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A SELECTION OF FORMULAE AND DATA USEFUL FOR THE DESIGN OF A.G. SYNCHROTRONS

C. Bovet, R. Gouiran, I. Gumowski, K.H. Reich

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LIST OF FREQUENTLY OCCURING SYMBOLS, THEIR MEANINGS AND UNITS *)

A	RF "bucket" area (in longitudinal phase plane **) [see page 31]
A_H	acceptance in horizontal phase plane** [see page 18] (= area of largest acceptable ellipse/π)
A_V	acceptance in vertical phase plane** [see page 18]
B, B_o	magnetic flux density, in teslas [T], nominal value
C, C_o	length of orbit [m], nominal value
c	velocity of light [m/s]
e	electronic charge [C]
eV	maximum energy gain per turn [keV] (unless units stated otherwise)
E	total energy of particle [GeV] (unless units stated otherwise)
E_o	rest energy of particle [GeV] (unless units stated otherwise)
f_a	accelerating frequency [Hz]
f	revolution frequency [Hz]
f_∞	asymptotic value of f_a reached at $\beta = 1$
g	gradient of magnetic field, in teslas per metre [Tm^{-1}]
h	harmonic number = f_a/f
K	focal constant [m^{-2}]
m	mass of particle [GeV/c^2]
m_p	mass of proton [GeV/c^2]
n	field index = $(-\rho_o/B_o)(\partial B/\partial x)$
p, p_o	momentum of particle [GeV/c], nominal value
Q	number of betatron oscillations per revolution
R, R_o	mean orbit radius (= $C/2\pi$), nominal value, [m] unless stated otherwise
T	kinetic energy of particle [GeV]
V	peak accelerating voltage per turn [kV]

---

*) In square brackets

**) With these definitions the available six-dimensional hypervolume is $A_6 = \pi^2 A_H A_V RA$ where A is in $(\Delta p/m_o c) - \varphi$ coordinates.

α_p	momentum compaction factor
$\alpha(s)$	Twiss parameter
β	ratio of particle velocity to that of light (= v/c)
$\beta(s)$, $\beta_{H,V}$	betatron amplitude function
Γ	= sin φ_s where φ_s refers to the synchronous particle
$\gamma(s)$	Twiss parameter
ϑ	deflection angle
ϵ	emittance in transverse plane [see page 18] (= area of ellipse/π occupied by beam in respective plane)
ϵ_H	horizontal beam emittance* [see page 18]
ϵ_V	vertical beam emittance* [see page 18]
Θ	azimuthal angle
μ	phase shift of betatron oscillation for one focusing period
ρ	bending radius [m], positive from centre towards outside
φ	"phase angle" between particle and zero crossing of RF voltage
φ_s	"phase angle" for synchronous (phase stationary) particle
$\psi(s)$	phase advance of the betatron oscillation

Other symbols are defined as they occur.

Coordinate system of particle: (Definitions of s, x, y, z as in Courant and Snyder[1.3**])

s	distance along beam axis
x	horizontal transverse coordinate, same sign as ρ
z	vertical transverse coordinate, positive towards sky
y	general transverse coordinate
$\bar{x}$	arithmetic mean of x
$\langle x \rangle$	r.m.s. value of x

A prime denotes differentiation with respect to s.

A dot denotes differentiation with respect to time.

---

*) With these definitions the six-dimensional invariant hypervolume occupied by the beam is $V_6 = (\pi\ \beta\ \gamma)^2\ \epsilon_H\ \epsilon_V\ R\ S_\varphi$ where S_φ is the area [in ($\Delta p/m_0 c$) - φ coordinates] occupied by one bunch in the longitudinal phase plane

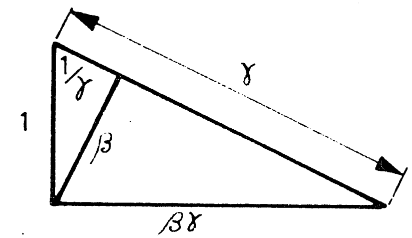
**) See page 43 for references

P A R T I

BASIC RELATIONS

1. PARTICLE VELOCITY, MOMENTUM AND ENERGY

1.1 Relations between β, cp, E_o, T, E, γ



In terms of / wanted	β	cp	T	E	γ
$\beta =$	β	$[(E_o/cp)^2 + 1]^{-1/2}$ cp/E	$[1 - (1 + T/E_o)^{-2}]^{1/2}$	$[1 - (E_o/E)^2]^{1/2}$ cp/E	$(1 - \gamma^{-2})^{1/2}$
$cp =$	$E_o(\beta^{-2} - 1)^{-1/2}$ $E\beta$	cp	$[T(2E_o + T)]^{1/2}$ $T[(\gamma + 1)/(\gamma - 1)]^{1/2}$	$(E^2 - E_o^2)^{1/2}$ $E\beta$	$E_o(\gamma^2 - 1)^{1/2}$
$E_o =$	$cp/\beta\gamma$ $E(1 - \beta^2)^{1/2}$	$cp(\gamma^2 - 1)^{-1/2}$	$T/(\gamma - 1)$	$(E^2 - c^2p^2)^{1/2}$	E/γ
$T =$	$[(1 - \beta^2)^{-1/2} - 1]E_o$	$[E_o^2 + c^2p^2]^{1/2} - E_o$ $cp[(\gamma - 1)/(\gamma + 1)]^{1/2}$	T	$E - E_o$	$E_o(\gamma - 1)$
$\gamma =$	$(1 - \beta^2)^{-1/2}$	$cp/E_o\beta$ $[1 + (cp/E_o)^2]^{1/2}$	$1 + T/E_o$	E/E_o	γ

In a synchrotron:

$\beta = 2\pi Rf/c \qquad (f = f_a/h)$

and $p[GeV/c] = 0.2997925\ B\rho\ [Tm]$, or $p\ [VAs^2m^{-1}] = eB\rho\ [As\ Tm]$.

