#5: EMITTANCES & PHASE SPACE 1/23/19

"1 particle, many particles, or none?"

One-turn matrix

$$M = \begin{pmatrix} \cos(\mu) + \alpha \sin(\mu) & \beta \sin(\mu) \\ -\gamma \sin(\mu) & \cos(\mu) - \alpha \sin(\mu) \end{pmatrix}$$

$\mu \equiv 2\pi Q_x$

$\boxed{\text{NO PARTICLES}}$ define Twiss parameters β, α, γ

$\Longrightarrow$ optical property of entire lattice

- But how about a single pass system, e.g. linac or a transfer line?

$\boxed{\text{1-PARTICLE}}$ turn n in a ring:

$$\text{(A)} \quad \boxed{\begin{aligned} x_n &= \sqrt{2J_x\beta_x}\,\sin(\phi_x) \\ x_n' &= \sqrt{\frac{2J_x}{\beta_x}}\left[\cos(\phi_x) - \alpha_x \sin(\phi_x)\right] \end{aligned}}$$

J - ACTION
ϕ - ANGLE
"Action-angle"
co-ordinate

where J_x is a constant

$$\phi_x = 2\pi\, Q_x \cdot n + \phi_0$$

Recall the Twiss identity $\beta\gamma = 1 + \alpha^2$ $\left(\det M = 1\right)$
and invert (A) : For ANY n

$$\boxed{2J_x = \beta_x x'^2 + 2\alpha_x x x' + \gamma_x x^2}$$

Area of the ellipse is

$$\boxed{A_x = 2\pi J_x}$$

EVERYWHERE around the ring, even though the ellipse stretching & tilting according to β, α, γ as functions of s

RMS displacement of 1 particle:

$$\sigma_{x,1} \equiv \langle x^2 \rangle^{\frac{1}{2}} = 2\sqrt{J\beta_x}\,\langle \sin^2(\phi) \rangle$$

$$\boxed{\sigma_{x,1} = \sqrt{J_x \beta_x}}$$ Very simple!!

$\boxed{\text{MANY PARTICLES}}$ with an action distribution $\rho(J_x)$

MS size of the bunch is

$$\sigma_x^2 = \frac{1}{N}\int_0^\infty \sigma_{x,1}^2\, \rho(J_x) . dJ_x$$

$$= \frac{\beta}{N}\int_0^\infty J_x \rho(J_x) . dJ_x$$

$N \sim 10^9$
10^{11}
particle

$$\boxed{\sigma_x^2 = \beta_x \langle J_x \rangle}$$

Simple ... so long as we stick to RMS measures

__DEFINE__ UNNORMALISED emittance as $\boxed{\epsilon_{x,u} = \langle J_x \rangle}$

And generally

$$\textcircled{A} \quad \begin{pmatrix} \langle x^2 \rangle \\ \langle x x' \rangle \\ \langle x'^2 \rangle \end{pmatrix} = \epsilon_{x,u} \begin{pmatrix} \beta_x \\ -\alpha_x \\ \gamma_x \end{pmatrix}.$$

$$\boxed{\sigma_x^2 = \epsilon_{x,u} \cdot \beta_x}$$

shows how Twiss parameters connect to beam distribution moments! (Not necessarily Gaussian).

WHAT ABOUT DISPERSION?

$$x_{TOT,n} = \sqrt{2J_x \beta_x}\, \sin(\phi_{x,n}) + M\, \delta_n$$

$$y_n = \sqrt{2J_y \beta_x}\, \sin(\phi_{y,n})$$

where

$$\delta_n = a_\delta \sin(\phi_{s,n})$$

IF $\phi_{x,n}$ & $\phi_{s,n}$ are uncorrelated.

$$\boxed{\sigma_{x,TOT}^2 = \beta_x \epsilon_{x,n} + M \left(\frac{\sigma_p}{p}\right)^2}$$

