H \rangle} = \frac{\Phi_{max}}{\langle H \rangle}$$

(5.138)

The height Φ_{max} of the potential maximum can be evaluated by performing the integration of Eq. (3.53) — or for a sinusoidal rf voltage, from Eq. (3.58).

The potential Φ_{max} was introduced in order to obtain the magnitude of the "aperture" of the energy oscillations. It is related to the maximum acceptable energy deviation ϵ_{max} — see Eq. (5.57) —

$$\Phi_{max} = \frac{\alpha^2}{2} \frac{\epsilon_{max}}{E_0}$$

(5.139)

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So ξ has the conceptually simple form

$$\xi = \frac{\epsilon_{max}^2}{2\sigma_\epsilon^2} .$$ (5.140)

The potential Φ_{max} and, therefore, the number ξ depends on the magnitude of the rf voltage which must always be sufficiently large to give a quantum lifetime greater than the desired storage time of the beam. Typically ξ must be at least as large as 18 or so, requiring that $(\epsilon_{max}/\sigma_\epsilon)$ be about 6.

For the particular (but very common) case of a storage ring with an isomagnetic guide field <u>and</u> a sinusoidally varying rf voltage, the parameter ξ can be expressed rather simply in terms of the ring parameters. Bringing together the results obtained in earlier sections for ϵ_{max} and σ_ϵ you can show that

$$\xi = \frac{J_\epsilon E_0}{\alpha k E_1} F(q) \qquad \text{(isomag)} .$$ (5.141)

where E_1 is a constant with the dimensions of an energy:

$$E_1 = \frac{3 m c C_q}{2 r_e} = \frac{55 \sqrt{3}}{64} \frac{\hbar c}{r_e} \approx 1.08 \times 10^8 \text{ eV},$$ (5.142)

and $F(q)$ is the energy aperture function which was defined in Eq. (3.60) and is shown in Fig. 38. The parameter q is the rf overvoltage — namely the ratio of the peak rf voltage to the energy lost in one turn. For large overvoltages $F(q)$ is approximately $(2q - \pi)$ and the quantum lifetime increases exponentially with increasing rf voltage.

Notice that in a storage ring with a given guide field (that is with a fixed α, J_ϵ, τ_ϵ, and E_0) the overvoltage required for any particular quantum lifetime (that is for a particular ξ) depends on the harmonic number k of the rf system. For large harmonic numbers the overvoltage required varied approximately as $\sqrt{k}$.

VI. THE LUMINOSITY OF A HIGH ENERGY STORAGE RING

6.1.   Recapitulation

The usefulness as an instrument for high energy physics of a colliding beam storage ring of any particular energy depends on its luminosity at each energy. In the first part of this report some general expressions for the luminosity were derived. (See Sections 1.5 through 1.8.) They depend explicitly on certain characteristics of the guide field in which the beams are stored and on the dimensions of the colliding beams at the collision point. The succeeding parts of this report have been devoted to an analysis of the behavior of the electrons in stored beams. We now have all of the information needed for relating the design characteristics and operating conditions of storage rings to their expected luminosity. I wish to show how all of the pieces can be brought together to understand the performance that may be expected of a <u>high energy</u> storage ring.

In doing so it will be useful to consider not only the general results that can be obtained, but also to relate — to the extent possible — the performance of a ring to a small number of parameters that may be used to describe the general nature of a ring. Such an exercise will serve to illuminate the most significant aspects of a ring design and will, therefore, permit some comparisons of apparently dissimilar designs.

Let's review where we stand. The luminosity has been written — in Eq. (1.17)— as

$$\mathscr{L} = \frac{f}{4} \frac{N^2}{A_{int}} \; ; \tag{6.1}$$

where N is the number of electrons stored in each beam, f is their frequency of revolution and A_{int} is the "effective interaction area." This area was defined — in Eq. (1.10) — by

$$A_{int} = \frac{\pi}{4} B w^*_{eff} h^*_{eff} \tag{6.2}$$

where B is the number of interacting bunches in each beam and w_{eff} and h_{eff} are the effective projected width and height of the circulating bunches at the interaction region. The latter were defined in Eqs. (1.6) and (1.8) in terms of the width w^* and height h^* (actually the double standard deviations) of the Gaussian distributions of the transverse density of the beam at the intersection. And in terms of the

bunch length ℓ and the half-angle δ between the trajectories of the two beams.[†] Specifically, if the two beams intersect in a horizontal plane with the half-angle δ_H between design orbits

$$w^*_{eff} = \left(w^{*2} + \ell^2 \delta_H^2 \right)^{1/2},$$

$$h^*_{eff} = h^*; \tag{6.3}$$

or if they cross in a vertical plane with the half-angle δ_V between the design orbits,

$$w^*_{eff} = w^*$$
$$h^*_{eff} = \left(h^{*2} + \ell^2 \delta_V^2 \right)^{1/2} \tag{6.4}$$

The beam width and height may be taken as their "natural" values (with or without coupling) which were evaluated in Sections 5.5 and 5.6. Or they may be increased by "artificial" stimulation to any size permitted by the transverse aperture.

The bunch length ℓ can be found from the results of Section 5.4. It depends, among other things, on the nature of the rf accelerating voltage. (And may involve, perhaps a mysterious factor for anomalous bunch lengthening.) But remember that the rf system must be operated with an rf voltage which will insure an adequate lifetime of the stored beam — as discussed in Section 5.8.

The dimensions computed for the beam will be correct only so long as the intensity of the stored beams is not too large. Otherwise the macroscopic fields produced at the intersection of the two beams will disturb the trajectories causing an increase of the beam dimensions to grow, and the luminosity to fall. The limit on the beam intensity — which depends on the beam dimensions and on the properties of the guide field — was described in Section 1.6, and considered in more detail in Section 2.12. To simplify the discussion here, I shall continue to assume that intensity limit is set by the <u>vertical</u> "tune shift." Then the prescription given above may be used to obtain the luminosity so long as the number of stored particles in each beam does not exceed the limit set by (see Section 1.6).

$$\frac{N}{A_{int}} \leq D_c = \frac{2 \Delta \nu_0 E_0}{r_e mc^2 \beta_V} \tag{6.6}$$

[†]The bunch length does not vary around a storage ring so we do not need to specify where it is to be evaluated.

where $\Delta\nu_0$ is a constant (which I shall take to be exactly 0.025), E_0 is the nominal energy of the stored beam, and β_V is the value of the betatron function of Section 2.6 evaluated at the intersection point. This last relation sets a fundamental upper bound to the luminosity that can be reached by a storage ring of given design characteristics.[†]

I would like to round out this report by applying the results that have been obtained to an idealized model storage ring. Some of the results will be independent of many of the details of the model. Those which do depend on particular assumptions of the model can easily be adapted to alternative configurations.