1.2 First Derivatives

In terms of / Wanted	$d\beta$	$d(cp)$	$d\gamma = dE/E_0 = dT/E_0$
$d\beta =$	$d\beta$	$[1 + (cp/E_0)^2]^{-3/2}\, d(cp)/E_0$ $\gamma^{-3}\, d(cp)/E_0$	$\gamma^{-2}(\gamma^2 - 1)^{-1/2}\, d\gamma$ $\beta^{-1}\, \gamma^{-3}\, d\gamma$
$d(cp) =$	$E_0(1 - \beta^2)^{-3/2}\, d\beta$ $E_0\, \gamma^3\, d\beta$	$d(cp)$	$E_0\gamma(\gamma^2 - 1)^{-1/2}\, d\gamma$ $E_0\, \beta^{-1}\, d\gamma$
$d\gamma =$ $= dE/E_0 =$ $= dT/E_0 =$	$\beta(1 - \beta^2)^{-3/2}\, d\beta$ $\beta\gamma^3\, d\beta$	$[1 + (E_0/cp)^2]^{-1/2}\, d(cp)/E_0$ $\beta\, d(cp)/E_0$	$d\gamma$

1.3 Logarithmic first derivatives

In terms of / Wanted	$d\beta/\beta$	dp/p	dT/T	$dE/E = d\gamma/\gamma$
$d\beta/\beta =$	$d\beta/\beta$	$\gamma^{-2}\, dp/p$ $dp/p - d\gamma/\gamma$	$[\gamma(\gamma + 1)]^{-1}\, dT/T$	$(\gamma^2 - 1)^{-1}\, d\gamma/\gamma$ $(\beta\gamma)^{-2}\, d\gamma/\gamma$
$dp/p =$	$\gamma^2 d\beta/\beta$	dp/p	$[\gamma/(\gamma + 1)]\, dT/T$	$\beta^{-2}\, d\gamma/\gamma$
$dT/T =$	$\gamma(\gamma + 1)\, d\beta/\beta$	$(1 + \gamma^{-1})\, dp/p$	dT/T	$\gamma(\gamma - 1)^{-1}\, d\gamma/\gamma$
$dE/E =$ $d\gamma/\gamma =$	$(\beta\gamma)^2\, d\beta/\beta$ $(\gamma^2 - 1)\, d\beta/\beta$	$\beta^2\, dp/p$ $dp/p - d\beta/\beta$	$(1 - \gamma^{-1})\, dT/T$	$d\gamma/\gamma$

See page 1 for meaning of symbols.

2. ENERGY AVAILABLE IN COLLISION BETWEEN TWO PARTICLES

β, γ, and Θ_c measured in the laboratory frame.

2.1 General two-body collision along the same line

$$E_{c.m.} = [m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2\ (1 - \beta_1\beta_2)]^{1/2} ,$$

where β is counted algebraically, and $E_{c.m.}$ is the total energy in the centre-of-mass frame, i.e. the maximum available energy.

2.2 Two identical particles

i) One particle at rest: $\beta_1 = 0$, $\gamma_1 = 1$

$$E_{c.m.} = m\ (2 + 2\gamma_2)^{1/2} \approx m\ (2\gamma_2)^{1/2} \qquad \text{for } \gamma_2 \gg 1 .$$

ii) Two particles having velocities of the same magnitude but of opposite sign: $\gamma_1 = \gamma_2 = \gamma$; $\beta_1 = -\beta_2$:

$$E_{c.m.} = 2E = 2m\gamma .$$

If a proton colliding with another proton at rest can liberate the same energy as a collision between two protons with opposite velocities, its energy is defined by

$$\gamma_{\acute{e}q} = 2\gamma^2 - 1 \approx 2\gamma^2 \qquad \text{for } \gamma \gg 1 .$$

iii) Two particles having velocities of the same magnitude but making a small angle Θ_c:

$$E_{c.m.} = 2E\ (1 - \beta^2 \sin^2 (\Theta_c/2)]^{1/2}$$
$$\approx 2E \cos (\Theta_c/2) \qquad \text{for } \beta \approx 1 .$$

$$\gamma_{eq} = 2\gamma^2 \cos^2 (\Theta_c/2) - 1$$
$$\approx 2\gamma^2 \cos^2 (\Theta_c/2) \qquad \text{for } \gamma \gg 1 .$$

See page 1 for meaning of symbols

3. MAGNETIC AND ELECTRIC DEFLECTION

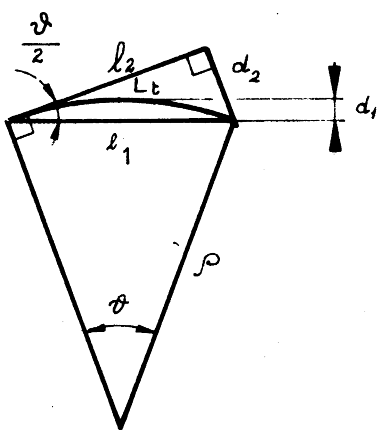
3.1 Magnetic deflection

a) Deflection angle ϑ [rad] = $BL_t/(B\rho)$ = $0.2997925\ BL_t/p$ [Tm/GeV/c)]

b) Beam rigidity (magnetic bending radius)

$$B\rho\ [\text{Tm}] = 3.3356\ p\ [\text{GeV/c}]$$
$$= 3.1297\ \beta\gamma \text{ for protons (refer to Table 1.1 for other expressions)}$$

c) Sagitta

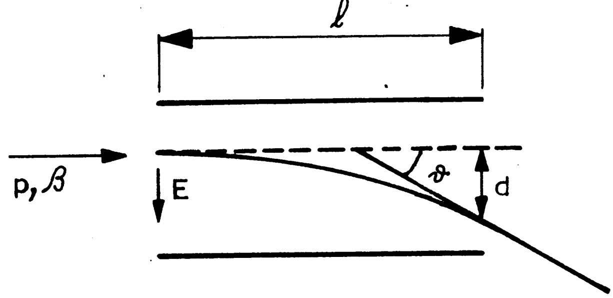


$$d_1 = \tfrac{1}{2}\,\ell_1 \tan(\vartheta/4) = 2\rho\sin^2(\vartheta/4)$$
$$\approx \frac{\rho\vartheta^2}{8} = \frac{L_t^2}{8\rho}$$

$$d_2 = \ell_2 \tan(\vartheta/2) = \rho(1 - \cos\vartheta)$$
$$\approx \frac{\rho\vartheta^2}{2} = \frac{L_t^2}{2\rho}$$

3.2 Electric deflection

a) Deflection angle



$$\vartheta\ [\text{rad}] = \arctan(E\ell/p\beta)\ [10^9\ \text{V/(GeV/c)}]$$

b) Sagitta

$$d[\text{m}] = E\ell^2/2\beta p\ [10^9\ \text{Vm/(GeV/c)}]\ .$$

3.3 Comparison of electric and magnetic deflection

For small ϑ,

$$B[T] \approx E/(300\ \beta)\ [MV/m] \text{ for the same deflection.}$$

Equivalent deflection for high fields, B = 2T, E = 10 MV/m corresponds to $\beta = 1/60$; and, for protons, p = 16 MeV/c, T = 0.13 MeV.