SUM OF SQUARES

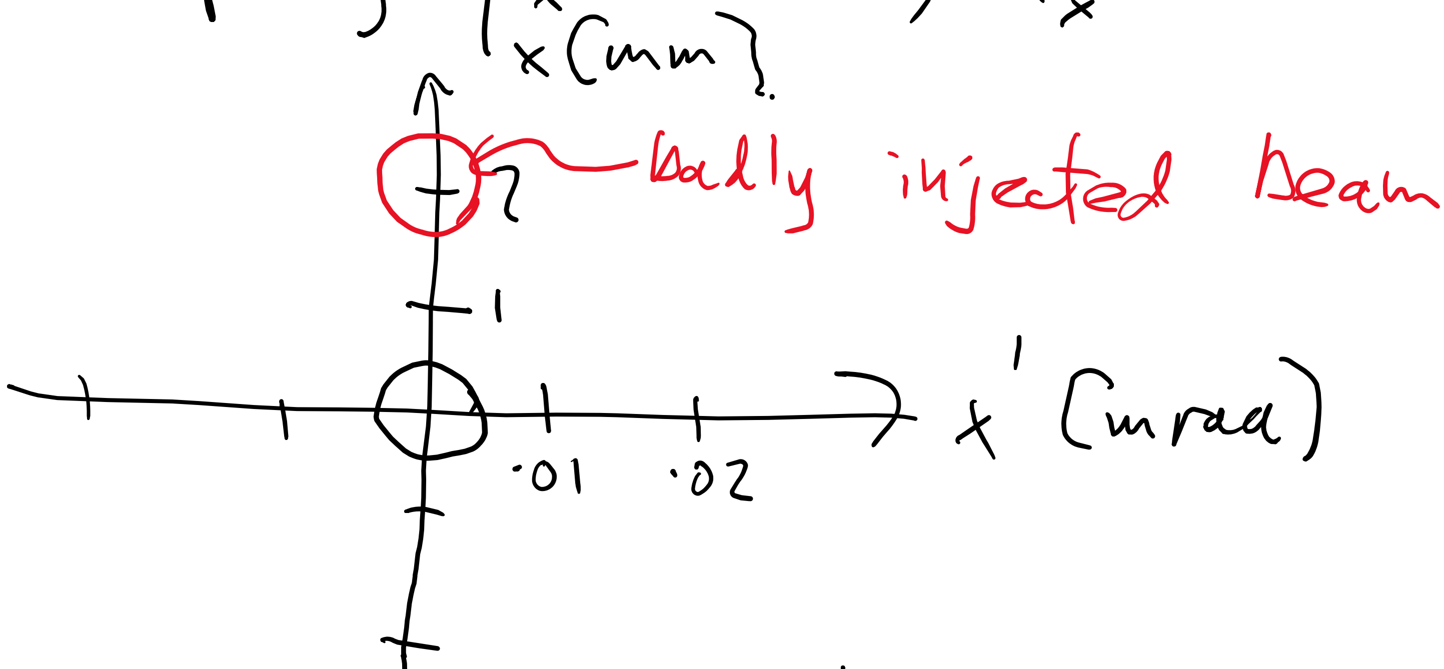
RMS measures are simple ... in a quasi-static storage ring.

WHAT HAPPENS DURING INJECTION TRANSIENTS?

FILAMENTATION + TUNE SPREAD

SCENARIO

For simplicity: $\beta_x = 100$ m, $\alpha_x = 0$



Badly injected particles have an action $J \sim \dfrac{\Delta x_{ERR}^2}{\beta_x}$

with a spread in tunes

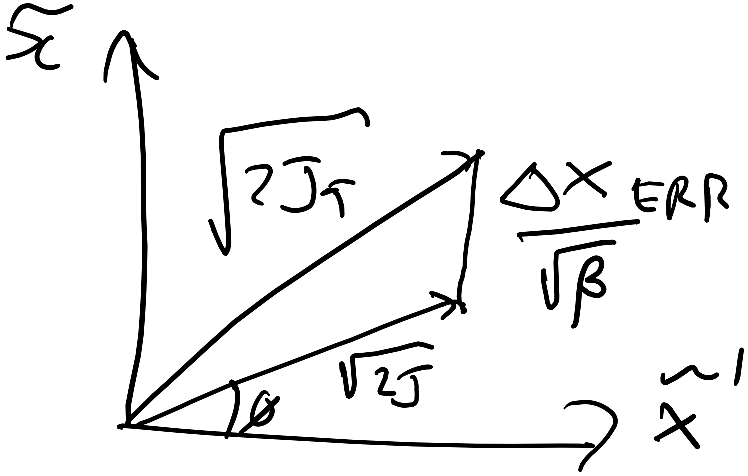
$$\boxed{Q(J) = Q(0) + \frac{dQ}{dJ}\cdot J}$$

Approximately constant

When transients die down (and filamentation is complete) the AREA is **FORMALLY** preserved ... but WHO CARES (for practical purposes)?