6.2. The Model Storage Ring

As a model storage ring we may take one for which the design orbit is in the form of a "racetrack" — consisting of two more-or-less semicircular "normal" sections joined by two "special" long straight segments. See Fig. 49. The normal
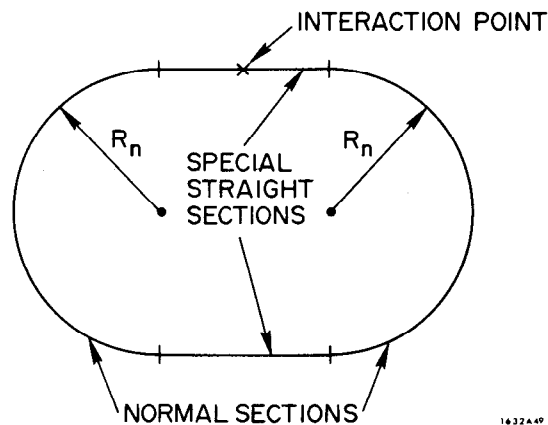
FIG. 49--Design orbit of the model storage ring.

semicircular sections of the guide field are assumed to consist of periodic arrays of bending magnetics and focussing quadrupoles — such as might be typical of a well-designed circular storage ring. These normal sections may be described by their mean radius R_n (defined as the total length of the normal sections divided by 2π)[††]

[†] As pointed out earlier, the intensity limit adopted here leaves out of consideration the interesting proposal of the Orsay group for storage rings in which the colliding beams are neutralized in the collision region.

[††] Notice that R_n is not the same as the parameter R sometimes used in earlier Sections.

by the typical value β_n of the betatron function (see Section 2.8). We may take that both the radical and vertical oscillations have about the same typical value of β so that $\beta_x \approx \beta_z \approx \beta_n$. It will also be convenient to write the ratio R_n/β_n as ν_n, the betatron number of the normal sections — which is equal to the betatron number one would have if there were no special long straight sections.

The analysis will be kept simple by assuming that the guide field is both separated function and isomagnetic (see Section 2.2). And the ratio of R_n to the bending radius ρ_0 will be called λ_n — which is a number usually fairly near to 3.

The long special straight sections we assume to have — at least approximately— the following characteristics. They are ideal "unity inserts" (see Section 2.13) so that the beam behavior in the normal sections of the ring will be essentially the same as if the long straight sections were not present and each of the straight sections will contribute 2π to the advance of the betatron oscillation phase. The center of the straight section is designed to be a beam collision point. At this point the vertical betatron function β_z takes on the value β_V and the horizontal betatron function β_x takes on the value β_H. The variation of β_z and β_x in the neighborhood of the collision point will be ignored.

The complete guide field will have a total length L of its design orbit which is $2\pi R_n$ plus the combined length of the two long straight sections. And the nominal frequency f of revolution of the electrons (the inverse period) will be c/L.

The total storage ring configuration is assumed to consist of two such idealized guide fields disposed so that two separate stored beams will collide at the centers of their respective long straight sections, but will be otherwise quite independent. It will be assumed that the crossing of the two beam orbits occurs in a horizontal plane with the crossing angle δ_H. Further δ_H will be taken always large enough that the term $\ell\delta_H$ dominates the term w* in Eq. (6.3). (I ignore any topological problem that may be implied by the configuration of the idealized storage ring!)

For definiteness, I shall assume that the model storage ring is equipped with a simple radio frequency system which produces a sinusoidal accelerating voltage whose peak value $\hat{V}$ is adjustable and whose frequency of oscillation is k times larger than the revolution frequency — where the harmonic number k is, of course,

an integer. It will be assumed also that the systems can deliver any power required to store the beams <u>up to a maximum</u> P_0 for <u>each</u> of the two stored beams. In a high energy ring it should generally true that the losses in the rf cavity systems are a small fraction of the total power available. I shall therefore, assume that P_0 is independent of the actual rf voltage used and therefore, of the energy of the electrons stored.

We shall see that the luminosity of the idealized rings can be expressed rather simply in terms of the parameters that have been chosen here to describe the physical characteristics of the storage ring. For future reference these parameters are listed in Table 6.1.

TABLE 6.1

PARAMETERS OF THE MODEL STORAGE RING

β_n	Typical value of the betatron functions in the normal sections.
R_n	Mean radius in the normal sections.
$\nu_n = R_n/\beta_n$	Betatron number of the normal sections.
ρ_n	Bending radius in the magnets.
λ_n	Elongation factor of the normal sections.
L	Total orbit length.
$f = c/L$	Electron revolution frequency.
P_0	Maximum radio frequency power available for each beam.
β_V, β_H	Vertical and horizontal betatron functions at the beam crossing point.
δ_H	Half-angle between the beam lines at the crossing point.

6.3. High Energy Luminosity

The maximum number N_{max} of electrons that can be stored in a beam is set by the maximum rf power P_0 available to accelerate each stored beam. Each electron in a stored beam radiates energy at the rate given by Eq. (4.9) so N_{max} is related to P_0 by

$$N_{max} = \frac{3(mc^2)^2}{4\pi r_e C} \frac{\rho_0 L P_0}{E_0^4} \tag{6.7}$$

The maximum stored intensity decreases as the fourth power of the energy.

I shall define the <u>high energy regime</u> of a storage ring as the range of energies in which the intensity of the stored beam is limited only by the available power — and so is determined by Eq. (6.7) — <u>and in which the effective interaction area</u> A_{int} <u>can always be adjusted to the value</u> A_c defined by

$$A_c = N_{max}/D_c \tag{6.8}$$

where D_c is the critical transverse density that was defined in Eq. (6.6). This high energy regime corresponds to "Case 2" of Section 1.7, so the optimum luminosity in this regime is given by Eq. (1.22). Retaining the notation $\mathscr{L}_2$ for this regime we have that

$$\mathscr{L}_2 = \frac{f}{4} D_c N_{max} = c_2 \frac{R_n P_0}{\lambda_n \beta_V E_0^3} \tag{6.9}$$

where L has been replaced by c/f, and ρ_0 by R_n/λ_n. The physical constant is

$$c_2 = \frac{3}{8\pi} \frac{(mc^2)^2 \Delta \nu_0}{r_e^2} \tag{6.10}$$

In convenient units, $c_2 = 6.125 \times 10^{29}$ (GeV)^3-meter^{-2}-watt^{-1}-sec^{-1}.

