#9: SEXTUPOLES & CHROMATICITY 1/28/19

CATCH-22: Sextupoles are (usually) necessary to avoid NON-linear resonances that are often driven by sextupoles themselves!

Recall motion of a particle with $\delta = \frac{\Delta p}{p}$ (constant)

Hill's equation

$$\left.\begin{aligned} x'' + \frac{K}{(1+\delta)} x &= 0 \\ y'' - \frac{K}{(1+\delta)} y &= 0 \end{aligned}\right\} \quad K \text{ "weaken" with } \delta$$

- CHROMATICITY measures the rate of change of tunes Q_x & Q_y with momentum

$$\boxed{\chi \equiv \frac{dQ}{d\delta}}$$

H & V

EG 1 FODO LATTICE

N identical cells, each with ϕ phase advance

$$Q = \frac{N}{2\pi}\phi$$

so the NATURAL (uncorrected) chromaticity is

$$\chi_{NAT} = \frac{N}{2\pi}\frac{d\phi}{d\delta}$$

but following Eq. 3.48

$$\frac{d\phi}{d\delta} = -2\tan(\phi/2)$$

so

$$\boxed{\chi_{x,NAT} = \chi_{y,NAT} = -Q\,\frac{\tan(\phi/2)}{\phi/2}} \sim -Q$$

This RULE OF THUMB often holds at factor of 2 level

EG 2 RHIC $\qquad \chi_{NAT} \approx -50$

RMS momentum spread $\frac{\sigma_p}{p} = \langle \delta^2 \rangle^{\frac{1}{2}} \approx 2 \times 10^{-3}$

UNCORRECTED tune spread $\boxed{\sigma_Q = \left|\chi_{NAT}\right| \frac{\sigma_p}{p} = 0.1}$!!

UNACCEPTABLY shrinks available locations in the tune plane

$\Rightarrow$ MUST DECREASE $|\chi|$

EG3 : SEXTUPOLE next to EVERY QUADRUPOLE

FIRST separate closed orbit & betatron oscillations:

$$x_{TOT} = \eta\delta + x$$

THEN expand the composite quad-sext kick:

$$\Delta x'_{TOT} = -q(1-\delta)(\eta\delta + x) - S(1-\delta)(\eta\delta + x)^2 + \dots$$

AND collect terms in $x^n \delta^m$

$$(\Delta\eta')\delta + \Delta x' = -q \cdot \eta\delta$$

$$- qx - Sx^2$$

$$+ q \cdot x\delta - S 2\eta x\delta + Sx^2 \cdot \delta$$

$$+ O(\delta^2) \dots$$

$x^0 \delta^1$

$x^m \delta^0$ GEOMETRIC

TO ORDER $x' \delta^m$

$$\boxed{\Delta x' = -q x + (q - 2\eta S) x \delta}$$

So the effective "quadrupole" strength is independent of δ IF

$\boxed{S = \frac{q}{2\eta}}$ at every quad-sext pair

- In a FODO lattice $\eta_F \approx 2\eta_D$ so

$$|S_F| \approx \frac{1}{2} |S_D|$$

Q: What happens if $\eta \approx 0$?

[Q: What do those geometric terms do?] later ...

EG 4 : CHROMATICITY CORRECTION : GENERAL CASE

Previously Equ. 8.29 : If a quad increases strength by Δq

$$\boxed{\Delta Q = \frac{\beta_0}{4\pi} \cdot \Delta q}$$

QUAD STRENGTH weakens with δ

$$\frac{dq}{d\delta} = -q$$

So natural chromaticities

$$\boxed{\begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix}_{NAT} = -\frac{1}{4\pi} \begin{pmatrix} \sum q \beta_x \\ \sum -q \beta_y \end{pmatrix}}$$

(Thick quads: use integrals)

where sums $\sum$ are over all (thin) quads.

Strength q is usually +ve (-ve) when β_x (β_y) is large

$\Rightarrow$ NATURAL CHROMATICITIES ARE NEGATIVE

SEXTUPOLES contribute a QUAD component

$$\frac{dq}{d\delta} = 2\eta S$$

So ONE FAMILY of sextupoles changes chromaticities

$$\Delta \begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix} = \frac{S_1}{2\pi} \begin{pmatrix} \sum \eta \beta_x \\ \sum -\eta \beta_y \end{pmatrix} \quad \begin{cases} \eta_x \text{ is H} \\ \eta_V = 0 \end{cases}$$

