#15: BEAM-BEAM 1-D RESONANCES:

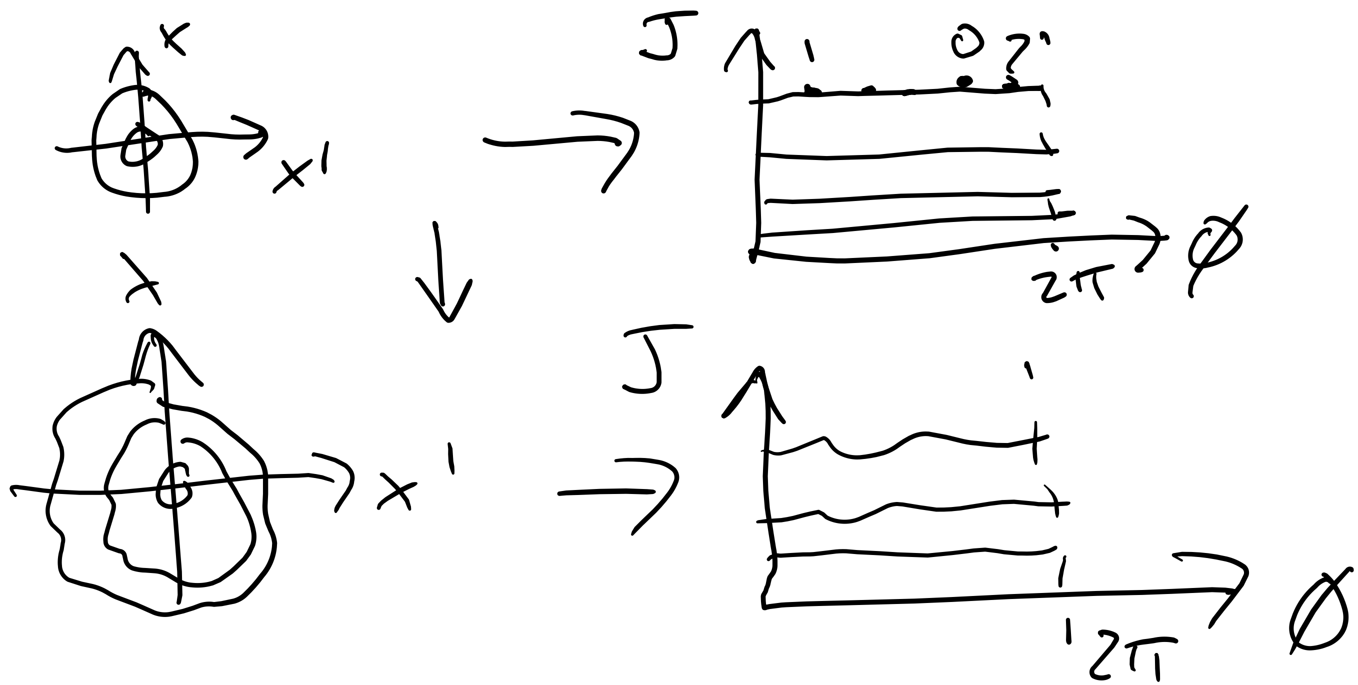
1/30/19

1st order theory that works!