Apart from the relatively unfree parameter λ_n (generally about 3) the optimum luminosity at each energy E_0 in the high energy regime depends only on the geometric scale of the ring, R_n, on the available rf power P_0, and on the betatron function β_V at the intersection. Notice that in this regime the luminosity <u>decreases</u> <u>as the cube of the operating energy</u> E_0. And it is inversely proportional to β_V.

The special long straight section in our model ring is intended to permit the introduction of a suitable magnet-free space for observation of the collision region and also to provide for the lowest possible value of β_V in that region. The general

properties of a low-beta insert were discussed in Section 2.13 although no attempt
was made there to illuminate the artistic aspects of their design. Let me say
here only that practical considerations seem to preclude the realization of a beta
lower than about 5 cm if a free space of about 5 meters is to be provided. (Also,
if the bunches in the stored beams are as long as 5 cm or so — which is not un-
typical for some convenient choices of the operating frequency — much smaller
values of β_V would not be useful since β_z would then vary by a large factor across
the beam intersection region, (see Section 2.13).)

I should remind you that β_V appears (as opposed to β_H) because I have assumed
that it is the vertical tune shift that sets the intensity limit. And I should also
point out that — from the results of Section 2.12 — the intensity limit of Eq. (6.6)
is strictly correct only if w_{eff} is appreciably larger than h_{eff}. If $w_{eff} \approx h_{eff}$, our
relations need some slight modification. And if w_{eff} h_{eff}, it is likely that the
horizontal tune shift will be dominant and β_V would then have to be replaced by
β_H in all of the formulas of this section.

The most important aspect of the result of Eq. (6.8) is perhaps the fact that
the optimum luminosity in the high energy region does not depend explicitly on the
dimensions of the beam at the intersection region, nor on the crossing angle. We
may then, say that all configurations of high energy storage rings are equivalent in
the high energy regime. If rings of different configurations have comparable
R_n, P_0 and β_V, (or β_H) then their optimum luminosities will all be comparable in
the high energy regime! And, as we have seen, this luminosity will decrease with
increasing operating energy as E_0^{-3}.

This behavior of electron storage rings is somewhat unfortunate. Cross
sections of any particular high energy, electron-positron interaction are likely to
decrease with energy at least as rapidly as E_0^{-2}. So the observed rate of interactions
of a given kind will have a downward dependence on energy that is rapid indeed!

We must expect that the high energy regime of a storage ring will extend over
a limited interval of energies — namely only over those energies for which the
stored current is determined by the available P_0 power according to Eq. (6.7) and
for which the beam area can be adjusted to satisfy Eq. (6.8). If we express Eq. (6.8)
in terms of the storage ring parameters — by using Eq. (6.7) for N_{max} and Eq. (6.6)
for D_c — we get that A_c is given by

$$A_c = \frac{3}{8\pi} \frac{(mc^2)^3}{c \Delta \nu_0} \frac{\rho_0 L P_0 \beta_V}{E_0^5} \tag{6.11}$$

For any particular storage ring (with a given P_0, L, ρ_0, and β_V) the beam area must vary as the <u>fifth</u> <u>power</u> of the energy if the optimum luminosity is to be obtained at each energy. The boundaries of the high energy regime are reached when it is no longer possible to satisfy Eq. (6.11).

For a given storage ring the upper boundary of the regime will be reached when the minimum achievable beam area is larger than the critical area of Eq. (6.11). (This circumstance may or may not occur before the maximum energy capability of the guide field is reached.) For any given ring the minimum effective interaction area that can be obtained will generally depend on the beam energy. Let's call this area $A_{min}(E_0)$. Then the upper boundary of the high energy regime will occur at the energy E_L (where L stands for "limit") for which the equation

$$A_{min}(E_L) = A_c(E_L) \qquad (6.12)$$

is satisfied — and $A_c(E_0)$ is the function defined by Eq. (6.11).

Above the energy E_L we have a new regime which we may call the <u>ultra</u> <u>high</u> <u>energy</u> <u>regime</u>. Let's write the luminosity in this regime $\mathscr{L}_3$. It will be obtained from Eq. (6.1) with N taken as the N_{max} of Eq. (6.7) and A_{int} taken equal to $A_{min}(E_0)$:

$$\mathscr{L}_3 = \frac{9}{64\pi} \frac{(mc^2)^4}{r_e^2 c} \frac{L\rho_0^2 P_0^2}{E_0^8 A_{min}(E_0)} \qquad (6.13)$$

It will generally turn out that $A_{min}(E_0)$ varies as either E_0 or as E_0^2 so the luminosity in the ultrahigh energy regime will decrease with energy as E_0^{-9} or E_0^{-10}! It is clear that a storage ring would probably not be very useful very far into this energy range. Indeed, we should expect a well-designed storage ring to have the energy E_L at or above the highest desired operating energy.

The lower boundary of the high energy regime will be found at that energy for which it is no longer possible to make the interaction area as large as the critical area demanded by Eq. (6.11). The maximum achievable interaction area may be a fixed number set by the aperture of the guide field and the geometry of the interaction region or may depend on the beam energy. Let's say that the maximum achievable area is some function $A_{max}(E_0)$. The lower boundary of the high energy regime will then occur at the energy E_T, which we may call the <u>transition</u> <u>energy</u>[†]

[†]Not to be confused with the so-called "transition energy" of proton synchrotrons.

for which

$$A_{max}(E_T) = A_c(E_T) \tag{6.14}$$

where $A_c(E_0)$ is the function defined by Eq. (6.11).

Below the transition energy the luminosity will no longer follow Eq. (6.9) and a new regime takes over. It is discussed in the next section.

6.4. Low-Energy Luminosity

A storage ring operated at an energy below E_T, the transition energy just defined may be said to be in its low-energy regime. The energy E_T was defined as the energy below which it is not possible — for some technical reason — to increase the effective interaction area to the size demanded by Eq. (6.8). It follows then that in the low-energy region the beam current must be held below what could be sustained by the rf power P_0 available if the beam density limit of Eq. (6.6) is not to be exceeded. The optimum luminosity will be obtained in this region if the effective interaction area A_{int} of Eq. (2.6) is adjusted to its maximum possible value at each energy — that is, to the value $A_{max}(E_0)$ defined in the preceding section — and the beam current is adjusted to the value N_c such that

$$N_c = D_c A_{max} \tag{6.15}$$

This situation corresponds to the Case 1 considered in Section 1.7, and the luminosity in the regime is the $\mathscr{L}_1$ given by Eq. (1.20):

$$\mathscr{L}_1 = \frac{f}{4} D_c^2 A_{max}. \tag{6.16}$$

You should remember that D_c and A_{max} will generally both be functions of energy.

