#4: LONGITUDINAL + OFF-MOMENTUM MOTION 1/22/19

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix} \qquad \delta = \frac{\Delta p}{p} \neq 0$$

Q: Why do we need (nonlinear) sextupoles?

$\boxed{\text{CONSTANT}}$ MOMENTUM OFFSET

$$\delta = \frac{p - p_0}{p} = \frac{\Delta p}{p}$$

Recall Hill's equations: for betatron oscillations:

$$\boxed{\begin{aligned} x'' + K(s)\cdot x &= 0 \\ y'' - K(s)\cdot y &= 0 \end{aligned}}$$

Quad strength K is periodic in s

$$K(s) = K(s + C)$$

$$0 \leq s < C$$

Bending radius R in a dipole becomes

$$\frac{1}{R} = \frac{1}{\rho} \cdot \frac{1}{(1+\delta)} = \underset{\substack{\downarrow \\ 1/\rho}}{G} \cdot \frac{1}{(1+\delta)}.$$

The TOTAL bend accumulated in a slice Δs is

$$\Delta x'_{TOT} = G\left(1 - \frac{1}{1+\delta}\right)\Delta s.$$

thanks to the rotating co-ordinate frame

Introduce TOT here: e.g.

$$x_{TOT} = \underset{\substack{\text{CLOSED} \\ \text{ORBIT}}}{x(\delta)} + \underset{\substack{\text{BETATRON} \\ \text{OSCILLATIONS}}}{x}$$

Adding in quadrupoles of strength $\frac{K}{1+\delta}$

GENERALISED HILL'S EQNS:

$$\boxed{\begin{aligned} x''_{TOT} + \frac{K}{1+\delta}\, x_{TOT} &= G\left(1 - \frac{1}{1+\delta}\right) \\ y''_{TOT} - \frac{K}{1+\delta} \cdot y_{TOT} &= 0 \end{aligned}}$$

- Assuming no vertical bending dipoles
- These equations are EXACT to all orders in δ
- $K(s)$ & $G(s)$ are periodic in s

To first order in δ :

(A)
$$\boxed{\begin{aligned} x''_{TOT} + K(1-\delta)\, x_{TOT} &\simeq G \cdot \delta \\ y''_{TOT} - K(1-\delta) \cdot y_{TOT} &= 0 \end{aligned}}$$

DISPERSION FUNCTION η

(B)
$$\boxed{x_{TOT} = \eta(s) \cdot \delta + x}$$

closed orbit

betatron osc.

Closed orbit (dispersion) respects Periodic Boundary Conditions (PBC)

Strictly speaking dispersion is a polynomial

$$\eta(s) = \eta_0(s) + \eta_1(s) \cdot \delta + \eta_2 \delta^2 + \ldots$$

usually we only care about $\eta = \eta_0$

Substitute (B) into (A) to get:

$$\boxed{\eta'' + K\eta = G}$$

First order dispersion with PBC

AND

$$\boxed{\begin{aligned} x'' + \frac{K}{(1+\delta)}x &= 0 \\ y'' - \frac{K}{(1+\delta)}y &= 0 \end{aligned}}$$

BETATRON OSCILLATIONS

Weaker/stronger quads perturb $\beta(\delta)$, $Q(\delta)$, ...

REVERT TO MATRICES

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix} = A(6\times 6)\begin{pmatrix} \\ \\ \\ \end{pmatrix} \quad 6-$$

In a long dipole with
$L = \rho\theta$ (and $K = 0$)

$$u'' = G = \frac{1}{\rho}$$

So
$$\begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_{OUT} = \begin{pmatrix} 1 & L & \rho(1-\cos(\theta)) \\ 0 & 1 & \sin(\theta) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_{IN}$$

OR in general

$$\boxed{\begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_2 = \begin{pmatrix} m_{11} & m_{12} & m_{16} \\ m_{21} & m_{22} & m_{26} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_1}$$

$\boxed{\text{EG 1:}}$ M is a one-turn 3×3 matrix

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = M \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

is easily solved for (η, η') at the reference point.

$\boxed{\text{EG 2:}}$ From thin quad D (center) to F (center) in a FODO cell

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_F = \begin{pmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_D$$

with $\underline{\eta'_F = \eta'_D = 0}$ (matched periodic solution) and so

$$\boxed{\begin{aligned} \eta_{MAX} &= L\theta\left(\frac{2+s}{2s^2}\right) \\ \eta_{MIN} &= L\theta\left(\frac{2-s}{2s^2}\right). \end{aligned}} \sim \frac{L^2}{\rho}$$

where $s \equiv \sin\left(\frac{\Delta\phi}{2}\right) = |qL|$

OSCILLATING MMTM / LONGITUDINAL MOTION

Q: A particle with $\delta > 0$ goes faster, but travels further: which wins? $\delta \neq$ CONST

(Clue: is $\gamma \gg 1$)