Henon/sextupole: Don't try 2nd order theory

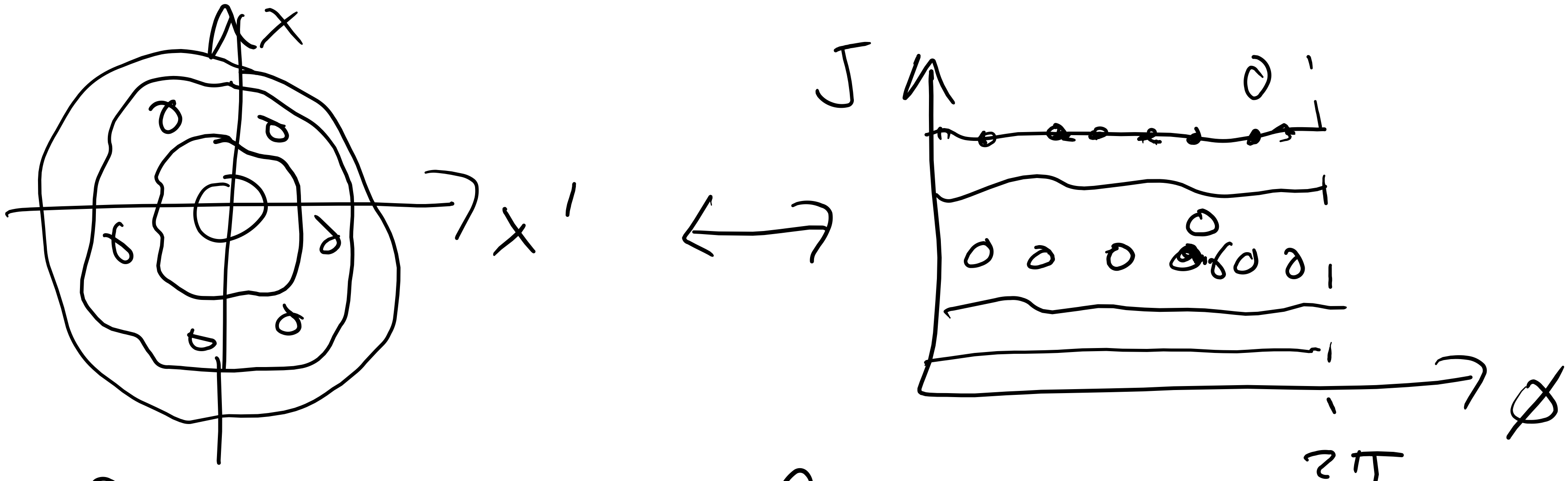
Recall "taxonomy"

① REGULAR NON-RESONANT



$$a = \sqrt{2J} \cdot \sigma$$

② REGULAR RESONANT



- Some particles skip from island to island
  - $\Rightarrow$ Only some phases are accessible: phase-locked
  - $\Rightarrow$ Tune (long time average) EXACTLY $Q = \frac{p}{n}$!!
- Note resemblance to STANDARD MAP

(③ RAPIDLY DIVERGENT)

④ CHAOS – Next lecture

SIMPLEST BEAM-BEAM

Test particle passes through a long Gaussian cylindrical bunch

$$\rho = \rho(z) \cdot \frac{e^{-r^2/2\sigma^2}}{2\pi\sigma^2}$$

LONG. $\sigma_z \gg \sigma$

Force $\bar{F} = q\left(\bar{E} + \bar{v} \times \bar{B}\right)$

where from Gauss' law

$$\boxed{F_E = \frac{q^2}{2\pi\epsilon_0} \cdot \rho(z) \cdot \frac{1}{r}\left[1 - e^{-r^2/2\sigma^2}\right]}$$

Also $F_B = \beta^2 F_E$

BEAM-BEAM $F_\perp = (1+\beta^2) F_E \approx 2 F_E$

[SPACE-CHARGE $F_\perp = (1-\beta^2) F_E = \frac{F_E}{\gamma^2}$]

INTEGRATE ONE PASSAGE

Assume $\beta^* \gg \sigma_z$

1-D: $y = y' = 0$

$$\boxed{\Delta x' = -\frac{4\pi \xi}{\beta^*} \cdot \frac{2\sigma^2}{x} \cdot \left(1 - e^{-x^2/2\sigma^2}\right)}$$

Where "BEAM-BEAM PARAMETER"

Ⓐ Bunch population

$$\boxed{\xi = \frac{N r_0}{4\pi \epsilon_N}}$$

classical radius

RMS NORMALISED emittance

SMALL $x \ll \sigma$: $\Delta x' = -\frac{4\pi \xi}{\beta^*} \cdot x$ LINEAR!

LARGE $x \gg \sigma$: $\Delta x' \sim 1/x$ WELL-BEHAVED

[Magnet at large x : $\delta x' \sim x^m \to \infty$]

SMALL X TUNE SHIFT

$$\boxed{\Delta Q = \frac{\beta^*}{4\pi} \cdot \frac{1}{f} = \xi}$$

ξ is SOMETIMES called "tune shift parameter"

Independent of β^* !! And of γ !!

Luminosity

$$\boxed{L = f_{REV}\, M\, \frac{(\beta\gamma)}{r_0} \cdot \frac{N\xi}{\beta^*}}$$

# of bunches

Maximise L by increasing ξ and N ($\sim \xi$), decrease β^*

ELECTRON/POSITRON beams have damping, so ...