The effective interaction area is defined by Eq. (6.2) and Eq. (6.3) or Eq. (6.4) in terms of the width, height, and length of a stored bunch of the collision point, the number of stored bunches in a beam, and the beam intersection angle. The maximum interaction area will be obtained when all these quantities are made as large as possible. Let's now look on what determines the limits on them.

First, it is clear that the introduction of a crossing angle will always increase the effective interaction area. The crossing angle may be fixed at some "large" angle — as in the recent SLAC design — or may be adjustable — as in the Frascati and DESY designs. In any case, there will probably be some upper limit set by the geometry of the ring design. If the angle is adjustable, the transition between the high-energy and low-energy regimes will presumably occur when the angle is

at its maximum. We may therefore take the crossing angle as fixed at its maximum value in the low-energy regime. The maximum angle is probably limited to some number not greater than 0.1 for various reasons. So long as it is no greater than that, the approximations I have made (which assume that the angle is much less than 1) are still valid.

I shall also assume — to simplify the formulas— that the crossing angle is "large" in the sense that the term $\ell \delta_H$ dominates w* in Eq. (6.4). (Remember that I have adopted a horizontal crossing angle for my model ring.) Then the effective interaction area of Eq. (6.2) becomes

$$A_{int} = \frac{\pi}{4} B\ell\delta_H h* \qquad (6.17)$$

And this is the quantity we want to be as large as possible.

Next, it is clear that we want B, the number of bunches, to be large. Since the rf voltage is periodic with the harmonic number k, the maximum number of bunches that can be stored is also k; with all bunches filled, B = k. It would at first sight appear that k should be as large as possible. That would be true if the crossing angle δ_H were zero. But with a large crossing angle it is the product Bℓ which appears and we must expect that ℓ will decrease with increasing k. It can indeed be shown that the maximum achievable value of Bℓ is nearly independent of k so long as k is not too small. Since the demonstration of that fact is a bit long, I defer it to the next section and just take here the result obtained there. To a good approximation, it is found that for k not too small, the maximum bunch length is just $1/3\pi$ of the minimum spacing between bunches, namely the ratio of the orbit length L to the harmonic number k. That is,

$$\ell_{max} = \frac{L}{3\pi k} \qquad (6.18)$$

We have the simple result that

$$(B\ell)_{max} = k\ell_{max} = L/3\pi \qquad (6.19)$$

The maximum of Bℓ in Eq. (6.17) is just a number proportional to the size of the storage ring.

The only remaining factor that appears in the effective interaction area is h*, the beam height at the collision point. We have seen in Part V that the minimum transverse dimensions of the beam are determined by the intrinsic quantum excitation of the radial oscillations and by the coupling between the vertical and horizontal

oscillations. It should be possible in principle to _increase_ the beam dimensions above this minimum size by exciting the betatron oscillations with specially applied external electromagnetic forces. This has in fact, been done in existing storage rings. But it is also found that such artificial beam enlargement does not lead to any increase in the maximum luminosity.

It can be shown that the techniques used for beam enlargement produce also some coherent transverse oscillations. And these coherent oscillations probably lead to a decrease in the current density that can be tolerated when two beams are brought into collision. It should be observed however, that in all existing rings the two beams are stored in the same guide field and have, therefore, identical betatron frequencies. We may expect that the coupling between the two beams would be greatly reduced if the beams had _different_ betatron frequencies — as could be arranged in future rings. It may then, be possible to get an increased luminosity in future storage rings by the artificial enlargement of the transverse dimensions. Since this possibility has however, not yet been studied in detail, I shall make the more pessimistic assumption that the current density limit of Eq. (6.6) must be applied only to beams in which the betatron oscillations in a beam are the completely incoherent "natural" oscillations produced by quantum fluctuations.

We must then ask what is the _maximum_ beam height that can be obtained using only the natural quantum-induced oscillations. This question has been considered in Section 5.6. Remembering that $h = 2\sigma_z$, we get from Eq. (5.115) that

$$h^{*2}_{max} = \frac{55}{64\sqrt{3}} \frac{\hbar c}{(mc^2)^3} \frac{\alpha R_n \beta_V E_0^3}{\rho_0 \nu_n} \tag{6.20}$$

For our present purposes we may use the approximation that $\alpha = 1/\nu_x^2 = 1/\nu_n^2$ — see Eq. (3.22). Also writing λ_n for R_n/ρ_0, we have that

$$h^{*2}_{max} = \frac{55}{16\sqrt{3}} \frac{\hbar c}{(mc^2)^3} \frac{\lambda_n \beta_V E_0^2}{\nu_n^3} \tag{6.21}$$

The maximum possible beam height varies directly with the beam energy E_0. (We are of course assuming that the aperture throughout the ring is always large enough to accommodate the beam width and height with the maximum coupling that we have postulated.)

We now have all of the information required for computing the maximum value of the interaction area of Eq. (6.17). Namely,

$$A_{max} = \frac{\pi}{4} (B\ell)_{max} \delta_H h^*_{max} , \tag{6.22}$$

where $(B\ell)_{max}$ is given in Eq. (6.19) and h^*_{max}, in Eq. (6.21). Putting the pieces together,

$$A_{max} = c_A \frac{\lambda_n L \beta_V^{1/2} \delta_H E_0}{\nu_n^{3/2}} , \tag{6.23}$$

where

$$c_A = \frac{1}{48\, mc^2} \left(\frac{55}{\sqrt{3}} \frac{\hbar c}{mc^2} \right)^{1/2} . \tag{6.24}$$

With this expression for A_{max} (and remembering that $L = c/f$) the low energy luminosity of Eq. (6.16) becomes

$$\mathscr{L}_1 = c_1 \frac{\lambda_n^{1/2} \delta_H E_0^3}{(\nu_n \beta_V)^{3/2}} , \tag{6.25}$$

with

$$c_1 = \frac{c \Delta \nu_0^2}{48\, r_e^2 (mc^2)^3} \frac{55}{\sqrt{3}} \frac{\hbar c}{mc^2}^{1/2} . \tag{6.26}$$

In convenient units, $c_1 = 1.29 \times 10^{37}$ $(GeV)^{-3}$ $-meter^{-1/2}$ $-sec^{-1}$.

We find that the optimum luminosity in the low-energy regime varies with the cube of the operating energy. In addition, it varies linearly with the crossing angle δ_H and inversely as the 3/2 power of the betatron function at the beam intersection.

