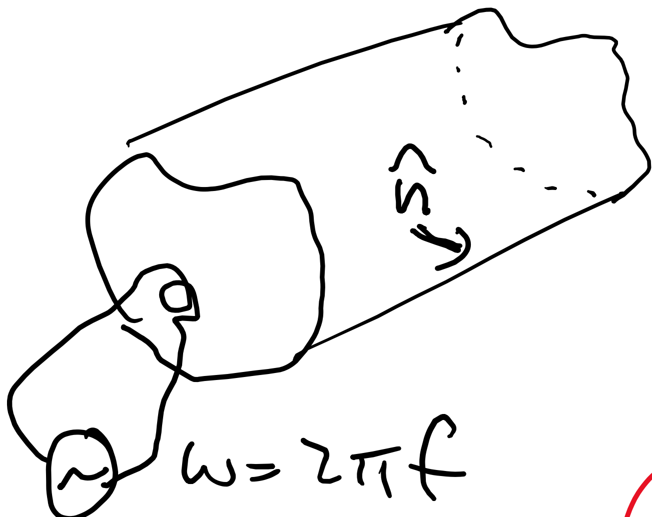
#7: RF CAVITIES

1/24/19



WAVEGUIDE:
- open at both ends
- constant (arbitrary) cross-section
- perfectly conducting

TRANSMITTING (?) fields oscillating at $\omega = 2\pi f$

Boundary conditions at the walls:

$\hat{n} \times \bar{E} = 0$ no parallel E-fields

$\hat{n} \cdot \bar{B} = 0$ no perpendicular B-fields

$\hat{n}$ - unit vector perpendicular to wall

Real parts of complex $\bar{B}$ + $\bar{E}$ vectors are physical

Consider $\underline{\underline{a}}$ mode labeled by WAVE NUMBER k

(A)

$$\boxed{\begin{aligned} \bar{E} &= \bar{E}(r,\theta) \cdot e^{i(kz-\omega t)} \\ \bar{B} &= \bar{B}(r,\theta) \cdot e^{i(kz-\omega t)} \end{aligned}}$$

where phase velocity in z-direction is

$$V_p = \omega / k$$

If k is imaginary then wave is damped
DOES NOT PROPOGATE

Q: How does k vary with ω ?

Q: WHAT happens when walls are added at waveguide ends, making a CAVITY

TRANSVERSE MODES

3 categories of mode solve (A)

1 **TRANSVERSE MAGNETIC** (TM)

$B_z = 0$ everywhere, $E_z = 0$ at walls
$\neq 0$ in general.

2 **TRANSVERSE ELECTRIC** (TE)

$E_z = 0$ everywhere, with $\frac{\partial B_z}{\partial n} = 0$ at walls

TM modes are most useful: accelerate/decelerate beam
(but crab cavities [TE] are state-of-art)

3 **TRANSVERSE ELECTROMAGNETIC** (TEM)

NO longitudinal fields: FREE-SPACE wave

$$\boxed{k = \sqrt{\mu_R \epsilon_R}\ \frac{\omega}{c}}$$

In a vacuum $\mu_R = \epsilon_R = 1$ ← TEM modes propagate @ "c" "UNAWARE" of walls because their wavelengths $\lambda = \frac{2\pi}{k}$ are much smaller than waveguide width.

SOLVE FOR TM (or TE) MODES

of a particular geometry identifies a family of cut-off frequencies ω_n with $n = 0, 1 \ldots \infty$

$$\boxed{k_n = \pm \frac{1}{c}\sqrt{\omega^2 - \omega_n^2}}$$

- Clearly mode n does NOT propagate if $\omega < \omega_n$
- $\pm k$ modes propagate forwards or backwards (or both)

- Many modes n can co-exist if ω is large enough
- It is advantageous to:
  1) Drive at $\omega_0 < \omega < \omega_1$ ONE MODE
  2) Use a geometry with $\omega_0 \ll \omega_1$ DYNAMIC RANGE
- The beam ITSELF can drive many unwanted HIGHER ORDER MODES (HOM's) up to frequencies

$$\boxed{\omega_{MAX} \simeq 2\pi \frac{c}{\sigma_s}}$$

where RMS bunch length may be (is) short.

- HOM's sometimes need to be explicitly damped...