PATH LENGTH (1 turn closed orbit)

increases by $\boxed{\Delta C_{PATH} = \delta \oint \eta \cdot d\theta}$

since $\frac{d\theta}{ds} = \frac{1}{\rho}$

$$\boxed{\Delta C_{PATH} = \delta \oint \frac{\eta}{\rho} \cdot ds}$$

Moves towards back of bunch

HIGHER SPEED MOVES FORWARD (per turn) by

$$\boxed{\Delta C_{SPEED} = C\,\frac{\Delta\beta}{\beta} = C\,\frac{\delta}{\gamma^2}}$$

Adding the 2 effects together:

(A) $$\boxed{z_{n+1} = z_n - \eta_S \cdot \delta_n}$$

where SLIP FACTOR $\rightarrow$ (COMPACTION FACTOR)

$$\boxed{\eta_S \equiv \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}}$$

$$\boxed{\frac{1}{\gamma_T^2} \equiv \frac{1}{C}\oint \frac{M}{\rho}\,ds}$$

Is $\gamma > \gamma_T$

HOW DOES δ_n CHANGE?

Going through an RF cavity ENERGY changes:

$$E_{n+1} = E_n + qV_{RF} \cdot \sin\left(2\pi \frac{z_n}{\lambda_{RF}}\right)$$

where

$$f_{RF} = \frac{c}{\lambda_{RF}} = h\, f_{REV}$$

h ← INTEGER HARMONIC NUMBER

and

$$\boxed{\frac{\Delta p}{p_0} = \frac{1}{\beta^2} \frac{\Delta E}{E_0}}$$

B

$$\boxed{\delta_{n+1} = \delta_n + \left(\frac{qV_{RF}}{\beta^2 E_0}\right) \cdot \sin\left(2\pi \frac{z_n}{\lambda_{RF}}\right)}$$

FOR SMALL OSCILLATIONS $|z| \ll \lambda_{RF}$

$$z_{n+1} = z_n - (\alpha_s C) \cdot \delta_n$$

$$\delta_{n+1} \simeq \delta_n + \left(\frac{qV_{RF}}{\beta^2 E_0}\right)\left(\frac{2\pi}{\lambda_{RF}}\right) \cdot z_{n+1}$$

$$\boxed{\begin{aligned} z_n &= a_z \sin(2\pi Q_s \cdot n + \phi_0) \\ \delta_n &= a_\delta \cos(2\pi Q_s \cdot n + \phi_0) \end{aligned}}$$

so long as the SYNCHROTRON TUNE

$$\boxed{Q_s = \sqrt{\frac{\alpha_s}{2\pi} \cdot \frac{C}{\lambda_{RF}} \cdot \frac{qV_{RF}}{\beta^2 E_0}}}$$

is much less than 1 !!

STANDARD MAP

Eqns. A + B are the same as the standard map

until finished {

$$\theta = \theta + \theta' \cdot \Delta t$$
$$\theta' = \theta' - \sin(\theta) \cdot \Delta t$$

}

with 1 — only 1 — control parameter:

$$\boxed{\cos(2\pi \cdot Q_s) = 1 - \frac{\Delta t^2}{2}}$$

- Clearly something goes "wrong" when $\Delta t > 2$
- What happens Q_s becomes large?

HAMILTONIAN SHORTHAND

IF the motion of 2 variables p & q (e.g. δ-z)

can be represented by $H(p,q)$

THEN by definition

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad , \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

so that

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial p}\cdot\frac{dp}{dt} + \frac{\partial H}{\partial q}\cdot\frac{dq}{dt} = 0}$$

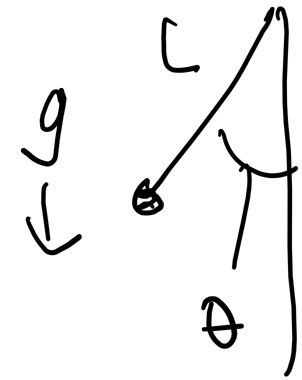
and H is a constant of the motion (e.g. ENERGY)

$\Rightarrow$ DRAW CONTOURS of H

EG GRAVITY PENDULUM

$p \Leftrightarrow \theta'$

$q \Leftrightarrow \theta$



Kinetic energy $\sim \frac{1}{2} L \theta'^2$

Potential energy $\sim -g \cos(\theta)$

So in shorthand

$$\boxed{H = \frac{1}{2}\theta'^2 - \cos(\theta)}$$

MOTION ALONG A CONTOUR

$\hat{p}, \hat{q}$ unit vectors.

$$\text{VELOCITY} = \frac{dp}{dt}\hat{p} + \frac{dq}{dt}\hat{q}$$

$$= -\frac{\partial H}{\partial q}\hat{p} + \frac{\partial H}{\partial p}\hat{q}$$

CONTOUR SLOPE

$$\bar{\nabla} H = \frac{\partial H}{\partial p}\hat{p} + \frac{\partial H}{\partial q}\hat{q}$$

So velocity + slope are perpendicular,
SAME LENGTH ...

$\Rightarrow$ SPEED is proportional to STEEPNESS

Chaos first appears near metastable equilibria