The low-energy regime in which the luminosity follows Eq. (6.25) extends from the energy E_T downward. (Above the energy E_T there is no longer enough current to match the interaction area.) The lower energy bound of this regime is less well determined. It will, most likely, occur when — for some reason or other — it is necessary to limit the stored current below the critical value defined by Eq. (6.15). The current limit may be set by the onset of some beam instability not considered here. Or in the most favorable circumstance the current will be limited at low energies only due to the loss of beam from the Touschek effect — see Section 1.3.

6.5. Maximum Bunch Length

We saw in the preceding section that the low-energy luminosity will, usually, be proportional to the bunch length ℓ. It is therefore, desirable that ℓ be as large as possible in the low-energy regime. The length of a stored bunch was discussed in Section 5.4; we saw there that it is $2c\,\sigma_\tau$, where σ_τ is the time spread. Using Eq. (5.66) for σ_τ

$$\ell^2 = \frac{4c^2 C_q}{(mc^2)^2} \; \frac{\alpha L}{J_\epsilon \rho_0} \; \frac{E_0^3}{e\dot{V}_0} \tag{6.27}$$

The bunch length is related to the various ring parameters, α, L, J_ϵ, ρ_0 — which we may take as fixed for a given ring — to the operating energy E_0 and to the slope $\dot{V}_0$ of the rf voltage — which is adjustable. I am assuming here that the rf voltage is sinusoidal. It is usually so for practical reasons. If the voltage is not sinusoidal the conclusions will be modified somewhat, although the qualitative features of the result will probably not be changed.

For a sinusoidal rf voltage $\dot{V}_0$ will vary with the peak rf voltage $\hat{V}$ and the bunch length will be longest when $\hat{V}$ is as small as possible. But as we saw in Section 5.8, the rf voltage determines also the beam <u>lifetime</u> due to energy fluctuations. We may say that the maximum bunch length is obtained when the rf voltage is set as low as possible compatible with an acceptable beam lifetime.

The voltage slope $\dot{V}_0$ has been written in terms of $\hat{V}$ in Eq. (3.39). It will be convenient to use, instead of $\hat{V}$, the "overvoltage" $q = \hat{V}/U_0$ that was found to be a useful parameter in Section 3.6. In terms of q, Eq. (3.39) becomes

$$\dot{V}_0 = \omega_{rf} U_0 (q^2 - 1)^{1/2}. \tag{6.28}$$

Now let's replace ω_{rf} by its equivalent $2\pi ck/L$ where L is the orbit length and k is the harmonic number of the system, and replace U_0 by the expression in Eq. (4.8). Then we have that

$$\dot{V}_0 = 2\pi c\, C_\gamma \; \frac{k E_0^4}{\rho_0 L} \; (q^2 - 1)^{1/2} \tag{6.29}$$

If we use this expression in Eq. (6.29) and express C_q and C_γ in terms of the constant E_1 defined by Eq. (5.140), we get for the bunch length the equation

$$\ell^2 = \frac{c^2 E_1}{\pi^2} \; \frac{\alpha L^2}{k J_\epsilon E_0} \; \frac{1}{(q^2 - 1)^{1/2}} \tag{6.30}$$

Now notice that the combination $J_c E_0 / \alpha E_1$ is also contained in the parameter ξ that appeared in our analysis of the quantum lifetime. See Eq. (5.141). So the bunch length can be expressed in terms of ξ and the overvoltage q. Using Eq. (5.141)

$$\ell^2 = \left(\frac{L}{\pi k}\right)^2 \frac{F(q)}{\xi (q^2 - 1)^{1/2}} , \qquad (6.31)$$

where $F(q)$ is the energy aperture function of Eq. (3.61). The total q-dependence of this expression for ℓ^2 is

$$\frac{F(q)}{(q^2 - 1)^{1/2}} = 2 - \frac{\cos^{-1}(1/q)}{(q^2 - 1)^{1/2}} \qquad (6.32)$$

For high energy rings q will generally be as high as 3 or more to get the desired quantum lifetime — provided only that the harmonic number is reasonably large, say greater than 20. For any q significantly greater than 1, the second term on the right-hand side is well approximated by $2/q \, (q+1)^{1/2}$, which will then also be quite a bit smaller than 1. We will not make a significant error if we ignore this term and write for Eq. (6.31),

$$\ell^2 \approx \left(\frac{L}{\pi k}\right)^2 \frac{2}{\xi} \qquad (6.33)$$

We have then, an expression for ℓ which depends only on L, k, and the lifetime parameter ξ.

We saw in Section 5.8 that the beam lifetime is an exceedingly rapid function of the rf voltage. The bunch length will therefore, not be very sensitive to the precise definition of an "acceptable" lifetime. I may, therefore, take the simple criterion that the quantum lifetime τ_q (from energy oscillations) shall be some large fixed multiple of the energy damping time constant τ_ϵ. Since τ_ϵ is typically 0.01 to 0.1 sec, we would probably be willing to accept a beam lifetime about 10^6 times larger.

In Eq. (5.135) the ratio of τ_q to τ_ϵ was expressed in terms of the parameter ξ:

$$\frac{\tau_q}{\tau_\epsilon} = \frac{e^\xi}{2\xi} \qquad (6.34)$$

Let's call ξ_0 the value of ξ that gives $\tau_q/\tau_\epsilon = 10^6$; then solving the transcendental equation for ξ_0, we find that $\xi_0 = 18$.

If we use this value for ξ in Eq. (6.33) we obtain the maximum bunch length

$$\ell_{max} \approx \frac{L}{3\pi k},$$ (6.35)

which is the expression that was adopted in the preceding section. We have the interesting result that, when the bunch length is a maximum, the total circumference occupied by the bunches, $k\ell_{max}$, is always a constant fraction — namely about $1/3\pi$ or 10% — of the ring circumference.

6.6. Optimum Luminosity Function

I have chosen to compute the optimum luminosity of any particular storage ring at the operating energy E_0 (assumed to be adjustable) in terms of six parameters (assumed to be fixed numbers) that describe the design characteristics of the ring. These parameters are R_n, ν_n, λ_n, β_V, δ_H, and P_0 — all described in Table 6.1. These six parameters can be chosen relatively independently in designing a ring. They would become interdependent however, if we were to take into account considerations external to the physical characteristics of the ring. In particular, any economic optimization would give relations among them all. For example, increasing R_n would mean greater costs for certain components which might, however, be compensated for by decreasing P_0 with a net overall increase in the high energy luminosity. It is, however, difficult to make a generally valid cost analysis and I shall not attempt to do so here. Rather, I shall only make some comments about the significance of some of the parametric dependences in the luminosity relations.