DISPLACEMENT ERROR

1-particle



$$J_T = J + \Delta x \sqrt{\frac{2J}{\beta}} \cos(\phi) + \frac{\Delta x^2}{\beta}$$

BUT, average over many particles (since $\langle \cos(\phi) \rangle = 0$)

$$\boxed{\epsilon_{u,T} = \epsilon_{u,0} + \frac{\Delta x_{ERR}^2}{\beta}} = \epsilon_{u,0}\left[1 + \frac{\Delta x^2}{2\sigma_\delta^2}\right]$$

MOMENTUM ERROR $\quad \delta_{ERR}$

Incoming beam off-momentum by δ_{ERR}

Since

$$x_{TOT} = x + \eta\,\delta_{ERR}$$

Analagous to

$$\Delta x_{ERR} = -\eta\,\delta_{ERR}$$

and

$$\boxed{\epsilon_{v,T} = \epsilon_{v,0} + \frac{\eta^2}{\beta}\delta_{ERR}^2}$$

showing why $\eta = \eta' = 0$ is <u>GOOD</u> for an injection location!

NORMALISED EMITTANCE / ADIABATIC DAMPING

When RF accelerates the beam by $\Delta(\beta\gamma)$

the angle of a particle

$$x' = \frac{p_\perp}{p_\parallel}$$

decreases $x' \to x' / \left(1 + \frac{\Delta(\beta\gamma)}{\beta\gamma}\right)$

Average over all particles with same action J

$$\langle J_{new} \rangle = \frac{J}{1 + \Delta(\beta\gamma)/\beta\gamma}$$

and the UNnormalized emittance shrinks:

$$\boxed{\epsilon_u = \frac{\epsilon_N}{\beta\gamma}}$$

NORMALISED emittance is constant

$$\boxed{\sigma^2 = \frac{\beta \epsilon_N}{(\beta\gamma)}}$$

Beam size shrinks during acceleration !!

BEWARE NOTATION !

PROTONS

$$\epsilon_{v,x} = \frac{\epsilon_N}{(\beta\gamma)}$$

ϵ_N CONSTANT

$$\epsilon_{v,x} \approx \epsilon_{v,y}$$

ROUND BEAMS

ELECTRONS

Radiate copious photons:

DAMPING ($\mathcal{H}$-ev)

EXCITATION ($\mathcal{H}$)

$$\epsilon_{v,x} = \epsilon_{v,x}(\gamma, \mathcal{H}, \text{OPTICS}\ldots)$$

Todd's lecture

$$\epsilon_{v,y} \ll \epsilon_{v,x}$$

FLAT

$$\sigma_y \ll \sigma_x$$

ϵ_N is not very useful....

$\boxed{\text{SINGLE PASS SYSTEMS}}$ LINACS, transfer lines

Can't use one-turn matrices to define Twiss parameters.

PARADIGM SHIFT: Beam distributions define β, α, γ (cf Ⓐ)

$$\begin{pmatrix} \beta_x \\ -\alpha_x \\ \gamma_x \end{pmatrix} = \frac{1}{\epsilon_{xA}} \begin{pmatrix} \langle x^2 \rangle \\ \langle x x' \rangle \\ \langle x'^2 \rangle \end{pmatrix}.$$

Where AREA EMITTANCE

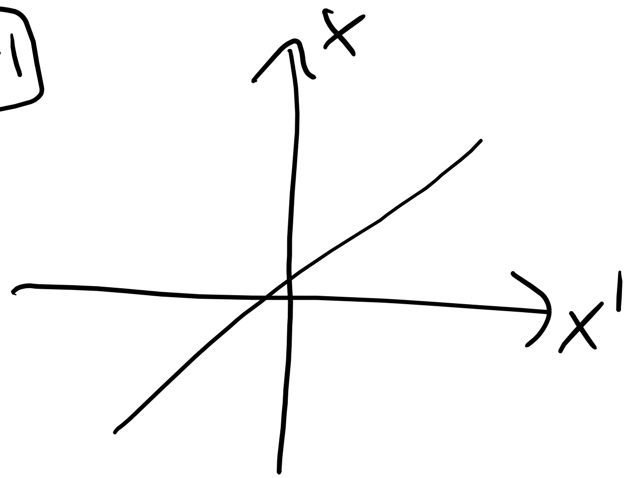
$$\boxed{\epsilon_{xA} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle}}$$

May have little to do with BEAM area in phase space!!