4. SOME FORMULAE FOR QUANTITIES RELATED TO SYNCHROTRONS

4.1 Mean machine radius

$R_0 = C_0/2\pi = (1 + k)\rho_0$, where k is the circumference factor.

4.2 Relations between p, R, B, f, β and their derivatives

4.2.1 p, R, B*)

a) Fundamental equation

$$p = e\rho_0\ (R/R_0)^{1/\alpha_p}\ B$$

b) Definition of α_p

$$\alpha_p = \frac{p}{R}\left(\frac{\partial R}{\partial p}\right)_B \quad \left(\approx \frac{1}{Q^2}\right)$$

4.2.2 f, β, R

$$f = \beta c/2\pi R\ .$$

4.2.3 Definition of transition energy $E_{tr} = \gamma_{tr} E_0$

$$\frac{p}{f}\left(\frac{\partial f}{\partial p}\right)_B = \frac{1}{\gamma_{tr}^2} - \alpha_p = 0$$

$$\gamma_{tr} = 1/\sqrt{\alpha_p}\ (\approx Q)\ .$$

---

*) B is defined on the nominal orbit C_0.

4.2.4 Differential relations

Variables	Equations
B, p, R	$\frac{dp}{p} = \gamma_{tr}^2 \frac{dR}{R} + \frac{dB}{B}$
f, p, R	$\frac{dp}{p} = \gamma^2 \frac{df}{f} + \gamma^2 \frac{dR}{R}$
B, f, p	$\frac{dB}{B} = \gamma_{tr}^2 \frac{df}{f} + \frac{\gamma^2 - \gamma_{tr}^2}{\gamma^2} \frac{dp}{p}$
B, f. R	$\frac{dB}{B} = \gamma^2 \frac{df}{f} + (\gamma^2 - \gamma_{tr}^2) \frac{dR}{R}$

4.3 Relation between currents and number of particles

a) Number of injected particles in terms of linac current:
In the case of multiturn injection

$$N = 1.3082 \times 10^{11} \; n_t \epsilon_a \epsilon_{tp} R I_L/\beta \quad [\mathrm{Am}]$$

where

N is the number of trapped particles in the synchrotron;
n_t is the number of injected turns;
I_L is the linac current $[A]$;
ϵ_a is the mean transverse phase space injection efficiency;
ϵ_{tp} is the longitudinal trapping efficiency.

b) Circulating current:

$$I\,[A] = (ec/2\pi)(N\beta/R) = 7.6441 \times 10^{-12} \; N\beta/R \; [m^{-1}] \;.$$

Number of charges passing per microsecond = $6.2418 \times 10^9 I$ [mA],
or I [mA] = 1.6021×10^{-10} × number of charges passing per microsecond.

ADDITIONAL FORMULAE

P A R T II

TRANSVERSE PHASE SPACE

1. MATRIX FORMULATION OF BEAM DYNAMICS

1.1 General form of matrices with dispersive terms

The general matrix M is defined by

[2,52]*
[3,52]

$$\begin{pmatrix} y \\ y' \\ \Delta p/p \end{pmatrix}_B = M(B|A) \begin{pmatrix} y \\ y' \\ \Delta p/p \end{pmatrix}_A$$

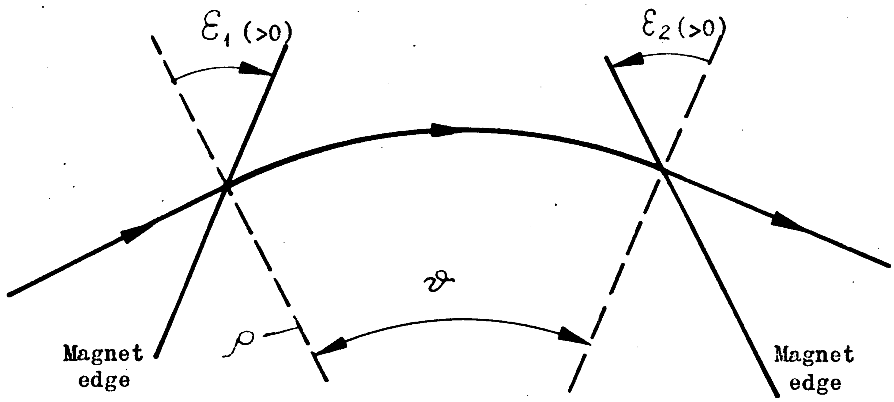
1.2 Drift length ℓ

$$M_\ell = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.3 Dipole magnet

a) Notation and definitions

[3,154]



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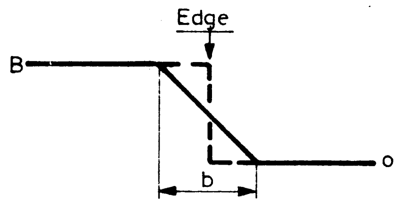
* See page 43 for references

Fringe field, linear approximation:



b) Pure sector magnet

$(\epsilon_1 = \epsilon_2 = 0,\ b = 0)$

$$M_H^S = \begin{pmatrix} \cos\vartheta & \rho\sin\vartheta & \rho(1-\cos\vartheta) \\ -\dfrac{\sin\vartheta}{\rho} & \cos\vartheta & \sin\vartheta \\ 0 & 0 & 1 \end{pmatrix}$$ in horizontal plane (plane of deflection)

$$M_V^S = \begin{pmatrix} 1 & \rho\vartheta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ in vertical plane

c) Magnet with parallel faces

$$M_H^R = \begin{pmatrix} 1 & \rho\sin\phi \\ 0 & 1 \end{pmatrix} \qquad M_V^R = M_V^E\, M_V^S\, M_V^E$$

d) Edge effect with linear fringe field

[4,100]* $$M_H^E = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{\tan\epsilon}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $\epsilon = \epsilon_1$ for entrance

$\epsilon = \epsilon_2$ for exit

$$M_V^E = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{1}{\rho}\left(\dfrac{b}{6\rho\cos\epsilon} - \tan\epsilon\right) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $\epsilon = \epsilon_1$ for entrance

$\epsilon = \epsilon_2$ for exit

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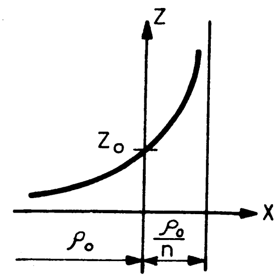
*) See page 43 for references

1.4 Gradient sector magnet



a) Equation of profile

$$\left(\frac{\rho_o}{n} - x \right) z = \frac{\rho_o}{n} z_o$$

z_o is the half-aperture where $\rho = \rho_o$.