in different directions

Two FAMILIES near F + D are quadrupoles

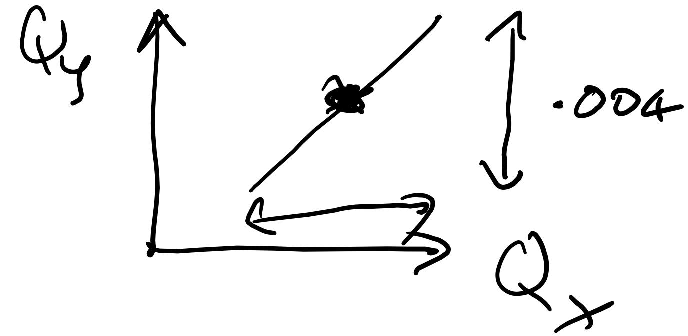
$$\boxed{\begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix}_{GOAL} = 2 \begin{pmatrix} \sum_F \eta \beta_x & \sum_D \eta \beta_x \\ \sum_F -\eta \beta_y & \sum_D -\eta \beta_x \end{pmatrix} \begin{pmatrix} S_F \\ S_D \end{pmatrix} + \begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix}_{NAT}}$$

SOLVE for S_F + S_D by inverting the matrix !!

EG5 RHIC (again) $\chi_{GOAL} = +2$

$\frac{\sigma_p}{p} = 2\times10^{-3}$

$\Rightarrow |\chi_{GOAL}| \frac{\sigma_p}{p} \Rightarrow \sigma_Q = .004$



Q: Why is TUNE FOOTPRINT a blob not a line?

Q: Why are χ goals slightly positive?
(collective effects)

[Q: What drives the resonance lines?]

HENON MAP : GEOMETRIC TERMS $(\delta = 0)$

"... exhibits ALL TYPICAL PROPERTIES of .. dynamical systems"

until finished {

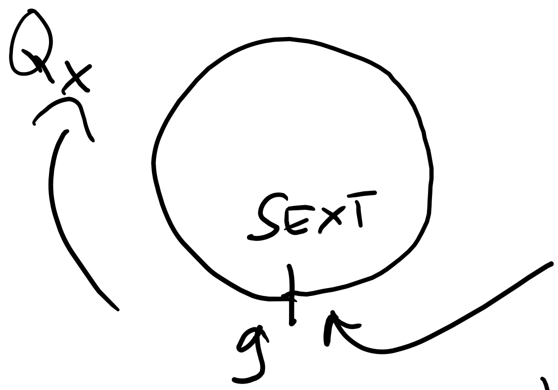
$$\begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(\ \ ) & \cos(\ \ ) \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix} + \begin{pmatrix} 0 \\ z^2 \end{pmatrix}$$

}

The ONLY control parameter Q is like the (horizontal) tune

The nonlinear kick z^2 is like a sextupole.

ONE SEXTUPOLE (horizontal motion)



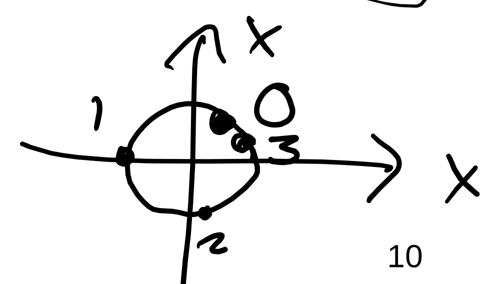
Reference point is JUST BEFORE the sextupole

KICK: $\begin{pmatrix} x \\ x' \end{pmatrix}_{0+c} = \begin{pmatrix} x \\ x' - g x^2 \end{pmatrix}_0$

ROTATE: $\begin{pmatrix} x \\ x' \end{pmatrix}_1 = R(2\pi Q) \begin{pmatrix} x \\ x' \end{pmatrix}_{0+c}$

EG 1 TUNE IS CLOSE TO 1/3 $\boxed{Q = \frac{1}{3} + \delta Q}$

NET MOTION after 3 turns is SMALL ....



$$\boxed{\begin{pmatrix} x \\ x' \end{pmatrix}_3 - \begin{pmatrix} x \\ x' \end{pmatrix}_0 \approx 3\mu \begin{pmatrix} x \\ -x' \end{pmatrix}_0 - g\left[\begin{pmatrix} S_3 \\ C_3 \end{pmatrix} x_0^2 + \begin{pmatrix} S_2 \\ C_2 \end{pmatrix} x_0^2 + \begin{pmatrix} S_1 \\ C_1 \end{pmatrix} x_0^2\right]}$$

WHERE $\mu = 2\pi\,\delta Q$, $C_k = \cos\left(k \frac{2\pi}{3}\right)$, $S_k = \sin\left(k\frac{2\pi}{3}\right)$.