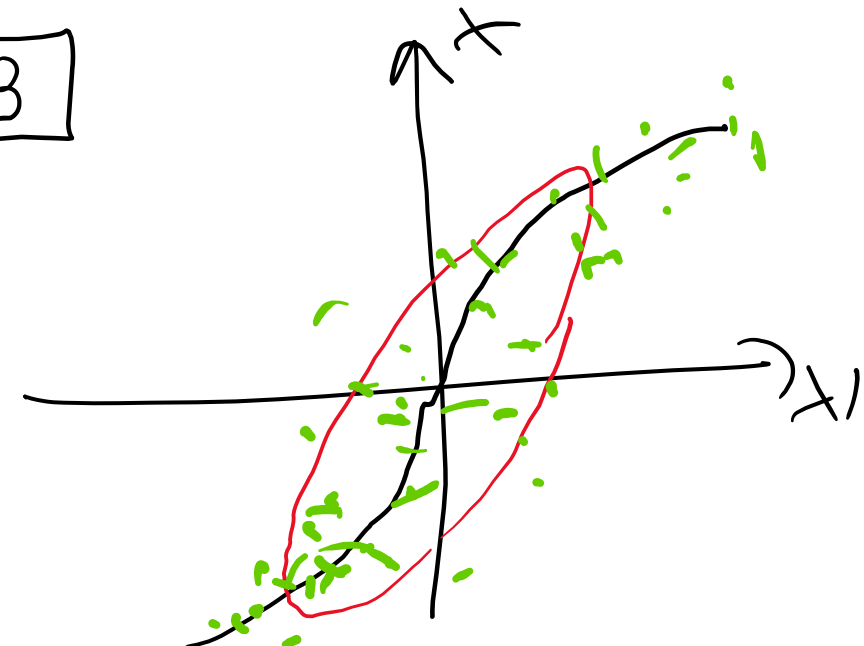
CONSIDER TWO "WANELER" DISTRIBUTIONS

$$x' = C x^m$$

$\boxed{m=1}$



$\boxed{m=3}$



$$\boxed{\epsilon_{x,A} = 0}$$

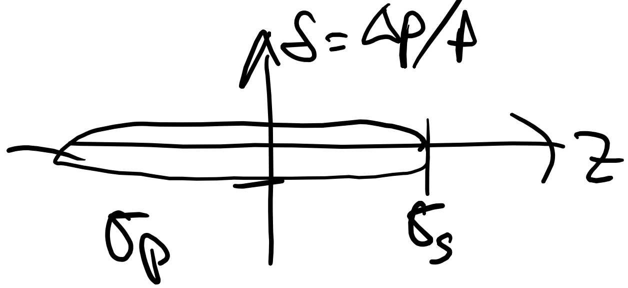
$$\boxed{\epsilon_{x,A} > 0}$$

$$\boxed{\epsilon_{x,A} = C\sqrt{\langle x^2\rangle\langle x^{2m}\rangle - \langle x^{m+1}\rangle^2}}$$

Twisted lines can be straightened: $\epsilon_{x,A}$ MAY DECREASE!!

LONGITUDINAL PARAMETERS

RMS bunch leng σ_s + momentum spread $\sigma_p = \frac{\langle \Delta p^2 \rangle^{\frac{1}{2}}}{p_0}$



constant around ring

ONE CONVENIENT longitudinal normalized emittance is ϵ_s

$$\sigma_s = \sqrt{\frac{\beta_s \epsilon_s}{(\beta\gamma)}} \quad , \quad \sigma_p = \sqrt{\frac{\epsilon_s}{\beta_s (\beta\gamma)}}$$

$$\boxed{\beta_s [m] = \frac{\sigma_s}{\sigma_p} = \frac{C}{2\pi} \cdot \frac{|\eta_s|}{Q_s}}$$

$\eta_s = \frac{1}{\gamma^2} - \frac{1}{\gamma_T^2}$ .... Exciting thing happen when $\gamma \to \gamma_T$