PILL BOXES - CYLINDRICAL RESONANT CAVITIES

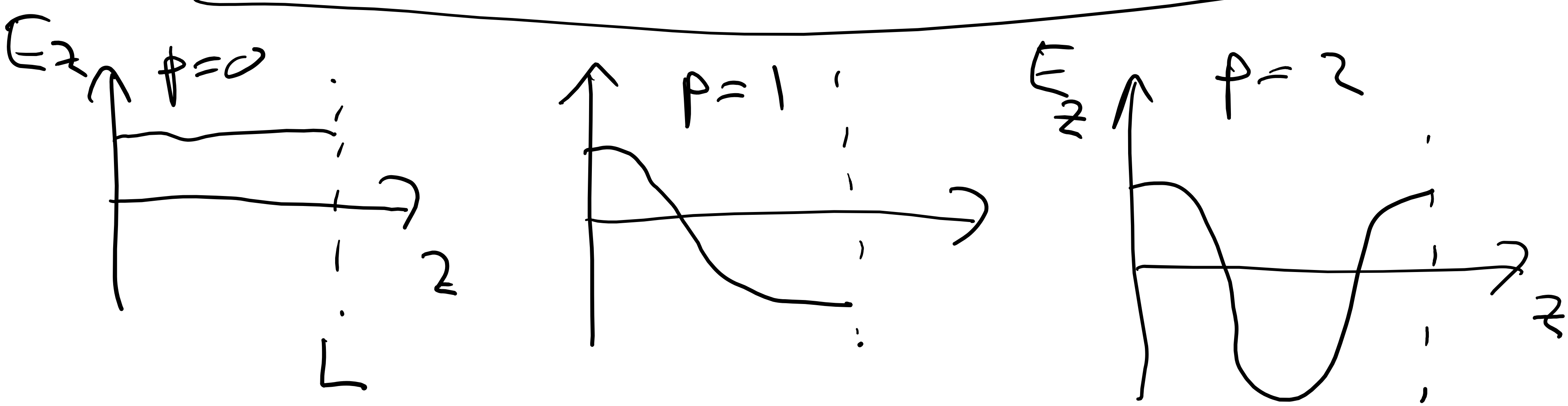
ADD FLAT ENDS at $z=0$ & L

Take a pair of $\pm k$ modes with frequency ω

$$k = \pm k_p = \pm p\frac{\pi}{L} \qquad p = 0, 1, \ldots \infty$$

and add together to form a TM mode

$$\boxed{E_z = \psi(r,\theta) \cdot \cos\left(p\pi \frac{z}{L}\right) \cdot e^{-i\omega_{RES} t}}$$



NEXT: SOLVE $\psi(r,\theta)$ for a circle

- adds 2 more indices: m, n

where the natural resonant frequencies for a pill-box are

(A) $$\boxed{\omega_{mnp} = c\sqrt{\left(\frac{U_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}} \quad [s^{-1}]$$

and $U_{mn} \geq 1$ is n'th root of m'th Bessel function

- Lowest resonant frequency has $p = 0$
- Simplify further with $m = 0$

(no azimutal structure)

TM_{0n0} MODE has only 2 non-zero field components

$$\boxed{\begin{aligned} E_z &= E_0 \; J_0\left(\nu_{0n} \frac{r}{R}\right) e^{-i\omega_{0n0} t} \\ B_\theta &= -B_0 \frac{\omega_{0n0} R}{\nu_{0n} c} \; J_1\left(\nu_{0n} \frac{r}{R}\right) e^{-i\omega_{0n0} t} \end{aligned}}$$

As illustrate (in book) for TM_{020}

- n counts the E_z zero crossing between $r = 0$ & R
- Note that B_θ is non-zero at $r = R$

$$\boxed{\omega_{0n0} = \frac{\nu_{0n} c}{R}} \qquad \begin{pmatrix} \nu_{01} \\ \nu_{02} \\ \nu_{03} \end{pmatrix} = \begin{pmatrix} 2.41 \\ 5.52 \\ 8.65 \end{pmatrix}$$

ADD HOLES to let beam through ...

- This let's field leak a little way into the beam pipe, but not far if ω_{010} is much smaller ~~the~~ than the pipe cut-off frequency
  - if pipe radius $\ll R$
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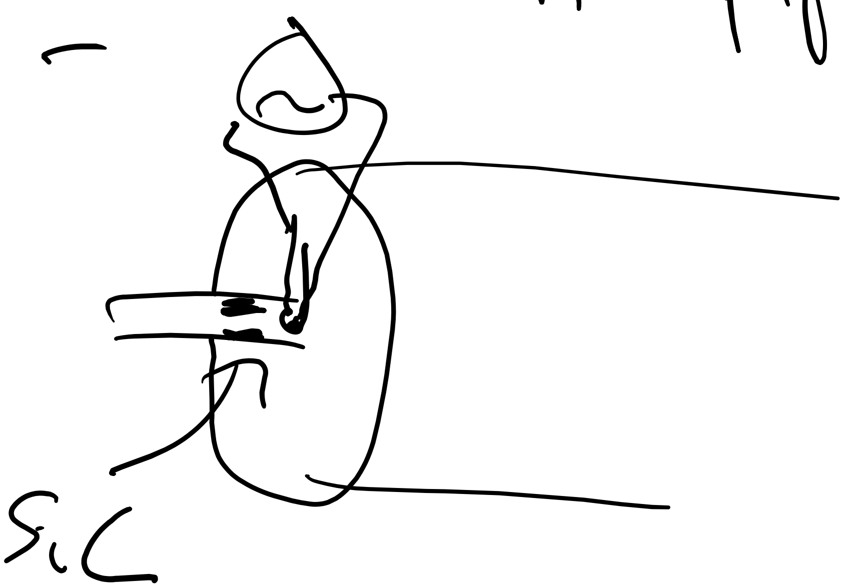


ADJUST R to get the right frequency: (see A)

How LONG should L be?