MORE SUCCINCTLY: 3-turn discrete

KOBAYASHI HAMILTONIAN:

$$\boxed{H_3 = \frac{\mu}{2}\left(x^2 + x'^2\right) + \frac{g}{9}\sum_{k=1}^{3}\left(C_k x + S_k x'\right)^3}$$

$$\Delta x \approx \frac{\partial H_3}{\partial x'} \cdot \Delta t$$

$$\Delta x' \approx -\frac{\partial H_3}{\partial x} \cdot \Delta t$$

with $\Delta t = 3$

[Q: Cross terms?]

- H_3 is approximately constant along a trajectory

- Factorising H_3 gives

$$\boxed{H_3 = \frac{g}{3}\left(\frac{2\mu}{g}\right)^3 + \frac{g}{12}\underbrace{\left(x+\sqrt{3}x' + \frac{4\mu}{g}\right)}_{0}\underbrace{\left(x-\sqrt{3}x'+\frac{4\mu}{g}\right)}_{0}\underbrace{\left(x-\frac{2\mu}{g}\right)}_{0}}$$

So, if $H_3 = \frac{g}{3}\left(\frac{2\mu}{g}\right)^3$

then

3 straight lines !!

$\boxed{\text{EG } Q = 0.324}$

- 3 straight lines!
- Regular "circular" motion near origin
- Rapid escape along divergent arms
- 3 outer "islands" unpredicted by perturbation theory

$\boxed{\text{EG } Q = .2516}$ $\simeq \frac{1}{4}$

- 4 RESONANCE ISLANDS!
- Island hopping: only some phases are accessible.
- Chains of many small islands also appear

$$Q = \frac{p}{q}$$

$q \leftarrow$ 33? (large number).

$\boxed{\text{EG} \quad Q = 0.211}$ $\approx 1/5$

- CHAOTIC "fly specks" DOTS are visually distinctive
- 5 islands are more rotationally symmetric

$\boxed{\text{EG} \quad Q = 0.185}$ $\simeq \frac{1}{6}$

6 smaller islands, more rotationally symmetric, closer to the <u>DYNAMIC APERTURE</u>

HENON's TAXONOMY of typical behavior:

1. REGULAR NON-RESONANT
   - Roughly circular motion
   - Enough dots (turns) make "continuous" lines
2. REGULAR RESONANT
   - Island hopping: only some phases accessible ($Q = .324$)
3. RAPIDLY DIVERGENT
   - Amplitude increases rapidly & regularly
4. CHAOS
   - Visually distinctive, but WHAT IS IT?

LYAPUNOV EXPONENT

- Start 2 particles VERY close together (e.g. one digit apart in double precision)

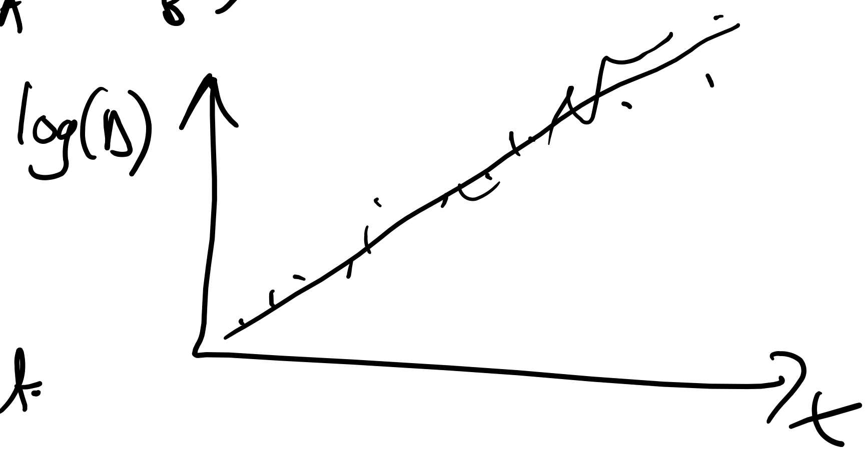
Q: How do they BEGIN to diverge in time?

A1: LINEARLY $\Rightarrow$ Regular
A2: EXPONENTIALLY $\Rightarrow$ Chaotic

e.g. $D = (x_A - x_B)^2 + (x_A' - x_B')^2$

$$\boxed{D(t) \sim D_0 e^{\lambda t}}$$

Lyapunov exponent



DYNAMIC APERTURE (DA)

- IN SIMULATION there is (usually) a clear cut maximum amplitude beyond which particles rapidly get lost – hit the beam pipe
- IN REALITY there is noise/excitation/diffusion drives particles across the DA
- beam current decreases with time

EVEN IN 1-D (not 3-D) WITH 1 SEXTUPOLE this is a complex story (and NAIVE)