$e^+ - e^-$ $\quad \xi_{MAX} \sim 0.1$

$h - h$ $\quad \xi_{MAX} \sim 0.01$

SOLUTION (1-D ROUND BEAM-BEAM INTERACTION)

Kobayashi Hamiltonian H_n with $Q_0 \approx \frac{p}{n}$

$$\boxed{H_n/2\pi = \left(Q_0 - \frac{p}{n}\right)J + \xi \cdot U(J) - \xi \, V_n(J) \cos(n\phi)}$$

Detuning function

Resonance function

and

$$U'(J) = \frac{2}{J}\left[1 - e^{-J/2} I_0(J/2)\right]$$

$$V_n'(J) = (-1)^{n/2} \cdot \frac{4}{J} \cdot e^{-J/2} I_{n/2}(J/2)$$

NOTE:

① Betatron amplitude $a = \sqrt{2J} \cdot \sigma$

② I_m is a modified Bessel function

③ Prime means differentiation w.r.t. J

HOW DOES THIS "SOLUTION" BEHAVE?

$$\Delta\phi = n\frac{\partial H_n}{\partial J}, \quad \Delta J = -n\frac{\partial H_n}{\partial \phi}$$

① $\xi = 0$ $\quad H_n = 2\pi\left(Q_0 - p/n\right)J$

$$\phi_{t+n} - \phi_n = 2\pi\underbrace{(nQ_0 - p)}_{\text{SMALL}}$$

② TUNE : average phase advance ($/2\pi$)

$$Q(J) \equiv \frac{1}{2\pi . n}\langle \phi_{t+n} - \phi_t \rangle$$

$$\boxed{Q(J) = Q_0 + \xi . U'(J)}$$

IF $\langle \cos(n\phi)\rangle = 0$

③ RESONANT ACTION J_R

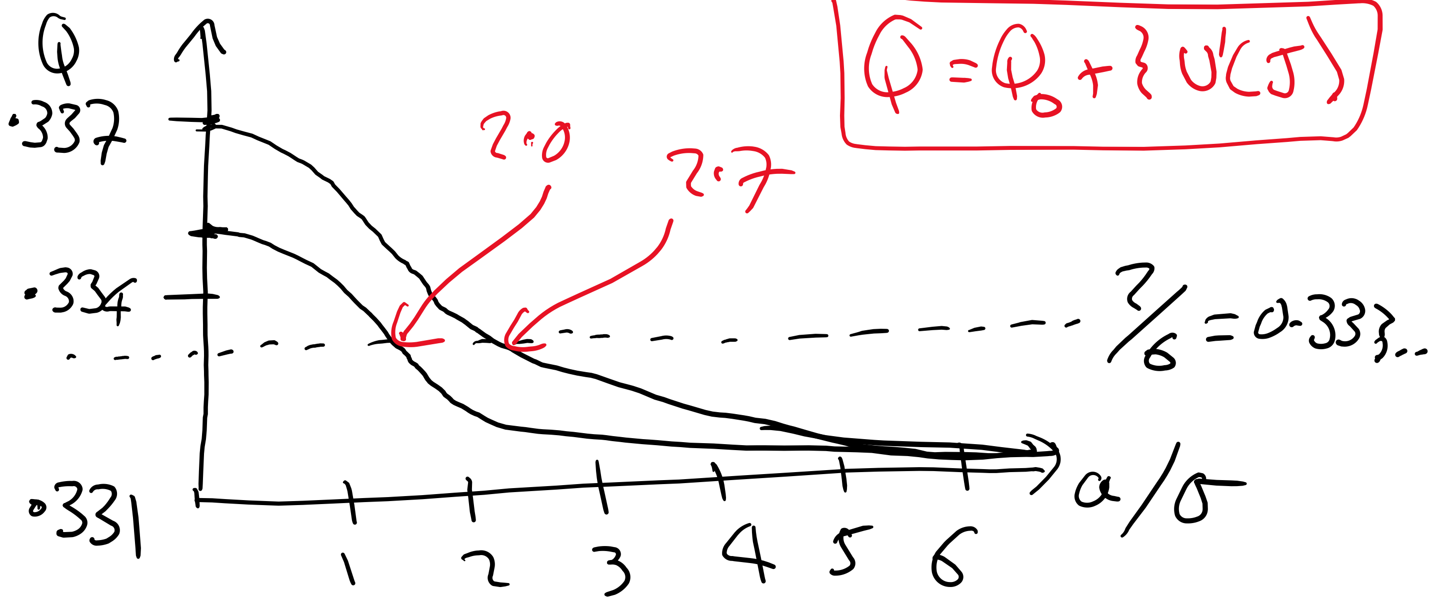
DEFINE by $\quad Q = Q_0 + \xi . U'(J_R) = p/n$

EXPAND nearby $\quad J = J_R + I$ ~ SMALL ...