Also $g[Tm^{-1}] = - nB_o/\rho_o$ $[Tm^{-1}]$

$= - 3.3356\ np_o/\rho_o^2$ $[(GeV/c)m^{-2}]$.

b) Focusing plane

[3,53] $$M_F^* = \begin{pmatrix} \cos \zeta & \frac{1}{\sqrt{K}} \sin \zeta & \frac{1}{\rho K}(1 - \cos \zeta) \\ -\sqrt{K} \sin \zeta & \cos \zeta & \frac{1}{\rho\sqrt{K}} \sin \zeta \\ 0 & 0 & 1 \end{pmatrix},$$

where $K[m^{-2}] = (|n| + 1)/\rho_o^2$ (horizontal plane)

$= |n|/\rho_o^2$ (vertical plane)

$\zeta = \ell_m \sqrt{K}$, ℓ_m is the magnetic length.

c) Defocusing plane

$$M_D^* = \begin{pmatrix} \cosh \zeta & \frac{1}{\sqrt{K}} \sinh \zeta & \frac{1}{\rho K}(\cosh \zeta - 1) \\ \sqrt{K} \sinh \zeta & \cosh \zeta & \frac{1}{\rho\sqrt{K}} \sinh \zeta \\ 0 & 0 & 1 \end{pmatrix},$$

where $K[m^{-2}] = (|n| - 1)/\rho_o^2$ (horizontal plane)

$= |n|/\rho_o^2$ (vertical plane)

$\zeta = \ell_m \sqrt{K}$, ℓ_m is the magnetic length

d) Pure quadrupole lens

The same matrices as in points (b) and (c) are used, with $1/\rho = 0$ and $K[m^{-2}] = 0.2997925\ g/p_o$ $[Tm^{-1}/(GeV/c)]$

$= - n/\rho_o^2 = g/(B\rho)$ $[m^{-2}]$

$= 0.31952\ g/\beta\gamma$ $[Tm^{-1}]$ for protons.

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*) Valid for coordinate system defined on page 2.

2. DESCRIPTION OF SINGLE PARTICLE MOTION IN A SYNCHROTRON

2.1 Equation of motion

[2,33] [5,55] [1,3]

$$d^2x/ds^2 + K_x(s)x = (\Delta p/p)/\rho(s)$$

$$d^2z/ds^2 + K_z(s)z = 0.$$

2.2 Solution of the equation of motion

[1,11] [2,79] [6,4]

$$\begin{cases} y(s) = \sqrt{\epsilon\beta(s)}\cos[\psi(s) + \delta] \\ y'(s) = -\sqrt{\epsilon/\beta(s)}\{\alpha(s)\cos[\psi(s)+\delta] + \sin[\psi(s)+\delta]\} = \sqrt{\epsilon\gamma(s)}\cos[\chi(s)+\delta] \end{cases}$$

$$\psi(s) = \int_0^s ds/\beta(s)$$

$$\beta(s)\gamma(s) = 1 + \alpha^2(s)$$

$$\beta'(s) = -2\alpha(s)$$

$$\tan[\psi(s) - \chi(s)] = 1/\alpha(s)$$

$$\sin[\psi(s) - \chi(s)] = -[\beta(s)\gamma(s)]^{-1/2}.$$

Envelope equation:

$$\sqrt{\beta}'' + K(s)\sqrt{\beta} - \beta^{-3/2} = 0$$

Initial conditions:

$$\epsilon = \gamma(o)y^2(o) + \beta(o)y'^2(o) + 2\alpha(o)y(o)y'(o)$$

$$\cos\delta = y(o)/\sqrt{\epsilon\beta(o)}$$

$$\tan\delta = -[\alpha(o) + \beta(o)y'(o)/y(o)].$$

For $y(o) = 0$:

$$y(s) = y'(o)\sqrt{\beta(o)\beta(s)}\sin\psi(s)$$

$$y'(s) = -y'(o)\sqrt{\beta(o)/\beta(s)}[\alpha(s)\sin\psi(s) - \cos\psi(s)].$$

$$Q = \psi(C)/(2\pi)$$

Betatron wavelength $\qquad \lambda = 2\pi R/Q$

Form factor $\qquad F = \beta_{max}Q/R.$

2.3 Sinusoidal approximation

$$\psi(s) \approx 2\pi s/\lambda = Q\, 2\pi s/C \;;\quad \beta(s) \approx R/Q$$

$$y(s) \approx \sqrt{\epsilon R/Q}\cos(Qs/R + \delta).$$

2.4 Motion through one period (cell) of length L_p

$K(s)$, $\alpha(s)$, $\beta(s)$, $\gamma(s)$ are periodic with period L_p,
and $\psi(s + L_p) = \psi(s) + \mu$, $\chi(s + L_p) = \chi(s) + \mu$.

The transfer matrix through one period may be written as

[1,6] $$M(s + L_p|s) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix}$$

$$\cos\mu = \tfrac{1}{2}(a_{11} + a_{22}) \qquad \beta(s) = a_{12}/\sin\mu$$
$$\gamma(s) = -a_{21}/\sin\mu \qquad \alpha(s) = \tfrac{1}{2}(a_{11} - a_{22})/\sin\mu .$$

2.5 Transfer matrix through any section

a) If the Twiss parameters at points s_1, s_2 are $(\beta_1, \alpha_1, \gamma_1)$ and $(\beta_2, \alpha_2, \gamma_2)$, respectively, the 2×2 transfer matrix from s_1 to s_2 can be written as

[1,9] $$M(s_2|s_1) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\psi + \alpha_1\sin\Delta\psi) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\left[\frac{(1 + \alpha_1\alpha_2)\sin\Delta\psi + (\alpha_2 - \alpha_1)\cos\Delta\psi}{\sqrt{\beta_1\beta_2}}\right] & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\psi - \alpha_2\sin\Delta\psi) \end{pmatrix},$$

where $\Delta\psi = \psi(s_2) - \psi(s_1)$.

b) Transformation of the Twiss parameters through a beam transfer section:

[7,100] $$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{21}m_{11} & 1 + 2m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} .$$

$$\tan\Delta\psi = m_{12}/[m_{11}\beta(s_1) - m_{12}\alpha(s_1)]$$

Example: drift length ℓ,

[8,2] $$\beta(s_2) = \beta(s_1) - 2\alpha(s_1)\ell + \gamma(s_1)\ell^2$$
$$\alpha(s_2) = \alpha(s_1) - \gamma(s_1)\ell$$
$$\gamma(s_2) = \gamma(s_1),$$

and $\psi(s_2) = \psi(s_1) + \arctan\{\ell/[\beta(s_1) - \alpha(s_1)\ell]\}$.