The maximum achievable luminosity in the low-energy regime is given by Eq. (6.25) and in the high-energy regime by Eq. (6.9). We have that

$$\mathscr{L}_1 = c_1 \frac{\lambda_n^{1/2} \delta_H E_0^3}{(\nu_n \beta_V)^{3/2}} \qquad \text{(low energy)};$$

$$\mathscr{L}_2 = c_2 \frac{P_0 R_n}{\lambda_n^{1/2} \beta_V E_0^3} \qquad \text{(high energy)}.$$

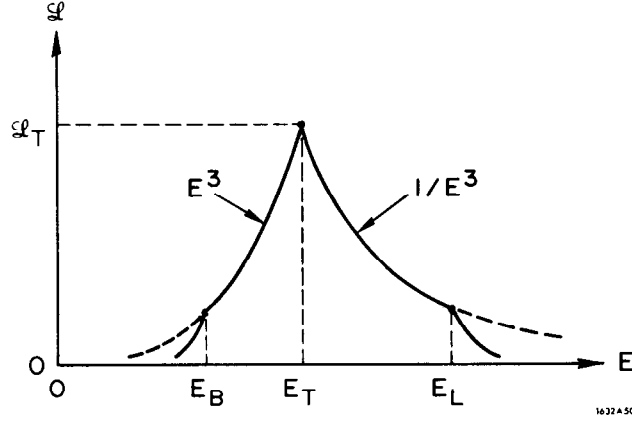The complete optimum luminosity function has the form shown in Fig. 50.

FIG. 50--The optimum luminosity function.

The transition between the two energy regimes occurs at the energy E_T which is evidently, the energy at which the two functions $\mathscr{L}_1$ and $\mathscr{L}_2$ are equal. Solving for this energy, we find that

$$E_T = C_T \left(\frac{\nu_n}{\lambda_n}\right)^{1/4} \left(\frac{P_0 R_n}{\delta_H}\right)^{1/6} \beta_V^{1/12} , \qquad (6.36)$$

with

$$C_T = \left(\frac{18}{\pi} \frac{(mc^2)^5}{c \Delta \nu_0}\right)^{1/6} \left(\frac{\sqrt{3}}{55} \frac{mc^2}{\hbar c}\right)^{1/12} ,$$

or

$$C_T = 6.18 \times 10^{-2} \text{ GeV -meter}^{1/4} \text{-watt}^{-1/6} .$$

You will notice that the transition energy varies exceedingly slowly with all of the significant parameters of the ring.

Below the transition energy E_T the luminosity goes as E^3. (One power comes from the beam height, a second from the corresponding increase of current permitted for a fixed current density limit and a third from the linear increase of the limiting current density with energy.) Above E_T the energy dependence is as $1/E^3$. (The current varies as $1/E^4$, but one power of E is compensated by the permitted linear decrease of the area with energy.)

In the low energy region the luminosity is proportional to the crossing angle δ_H and independent of the ring radius R_n and of the available rf power P_0 — although E_T, which sets the upper edge of the low energy region does depend on the rf power. At high energies, on the other hand the luminosity is proportional

to P_0 and to R_n, but <u>independent</u> of δ_H. This lack of dependence comes about because the beam dimensions are always adjusted to reach the limiting current density. The choice of δ_H is not coupled to the choice of other parameters (but will most likely be dictated by geometrical constraints imposed by the intersection region layout). The rf power P_0 costs money — both for itself and for associated costs of the vacuum system which must absorb the power. There are also costs associated with R_n — and perhaps roughly in proportion or stronger. (For a given λ_n, ν_n, and energy, the magnet aperture must increase with R_n although the magnetic field decreases; cabling and vacuum systems increase with R_n; and real estate costs go as R_n^2.)

A dependence on ν_n appears only in the low energy region — when the luminosity goes as $\nu_n^{-3/2}$. This is a good place to point out a possible criticism of the analysis for the low energy region. In obtaining the maximum possible beam area, I assumed that ν_n was kept a constant. One might argue that ν_n should be "tuned" to get a further increase in beam size — that is, that ν_n should be lowered until some other limit was reached. In principle, one could — by decreasing ν_n — always increase the natural beam size until beam was lost to the aperture limits. Such a method of operation would lead to a different energy dependence at low energies.

There are however, other constraints on ν_n. There are relatively few good operating points in the resonance diagram. So the choices of ν_n are a discrete set, and it did not appear appropriate to treat it as a continuously adjustable parameter. Perhaps more importantly, for a given geometry of the components it would generally not be possible to vary ν_n and maintain the appropriate matching conditions with the long straight sections. One would then expect that any change in ν_n would lead to some related variation of β_V. (This point is considered further below.) All in all, the analysis made here does not seem too inappropriate although one may wish in practice to consider the possibility of employing some alternate choices of ν_n for low energy operation.

Assuming a constant value of ν_n the primary consideration affecting the choice of its design value is an economic one. The aperture requirements of all magnets are related to ν_n. With the assumptions made here, ν_n places a requirement on the useful horizontal aperture at the maximum design energy and also one on the vertical aperture which must accommodate the beam when it takes on its maximum height — which will occur near E_T.

There is at all energies a relatively weak dependence on the elongation factor λ_n, for which there is anyway, generally little freedom of choice. Notice, also that the harmonic number does not appear explicitly in the optimum luminosity functions.

At both low and high energies the luminosity has a relatively strong inverse dependence on β_V. It would seem that β_V should always be made as small as possible. The lowest practical value may be set by various factors: the variation of β_z across the interaction zone; aberration in the lenses; practical limitations of quadrupole strengths and apertures; requirement for a long magnet-free space at the interaction, to name a few. In recent designs it has seemed feasible to make β_V as small as about 0.05 meters, or about 10^{-2} of β_n. But the quantitative nature of possible inherent constraints on β_V and their relation to other parameters is not clear to me at the present moment.