- A particle with (constant) speed βc passes through center at $t=0$, acquires a voltage

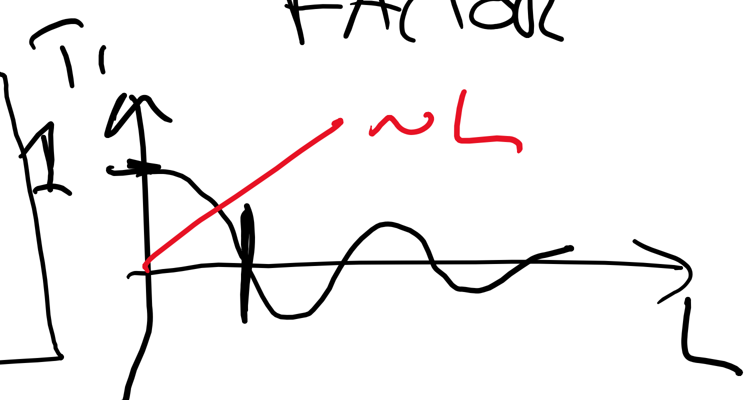
$$V_A = \int_{-L/2}^{L/2} E_z \cdot dz = \beta c \cdot E_0 \int_{-L/2\beta c}^{+L/2\beta c} e^{i\omega t} \cdot dt$$

With TM_{010} mode, then

$$V_A = E_0 L \cdot T_1$$

TRANSIT TIME FACTOR

$$\boxed{T_1(L) = \frac{\sin(\omega L/2\beta c)}{(\omega L/2\beta c)}}$$



and V_A has a maximum when

$$T = 2/\pi$$

$$\boxed{L_{OPT} = \pi \frac{\beta c}{\omega}}$$

EG in TM_{010} $\quad L_{OPT} = \frac{\pi}{2.41} \cdot \beta R$

VERY INEFFICIENT when $\beta \ll 1$!! Protons or ions

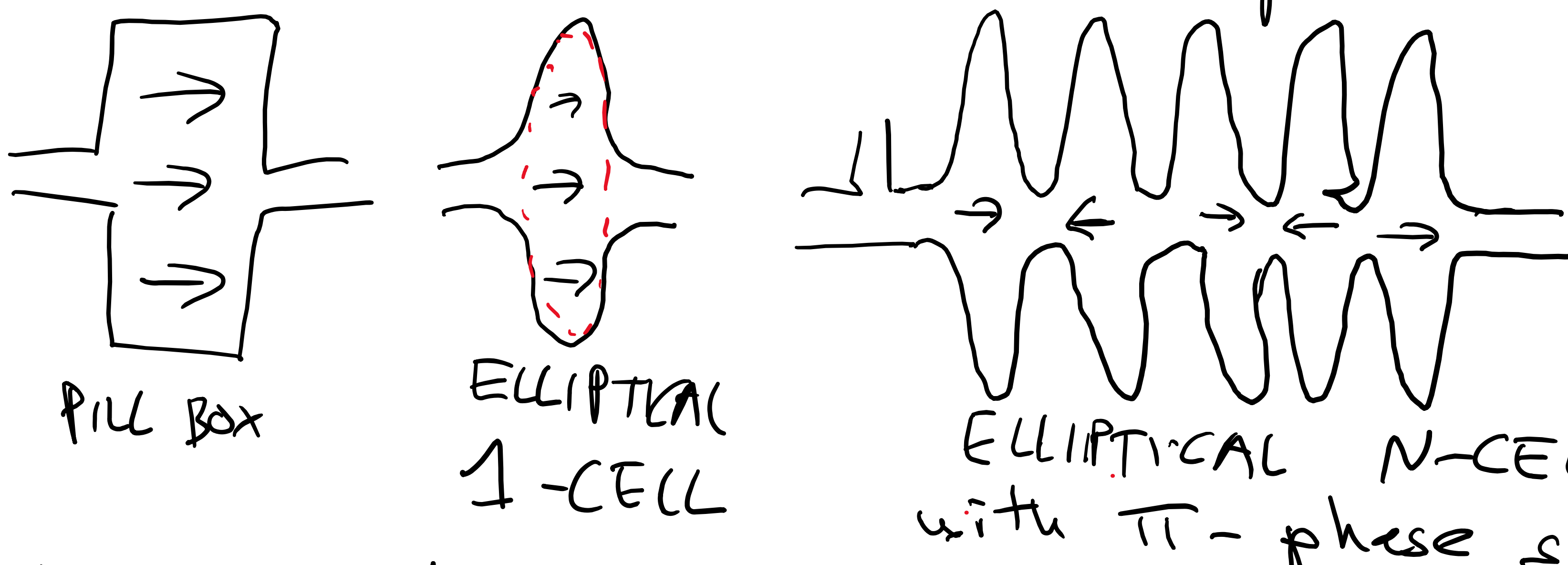
NON-RELATIVISTIC particles (say $\beta \leq 0.5$) need different cavity geometries.