2.6 Normalization

After a normalisation transformation $\binom{\eta}{\eta'} = M\binom{y}{y'}$ (with $\eta' = d\eta/d\psi$), the transverse phase space trajectories have the form of circles on which a phase advance $\psi(s)$ produces simply a rotation by ψ. Possible transformations matrices:

$$M = \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{pmatrix}; \text{ or, when } \alpha = 0, \; M = \begin{pmatrix} 1 & 0 \\ 0 & \beta(s) \end{pmatrix}$$

3. ELLIPSE REPRESENTATION IN TRANSVERSE PHASE SPACE (see page 18 for units)

3.1 The Courant-Snyder invariant

$$\gamma(s)y^2 + 2\alpha(s)yy' + \beta(s)y'^2 = \epsilon = \frac{\text{area}}{\pi} .$$

The largest area contained in the synchrotron is given by the acceptance $A_{H,V} = r^2/\beta_{max}$, where r is the half-aperture of the vacuum chamber at β_{max}.

3.2 Ellipse parameters

An ellipse centred at the origin of the phase plane is determined by three independent parameters. Depending on the particular problem, one can choose one of the following sets:

i) Twiss parameters α, β, γ and ϵ giving the emittance (see Section 3.1). These parameters transform with the matrix given in Section 2.5 (b);

ii) the elements c_i of a 2 × 2 matrix which transforms by multiplication with $M(s_2|s_1)$; $(c_3 y - c_1 y')^2 + (c_4 y - c_2 y')^2 = \epsilon^2$

iii) L, S, and ϵ where L is the ratio of the ellipse axes a/b at the waist, S is the distance of the waist along the beam (>0 if waist upstream).

The optical transformations from s_1 to s_2 are:

$$L(s_2) = \frac{L(s_1)}{[m_{21} L(s_1)]^2 + [m_{21} S(s_1) + m_{22}]^2}$$

$$S(s_2) = \frac{m_{11}m_{21} L^2(s_1) + [m_{11} S(s_1) + m_{12}][m_{21} S(s_1) + m_{22}]}{[m_{21} L(s_1)]^2 + [m_{21} S(s_1) + m_{22}]^2} .$$

Conversion from one set to the other is given in Table 3.3.

3.3 Conversion of ellipse parameters

given / wanted	$\alpha, \beta, \gamma, \epsilon$	$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$	L, S, ϵ
	$\beta\gamma - \alpha^2 = 1$	$\epsilon = c_1c_4 - c_2c_3$	
α	α	$-(c_1c_3 + c_2c_4)/\epsilon$	$-S/L$
β	β	$(c_1^2 + c_2^2)/\epsilon$	$L + S^2/L$
γ	γ	$(c_3^2 + c_4^2)/\epsilon$	$1/L$
c_1	$\sqrt{\epsilon\beta}$	c_1	$\sqrt{L\epsilon}$
c_2	0	c_2	$S\sqrt{\epsilon/L}$
c_3	$-\alpha\sqrt{\epsilon/\beta}$	c_3	0
c_4	$\sqrt{\epsilon/\beta}$	c_4	$\sqrt{\epsilon/L}$
L	$1/\gamma$	$\epsilon/(c_3^2 + c_4^2)$	L
S	$-\alpha/\gamma$	$(c_1c_3+c_2c_4)/(c_3^2 + c_4^2)$	S

Comments on units:

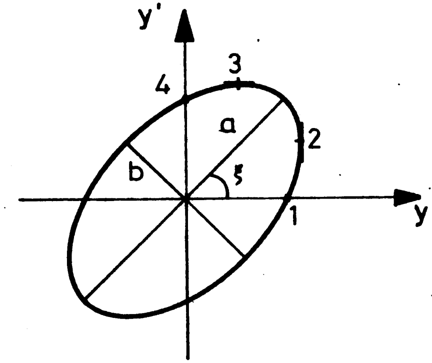
The equations of Sections 3.1 to 3.4 are valid for either of the two following sets of units:

i) All lengths in metres, all angles in radians, the emittances in rad m, L (= y_1/y_3') in m/rad , S(=y_3/y_3') in m/rad.

ii) All phase plane dimensions in mm and mrad, emittances in mrad mm, β = [mm/mrad] if defined as y_2/y_4' or β = [m] if defined as reduced betatron wavelength(similarly γ = [mm/mrad], or [m^{-1}], L = [mm/mrad], S = [mm/mrad] if defined as y_3/y_3' or S = [m] if defined as distance from waist.

N.B.: the values of α, β, γ, L and S do not depend on the choice between i) and ii).