WHERE ARE THE ISLANDS?

EG: $Q_0 = 0.331$, near $\frac{p}{n} = \frac{2}{6}$

$$\boxed{Q = Q_0 + \xi U'(J)}$$



a) $\xi = .0042$	: a_R/σ	2.0
b) $\xi = .006$		2.7

PREDICTED LOCATIONS ARE CORRECT!!

Islands get larger at larger amplitudes

RESONANCE HAMILTONIAN

Use $U(J) = U(J_R) + U'(J_R)\, I + \frac{1}{2} U''(J_R)\, I^2 + \ldots$

$$V_n(J) = V_n(J_R) + \ldots$$

Then

Ⓑ $$\boxed{H_{Rn} = 2\pi \left[\frac{1}{2}\left(\xi U''_R\right) I^2 - \left(\xi V_{Rn}\right) \cos(n\phi) \right]}$$

Compare to STANDARD MAP (pendulum) Hamiltonian
(Eqn. 4.40)

$$H(p,q) = \frac{1}{2} p^2 - \cos(q)$$

the fundamental difference is "n" !!

NOTE: $V_n(J) = 0$ for ODD $n \Rightarrow$ No ODD resonance.
(But $\frac{p}{5} \rightarrow \frac{2p}{10}$)

HOW BIG ARE THE ISLANDS?

Motion near a resonance described by H_{Rn} (B)

$$\phi_{t+n} - \phi_t = n\frac{\partial H_{Rn}}{\partial I}, \quad I_{t+n} - I_t = -n\frac{\partial H_{Rn}}{\partial \phi}$$

or, in a continuous time approximation

$$\boxed{\begin{aligned} \frac{d\phi}{dt} &= 2\pi \, \xi \nu_R' \cdot I \\ \frac{dI}{dt} &= -2\pi \, \xi V_{Rn} \cdot \sin(n\phi) \end{aligned}}$$

(C)

with fixed points at

$$I_{FP} = 0, \qquad \cos(n\phi_{FP}) = \pm 1$$

either STABLE (island center) or UNSTABLE (saddle points)

H_{Rn} is constant along separatrix

so $H_{Rn}(0,0) = H_{Rn}\left(\frac{\pi}{6}, I_{HW}\right)$

and island half-width

$$\boxed{I_{HW} \approx 2\left(\frac{\xi V_{Rn}}{\xi U''_{Rn}}\right)^{\frac{1}{2}}}$$

PREDICTED HALF-WIDTHS ARE ALSO ACCURATE

E.G. $a_{HW} = 0.317\ \sigma$ when $Q_0 = 0.331$

$\quad$ & $\xi = .0042$

ISLAND TUNES

Near an island center — with $|\sin(n\phi)| \ll 1$

(C) $\Rightarrow$

$$I = a_I \cos(2\pi Q_I \cdot t)$$

$$\phi = \phi_{FP} + a_\phi \sin(2\pi Q_I \cdot t)$$

where island tune

$$\boxed{Q_I = n\xi\left(-V_{Rn} U''\right)^{\frac{1}{2}}}$$

NOTE: 1) $Q_I \sim \xi$ increases with beam-beam strength

2) $Q_I/3$ is of order 1 !

Observe: Q_I (and therefore ξ) is a "control parameter" in considering external effects, e.g. CHROMATIC TUNE MODULATION at Q_s

ELECTRONS – FLAT BEAMS

- Electrons have DAMPING (good!) and quantum EXCITATION (can be bad: phase space diffusion)
- Usually e^+-e^- colliders are FLAT:

$$\epsilon_H \gg \epsilon_V$$

$$\beta_H^* \gg \beta_V^*$$

$$\sigma_H \gg \sigma_V$$

nonetheless with roughl commensurate H & V beam-beam parameter since with bi-Gaussian bunches

$$\boxed{\xi_{H,V} = \frac{N r_0}{2\pi(\beta\gamma} \cdot \frac{\beta^*_{H,V}}{\sigma^*_{H,V}(\sigma^*_H + \sigma^*_V)}}$$

But so what? What limits ξ?

We have seen NO unstable motion (this lecture)!

Where does CHAOS come from?

To be continued ....