RESONANT ZOO:

Spoke, split-ring, inter-digital, $\frac{1}{4}$ wave

$\frac{1}{2}$-wave, ... - - RFQ, ...

RELATIVISTIC CAVITIES: Geometries morph away from pill boxes:



The TM_{mnp} "language" survives if rotational symmetry is preserved

Q1: WHAT is the optimum # of cells in a cavity? (See later lecture)

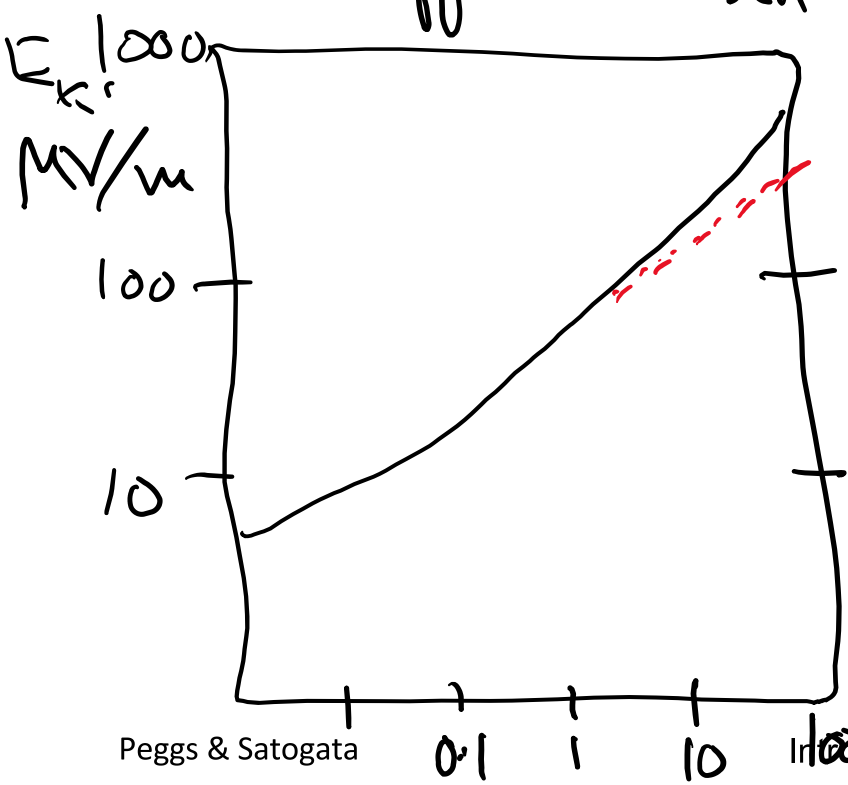
Q2: How BIG can gradient E_0 be? Copper cavities vs. superconducting?

KILPATRICK CRITERION (1950's)

Empirically: $\boxed{f = 1.64\, E_k^2\, e^{-(8.5/E_k)}}$

f: MHz E_k: MV/m

defines the frequency necessary to reach a gradient E_k on a copper wall without breakdown



- Conservative by ~ a factor of 2
- Suggests (falsely?) that copper cavities at f >> 10 GHz can outstrip superconducting cavities ($E_{sc} \gg 10$ MV/m)

SUPERCONDUCTING CAVITIES

- Many geometries are possible despite more difficult manufacture
- Lower operating costs: "no" heat dissipation in conducting walls
- Higher capital cost: complexity, material treatment, cryogenics, ...
- Very high Q values mean very small bandwidths

$$f_{BW} = \frac{f}{Q} \sim \frac{10^9}{10^{10}} \,[\mathrm{Hz}] \Rightarrow 0.1\,[\mathrm{Hz}]$$

and sensitivity to mechanical distortions:

1) microphonics (any noise, e.g. cryogenics)
2) Lorentz force detuning (in pulsed operation)

$\Rightarrow$ CW operations biases (more) towards SC technology.