3.4 Geometrical properties of the ellipse



	$\alpha, \beta, \gamma, \epsilon$	$c_1 \quad c_2$ $c_3 \quad c_4$	L, S, ϵ
	$\beta\gamma - \alpha^2 = 1$ $H = \frac{1}{2}(\beta + \gamma)$	$\epsilon = c_1c_4 - c_2c_3$ $H = \frac{1}{2}(c_1^2 + c_2^2 + c_3^2 + c_4^2)/\epsilon$	$H = \frac{1}{2L}(L^2 + S^2 + 1)$
y_1	$\sqrt{\epsilon/\gamma}$	$\epsilon/\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon L}$
y_2	$\sqrt{\epsilon\beta}$	$\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon/L}\sqrt{S^2 + L^2}$
y_2'	$-\alpha\sqrt{\epsilon/\beta}$	$(c_1c_3 + c_2c_4)/\sqrt{c_1^2 + c_2^2}$	$S\sqrt{\epsilon/L}/\sqrt{S^2 + L^2}$
y_3	$-\alpha\sqrt{\epsilon/\gamma}$	$(c_1c_3 + c_2c_4)/\sqrt{c_3^2 + c_4^2}$	$S\sqrt{\epsilon/L}$
y_3'	$\sqrt{\epsilon\gamma}$	$\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon/L}$
y_4'	$\sqrt{\epsilon/\beta}$	$\epsilon/\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon L}/\sqrt{S^2 + L^2}$
a		$\sqrt{\epsilon/2}\,(\sqrt{H+1} + \sqrt{H-1})$	
b		$\sqrt{2\epsilon}/(\sqrt{H+1} + \sqrt{H-1}) = \sqrt{\epsilon/2}\,(\sqrt{H+1} - \sqrt{H-1})$	
a/b > 1		$H + \sqrt{H^2 - 1}$	
$\tan \xi$	$[-\alpha(H + \sqrt{H^2-1})]/[\beta(H + \sqrt{H^2-1}) - 1]$	$[c_2 + c_3(H + \sqrt{H^2-1})]/[c_1(H + \sqrt{H^2-1}) - c_4]$	$S/[L(H + \sqrt{H^2-1}) - 1]$
$\sin 2\xi$	$-\alpha/\sqrt{H^2-1}$	$(c_1c_3 + c_2c_4)/\epsilon\sqrt{H^2-1}$	$S/L\sqrt{H^2-1}$
$\cos 2\xi$	$(\beta - \gamma)/2\sqrt{H^2-1}$	$(c_1^2 + c_2^2 - c_3^2 - c_4^2)/2\epsilon\sqrt{H^2-1}$	$(L^2 + S^2 - 1)/2L\sqrt{H^2-1}$
$\tan 2\xi$	$-2\alpha/(\beta - \gamma)$	$2(c_1c_3 + c_2c_4)/(c_1^2 + c_2^2 - c_3^2 - c_4^2)$	$2S/(L^2 + S^2 - 1)$

3.4.4 Two ellipses S_1, S_2 with same area S and centre

Common area S_c is given by

$$\frac{S_c}{S} = \frac{4}{\pi} \text{ arc tan } [D - \sqrt{D^2 - 1}]^{1/2}$$

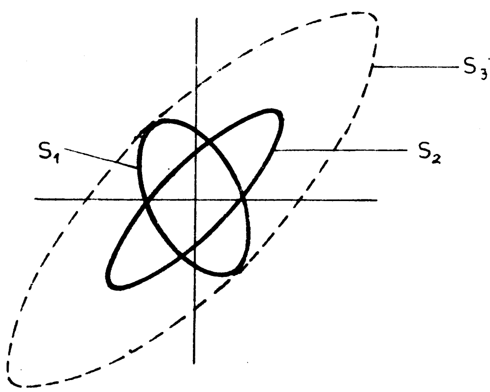
where $$D = \frac{1}{2} (\beta_2\gamma_1 + \gamma_2\beta_1 - 2\alpha_1\alpha_2)$$

$$= 1 + \frac{(L_2 - L_1)^2 + (S_2 - S_1)^2}{2L_1L_2} .$$

For meaning of $\alpha, \beta, \gamma. L, S$ see 3.2 .

3.4.5 Three ellipses

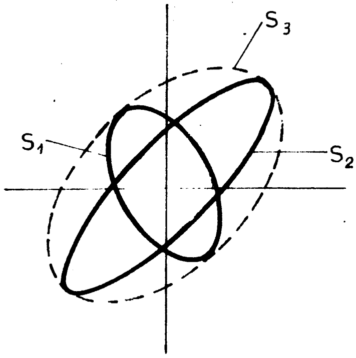
a) Area of ellipse S_3, similar to S_2, such that S_3 circumscribes S_1: (area S_1 = area S_2 = S)



$$\frac{S_3}{S} = D + \sqrt{D^2 - 1}$$

See 3.4.4 for meaning of D

b) Area of ellipse S_3 circumscribing two ellipses S_1, S_2 of same area S:

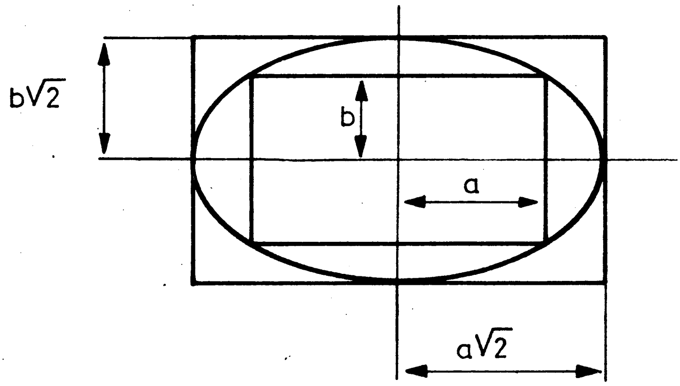


$$\frac{S_3}{S} = (D + \sqrt{D^2 - 1})^{1/2}$$

See 3.4.4 for meaning of D

3.5 Relations between areas of rectangles, ellipses and circles

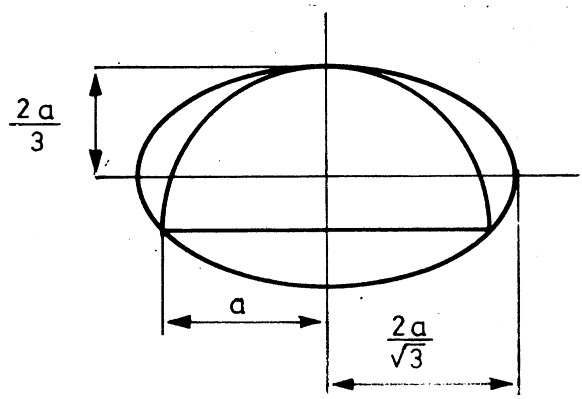
3.5.1 Rectangle and ellipse



$$\frac{S_{\text{min ellipse}}}{S_{\text{inscribed rect.}}} = \frac{\pi}{2}$$

$$\frac{S_{\text{circumscribed rect.}}}{S_{\text{ellipse}}} = \frac{4}{\pi}$$

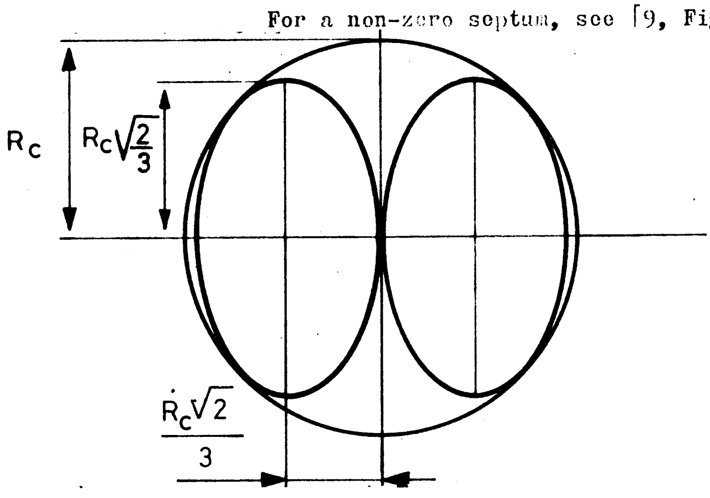
3.5.2 Semi-circle and ellipse



$$\frac{S_{\text{min ellipse}}}{S_{\text{semi-circle}}} = \frac{8\sqrt{3}}{9} \approx 1.54$$