If we evaluate either $\mathscr{L}_1$ or $\mathscr{L}_2$ at the transition energy, we can find the peak value $\mathscr{L}_T$ of the luminosity function.

$$\mathscr{L}_T = c_0 \frac{(R_n \delta_H P_0)^{1/2}}{\lambda_n^{1/2} \nu_n^{3/4} \beta_V^{5/4}}, \qquad (6.37)$$

with

$$c_0 = \left(\frac{\Delta \nu_0^2}{r_e mc^2}\right)^2 \frac{1}{C_T^3} = 2.81 \times 10^{33} \text{ meter}^{-1/4} \text{-watt}^{-1/2} \text{-sec}^{-1}.$$

The optimum luminsoity in the two energy regions can be expressed conveniently in terms of $\mathscr{L}_T$. Clearly,

$$\mathscr{L}_1 = \left(\frac{E_0}{E_T}\right)^3 \mathscr{L}_T,$$

$$\mathscr{L}_2 = \left(\frac{E_T}{E_0}\right)^3 \mathscr{L}_T. \qquad (6.38)$$

The upper limit E_L of the high energy regime is reached when the effective interaction area can no longer be reduced to satisfy Eq. (6.8). Recall that the current is decreasing as E_0^{-4}, and that the critical current density decreases as

E_0^{-1}. The interaction area must, above E_T decrease as E_0^{-5}. In our model ring, the only significant control on the beam area is through the product of the bunch number B with the height h*. The product Bh* must, therefore be decreased as E_0^{-5} through the high-energy regime. We have assumed that the beam height may be controlled by varying the coupling between the vertical and radial oscillations. Let me assume that with the minimum achievable coupling the beam height is reduced by some constant factor μ below the height that would occur with <u>maximum</u> coupling. Then the <u>minimum</u> height is μh^*_{max} at each energy. Now recall that h^*_{max} is proportional to the beam energy; so in going from E_T to E_L, the beam height can be reduced by the factor $\mu E_L/E_T$. Finally, the number of bunches B can be reduced by filling only some of the available bunch positions. At E_T all of the k possible bunches are filled. To get the minimum area we should fill only one bunch. This gives us a reduction by the factor k. The total reduction possible is the $\mu E_L/kE_T$, and this must equal $(E_L/E_T)^{-5}$. We get that

$$\frac{E_L}{E_T} = \left(\frac{\mu}{k}\right)^{1/6} \tag{6.39}$$

Probably μ is about 0.02 or so; clearly a high harmonic number is needed if E_L is to be significantly higher than E_T.

Some further decrease in effective area can also be obtained decreasing the bunch length — which can be done by increasing the rf voltage above that assumed in deriving A_{max}. The length decreases only as the square root of the voltage however; and, since there is likely to be little excess voltage capability at the highest energy, there is not much to be gained by this procedure.

One further remark. I have assumed all along that the critical current density was due to the <u>vertical</u> tune shift. We should now check up on this assumption. So long as β_V is somewhat less than β_H, the beam height at the interaction point will always be less than its width. Applying the results of Section 2.12, to the crossing geometry of our model storage ring, the ratio of the horizontal tune shift to the vertical one is

$$\frac{\Delta \nu_x}{\Delta \nu_z} = \frac{\beta_H}{\beta_V} \frac{h^*}{\ell \delta_H} \tag{6.40}$$

If this ratio is never to exceed 1 we have the condition that

$$\frac{\beta_H}{\beta_V} \leq \frac{\ell \delta_H}{h^*} \qquad (6.41)$$

We have until now not considered β_H at all. We see however, that it must not be too large for our calculations to be valid. For our model ring $\ell \delta_H$ will generally (unless k is extremely large) be much larger than h*, so the condition on β_H will not usually be difficult to satisfy.

6.7. Luminosity Function for Project SPEAR

The model storage ring of Section 6.2 corresponds very closely to a design proposed recently by the Storage Ring Group[†] at the Stanford Linear Accelerator Center. The design is called SPEAR — for Stanford Positron Electron Asymmetric Rings. (The individual rings are asymmetric, because the special straight sections are not placed symmetrically in the otherwise circular guide field.) The essential parameters proposed for SPEAR are:

$$R_n = 34.3 \text{ meters} \qquad \lambda_n = 2.7$$

$$\nu_n = 3.2 \qquad \beta_V = 0.05 \text{ meters}$$

$$\delta_H = 0.10 \qquad P_0 = 5.5 \times 10^5 \text{ watts}$$

With these parameters the peak luminosity given by Eq. (6.37) and the transition energy evaluated from Eq. (6.36) are

$$E_T = 1.19 \text{ GeV}$$

$$\mathscr{L}_T = 5.1 \times 10^{33} \text{ cm}^{-2} \text{ sec}^{-1}$$

From these two numbers — and using the equations of (6.38) — the luminosity curve of Fig. 51 is obtained.

In the SPEAR design the rf harmonic number k is 36 so the upper limit of the high-energy regime expected from Eq. (6.35) is at about 3.2 GeV — which is above the top design energy of the rings at 3 GeV. No estimate has been made of the lower bound of the low-energy regime.

[†]The design was developed in January 1969 by B. Gittelman, B. Richter, D. Ritson, and M. Sands.
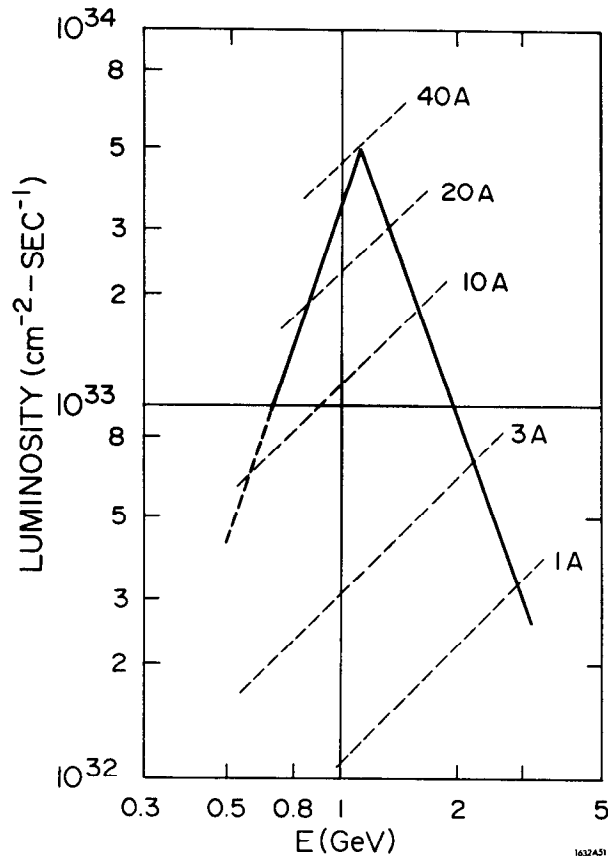
FIG. 51--Luminosity function for Project SPEAR.

The stored beam currents required to reach the luminosities shown in Fig. 51 are rather high. If some effect that has not been taken into account limits the current below what is required to reach the optimum luminosity, the realizable luminosity will of course, be decreased. I show by the broken lines in the figure, the luminosities that can be reached with any arbitrarily specified current — provided only that the interaction area can still be adjusted to get the best luminosity. To get the peak luminosity $\mathscr{L}_T$, a current of 40 amperes must be stored in each beam! It remains to be seen whether such high beam currents can indeed be stored in an electron-positron, colliding beam storage ring.