3.5.3 Two maximum ellipses circumscribed by a circle

For a non-zero septum, see [9, Fig. 9]



$$\frac{S_{\text{circle}}}{S_{\text{max ellipse}}} = \frac{3\sqrt{3}}{2} \approx 2.60$$

4. CLOSED ORBIT

4.1 Closed orbit for a momentum deviation $\Delta p/p$

The closed orbit for a given $\Delta p/p$ is given by its phase space coordinates:

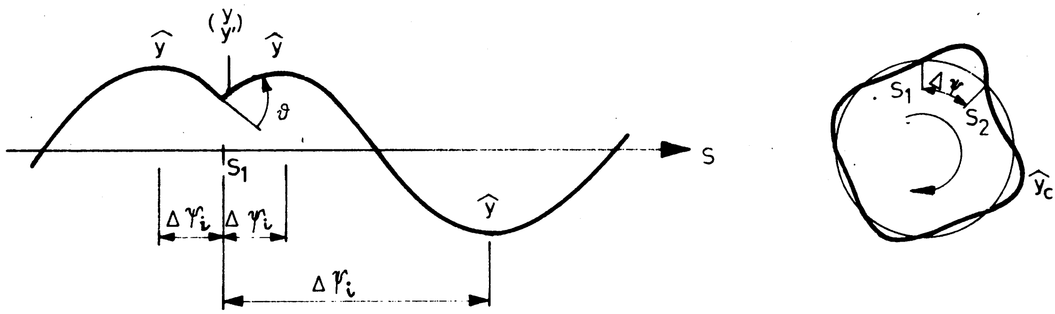
[6,19] $$\begin{pmatrix} e(s) \\ e'(s) \end{pmatrix} = \frac{\Delta p/p}{2(1-\cos\mu)} \begin{pmatrix} m_{13} + m_{12}m_{23} - m_{22}m_{13} \\ m_{23} + m_{21}m_{13} - m_{11}m_{23} \end{pmatrix} ,$$

the m_{ij} being the elements of the 3×3 transfer matrix through one period.

The transformation of the closed orbit vector through the machine is as in Section II.1.1., page 12.

4.2 Orbit deformations *)

4.2.1 One and two dipoles



a) One dipole producing a deflection angle ϑ

$$y\,(s_1) = 0.5\beta(s_1)\vartheta(s_1)\cot(\pi Q)$$

$$y'(s_1) = 0.5\vartheta(s_1)[1-\alpha(s_1)\cot(\pi Q)] .$$

The maximum orbit deviations are

$$\hat{y}(s) = 0.5\vartheta(s_1)[\beta(s_1)\beta(s)]^{1/2}/\sin(\pi Q)$$

and occur approximately at distances (in betatron oscillation phase) of

$$\Delta\psi_i \approx \pm\pi(Q-m) , \quad m = 1, 2, 3, \ldots < Q .$$

*) Complete decoupling between betatron and synchrotron oscillations, i.e. $\nu_{\text{betatron}} >> \nu_{\text{synchrotron}}$, is assumed throughout.

b) Reduction of the deformation by a second dipole

For best reduction between s_2 and s_1 of a deformation described in a), a second dipole positioned at s_2 and spaced by $\Delta\psi$ should provide a deflection

$$\vartheta(s_2) = -\cos\Delta\psi[\beta(s_1)\beta(s_2)]^{1/2}\,\vartheta(s_1).$$

The remaining relative deformations of the corrected orbit are:

between s_2 and s_1 ("outside" dipoles):

$$\hat{y}_c/\hat{y} = \sin\Delta\psi$$

between s_1 and s_2 ("between" dipoles):

$$\hat{y}_c/\hat{y} = [\cos\Delta\psi + 2\sin^2\pi Q - \cos 2(\pi Q - \Delta\psi)]^{1/2} .$$

4.2.2 <u>Distortions due to random errors</u>

The number of magnet units m is assumed to be

[10,6]* $$m > 3Q .$$

a) r.m.s. value of y(s)

$$\langle y\rangle = \frac{\pi}{\sqrt{2}|\sin\pi Q|}\;\frac{R}{Q}\;\frac{|n|}{\rho}\;\frac{\langle\delta\rangle}{\sqrt{m}} ,$$

where δ is the position error of one of the m gradient magnets.

More generally, one has

$$\langle y\rangle = \frac{1}{2\sqrt{2}|\sin\pi Q|}\sqrt{\bar\beta}\sqrt{\sum_i m_i\,\beta_i\,\Psi_i^2}$$

where $\bar\beta = (1/C)\int_0^C \beta(s)ds$ and the equivalent kicks Ψ_i are for the various cases of interest:

*) See page 43 for references

Type of element	Source of kick	r.m.s. value	Ψ_i	Directions
Gradient element	Displacement	$<\Delta y>$	$K_i \ell_i < \Delta y >$	x and z
Bending elements	Tilt	$<\Delta\Theta_e>$	$\vartheta_i < \Delta\Theta_e >$	z only
Bending elements	Field error	$<\Delta B/B>$	$\vartheta_i <\Delta B/B>$	x only
Straight sections	Stray field	$<\Delta B_s>$	$\ell_i <\Delta B_s>/\rho B_{inj}$	x and z
Gradient and bending elements	Displacement and field error	$<\Delta y>$ and $<\Delta B/B>$	$K_i \ell_i \left[<\Delta y^2> + \rho^2/n^2 <(\Delta B/B)^2>\right]^{1/2}$	x and z

b) Value $\hat{y}$ not exceeded with a probability P

$$\hat{y}_P(s) = k(P)\left[1 + \frac{|\sin \pi Q|}{3}\right]^{*} \sqrt{\frac{\beta(s)}{\bar{\beta}}}\ \sqrt{2} < y >$$

with k(P) given by

[12, Fig. 1]

k \ P	50%	75%	90%	98%
rectangular vacuum chamber	1.11	1.41	1.72	2.14
elliptical vacuum chamber	1.28	1.63	1.95	2.40

[11, Fig. 2] *) This bracket takes into account the mean influence of higher harmonics.