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PLUS OR MINUS ?
ALGEBRAIC SIGNS IN THE STORAGE RING EQUATIONS OF
SLAC REPORT NO. 121[*]

When I was writing SLAC Report 121, I was making the implicit assumption
that the curved parts of the design orbit would always bend in the same direction.
The RLA design shows that such an assumption was short-sighted. I have, there-
fore, reviewed SLAC-121 to see which equations may need to be changed when the
design orbit has parts with a reverse curvature. Fortunately, very few changes
are required. In this note, I report the adjustments that should be made in SLAC-
121 so that the results will be applicable to rings of arbitrary curvature. In addi-
tion, I list other miscellaneous corrections — particularly of algebraic signs.

A.   Comments on Part II of SLAC-121

In describing the design orbit, it was assumed that the direction of rotation
was clockwise (looking down on the orbit). See Fig. 7. (I now feel that this was a
poor choice, but that's life.) That is, the orbit was assumed to curve toward the
right, while the positive direction of the horizontal (or radial) coordinate, x, was
taken to the left. The positive direction of the z-coordinate, of course, defines the
"upward" direction. The equations of SLAC-121 will, as we shall see, generally
hold with a minimum of tinkering if we maintain the convention that the x-coordinate
is taken as positive to the left of direction of travel, and if we insist that the net
curvature of the design orbit shall be toward the right, while permitting that parts
of the orbit may have opposite curvatures — namely, toward the left.

With these understandings, Eqs. (2.1) and (2.2) may be left as they are, but
Eqs. (2.3) and (2.4) need the following comments. It is convenient to define the
curvature function G(s) so that it is positive when the orbit curves to the right
(toward negative x), and is negative for the opposite curvature. Equation (2.3)
will give this result provided that we specify that e shall represent the electric
charge, with sign, of the circulating particle. (That is, e is a negative number
for electrons.) For consistency, e should be interpreted the same way in Eq. (2.4),
and, also in all subsequent equations in SLAC-121.

Equation (2.5) now defines the "radius of curvature" of the orbit ρ_s as an algebraic quantity. The radius is <u>positive</u> if the center of curvature is toward negative x, and negative of the center of curvature is toward positive x.

Equation (2.6) has a typographical error; it should read:

$$-d\theta_0 = \frac{ds}{\rho_s} = G(s)ds \qquad (2.6)$$

(It is intended here and later that angles in the plane of the orbit are measured with the usual convention — positive angular changes are counter-clockwise.)

With these adjustments all of the remaining equations of Part II need no change to take into account orbits that may have reverse bends. It is only necessary to keep in mind that e, $G(s)$, ρ_s, and $K_1(s)$ are all quantities with appropriate signs, and in particular, that $G(s)$ and ρ_s (and, of course, $K_1(s)$) may have both positive to negative values around the ring.

Notice, however, that the definition of an "isomagnetic" guide field in Eq. (2.9) intends that $G(s)$ shall have a unique value — <u>including the sign</u> — in all bending magnets. Our conventions then dictate that $G_0 = 1/\rho_0$ is necessarily a positive number.

While I am at it, I may as well point out some careless errors of sign in Part II that are not basically related to the present discussion.

Notice that K_x, K_z, and the generic K have been <u>defined</u> to be <u>positive</u> when they are <u>defocussing</u>. See Eqs. (2.19), (2.20) and (2.31).* Equation (2.32) is then wrong — it assumes the opposite definition. So Eq. (2.32) should read

$$K < 0: \quad x = a \cos(\sqrt{-K}\, s + b)$$
$$K = 0: \quad x = a\, s + b$$
$$K > 0: \quad x = a \cosh(\sqrt{K}\, s + b) \qquad (2.32)$$

Similarly, the matrices of Table I are wrong. Letting the conditions on the left of Table I stand as is (K < 0, K = 0, K > 0 in that order), the matrix elements need correction by replacing K everywhere with its negative. (Change K to -K, and -K to K.) Sorry about that.

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*Probably a bad choice. And, clearly, I was quite ambivalent, since I shifted ground and wrote some of the equations with the opposite convention.

A careless error of sign was also made in writing Eq. (2.84). Clearly, if δG is to be interpreted in the normal way as the change in G (with appropriate sign) Eq. (2.84) will follow from the immediately previous equation if it reads

$$\Delta x' = -\delta G \, \Delta s. \qquad (2.84)$$

The error made here was propagated in all subsequent equations, so all of the equations of Section 2.10 should be corrected by changing δG to $-\delta G$.

There is a typo in Eq. (2.60). The last integral should be preceeded by the factor $1/2\pi$.

B. Comments on Part III of SLAC-121

All of the numbered equations in this part are, I believe, correct as they stand — with G(s) an algebraic quantity with appropriate sign.

There are a few errors in the text. In the line above Eq. (3.5) the equation should read: $K_x = -G^2$. In the material above Eq. (3.6) δG should be replaced wherever it occurs by $-\delta G$.

C. Comments on Part IV of SLAC-121

The material of this part is OK. In particular, the integral for $\mathscr{D}$, Eq. (4.18) contains G(s) to the first power, so those parts of the orbit with reverse curvatures will (for the same sign of K_1) give an opposite contribution to the integral.

There are a few typos. In Eq. (4.13) ρ should read ρ_s. In Eq. (4.17) the large parenthesis which should preceed $1/\rho$ is broken. In Eq. (4.26) the long bar after the first z' should be an arrow ($\rightarrow$). In Eq. (4.48) the negative sign after the equal sign should be deleted.

D. Comments on Part V of SLAC-121

This part suffers considerably from the implicit assumption that the design orbit had a homogeneous (always positive) curvature. To make it apply generally to orbits with some segments of abnormal curvature the following changes are required.

Eqs. (5.3), (5.9): ω_c is a positive quantity, so in these equations ρ should be replaced by its absolute value $|\rho|$. Eq. (5.20): Replace ρ^3 by $|\rho^3|$.

It follows that the quantum excitation depends only on the magnitude of the orbit curvature. So the following changes should be made in the rest of Part V.
Eqs. (5.40), (5.41): Replace $\gamma_0^3 G$ by $\gamma_0^3 |G|$.
Eqs. (5.42), (5.44), (5.45), (5.47), (5.82), (5.83): Replace G^3 by $|G^3|$.

There are also a couple of typos. In the middle of page 129, $E_0^{2/3}$ should read $E_0^{3/2}$. In Eq. (5.71) the inner parenthesis should be squared — as in the preceding equation.