5. EFFECTS OF VARIOUS FOCUSING PERTURBATIONS ON THE FREQUENCY AND AMPLITUDE OF BETATRON OSCILLATIONS

5.1 Change in frequency due to tuning of quadrupole lenses or gradient errors

The frequency shift is given by

[1,23]* $$\cos(2\pi Q) - \cos(2\pi Q_0) = 0.5 \sin(2\pi Q_0) \int_0^C \beta(s)k(s)ds$$

where $2\pi Q_0$ is the unperturbed phase shift around the orbit of length C and k(s) the focal constant of the perturbation(s).

In the case of small frequency shifts this becomes

$$\Delta Q = (1/4\pi) \int_0^C \beta(s)k(s)ds \text{ .}$$

For m >> Q, random errors in m elements produce a shift

[1,27] $$\Delta Q = \frac{1}{4\pi} \sqrt{\sum_i m_i (\beta_i K_i \ell_i)^2} < \frac{\Delta K}{K} >$$

where the symbols are the same as in Section 4.2.2, pages 23 and 24.

5.2 Tuning of momentum dependent frequency shifts by means of sextupoles

To compensate a shift caused by $k(s) = -K(s)\Delta p/p$ one needs (in the case of an ideal closed orbit) a sextupole field $\partial^2 B_z/\partial x^2$ such that

$$\int_0^C \beta(s) \left[\frac{\partial^2 B_z}{\partial x^2} (s)e(s) - B\rho k(s) \right] ds = 0$$

where e(s) is defined in Section 4.1 on page 22.

5.3 "Beating" of amplitudes

The beat factor characterising the amplitude function modified by gradient errors is

[1,25] $$G = [\beta(\text{actual})/\beta(\text{ideal})]_{max} \text{ .}$$

---

*) See page 43 for references

In practice one is more interested in $(\Delta\hat{y}/\hat{y})_P = 0.5(G-1)$. Similarly as for the orbit (Section 4.2.2. page 23) one has:

$$(\Delta\hat{y}/\hat{y})_P = \frac{k(P)}{4}\left[\frac{1}{3} + \frac{1}{|\sin 2\pi Q|}\right] \sqrt{\sum_i m_i(\beta_i K_i \ell_i)^2} < \frac{\Delta K}{K} > .$$

5.4 Stopbands due to random gradient errors

The total width of the stopband is

$$\delta Q = 2\Delta Q$$

where the ΔQ is given in Section 5.1 on page 25.

6. SPACE CHARGE LIMIT

Symbols:

N	:	limit of the number of particles in the synchrotron
B_f	:	bunching factor (< 1)
b[m]	:	mean semi-minor beam axis (vertical)
a[m]	:	mean semi-major beam axis (horizontal)
$\Delta(Q^2)$	:	$Q_0^2 - Q^2 \approx 2Q_0\Delta Q$
r	:	classical particle radius (= $e/\{4\pi\varepsilon_0 m c^2_{[eV]}\}$, see p. 45)
2h[m]	:	vertical aperture of the vacuum chamber
2w[m]	:	horizontal aperture of the vacuum chamber
2v[m]	:	height of the magnet gap.

a) Individual particle limit (without neutralization)

[13,331]* $$N_{ind} = -0.5\,\pi\, b(a+b)(R\, r\, F)^{-1}\beta^2\gamma^3\, B_f\, \Delta(Q_i^2)$$

$$\approx -(\pi\,\epsilon_V\,\beta\gamma)(1+\sqrt{\epsilon_H/\epsilon_V})(rF)^{-1}\,\beta\gamma^2\, B_f\,\Delta Q_i$$

where

$$F = 1 + [b(a+b)/h^2]\{\varepsilon_1[1 + B_f(\gamma^2-1)] + \varepsilon_2\, B_f(\gamma^2-1)(h^2/v^2)\}$$

with $\varepsilon_1, \varepsilon_2$, the image force coefficients given on page 27, and $\epsilon_{H,V}$ in rad m .

---

*) See page 43 for references

w/h	1(circle)	5/4	4/3	3/2	2/1	∞(parallel plates)
ε_1	0	0.090	0.107	0.134	0.172	0.206
ξ_1	0.5	0.553	0.559	0.575	0.599	0.617

For parallel straight pole pieces, and to good approximation for wedged-shaped poles, the magnetostatic image coefficients have the values $\varepsilon_2 = 0.411$, $\xi_2 = 0.617$.

b) Coherent particle limit (without neutralization)

[13,342]*

$$N_{coh} = -\pi Q_0 h^2 (R\ r\ F)^{-1} \beta^2 \gamma^3 B_f \Delta Q_c$$

where near an integral resonance

$$F = \xi_1 [1 + B_f(\gamma^2 - 1)] + \xi_2 B_f(\gamma^2 - 1) h^2/v^2$$

and near a half-integral resonance

[13a,150]

$$F = \xi_1 + \varepsilon_1 B_f(\gamma^2 - 1) + \varepsilon_2 B_f(\gamma^2 - 1) h^2/v^2$$

with ε_1, ε_2, ξ_1 and ξ_2 as given above.

For $B_f \gamma^2 >> 1$, one has near an integral resonance

$$N_{coh} \approx -\pi Q_0 h^2 [R r (\xi_1 + \xi_2 h^2/v^2)]^{-1} \gamma \Delta Q_c$$

---

*) See page 43 for references

ADDITIONAL FORMULAE

P A R T III

LONGITUDINAL PHASE SPACE

1. ACCELERATING VOLTAGE

$$V \sin \varphi_s [\mathrm{kV}] = 10^{-3} \; [\, C(\rho \dot{B} + B \dot{\rho}) - \dot{B} S_F] \; [\mathrm{m}^2 \; \mathrm{T/s}]$$

where the index s refers to the synchronous particle and S_F is an equivalent area such that $B\, S_F$ is the total flux enclosed by C.

For $\dot{\rho} = S_F = 0$, one has

$$V \sin \varphi_s [\mathrm{kV}] = 0.020958 \; R \; \dot{p} \; [\mathrm{m(GeV/c)/s}]$$

(see Table I.1.1 for other expressions).

2. ACCELERATING FREQUENCY

$$f_a [\mathrm{Hz}] = h \; f = h \; c \; \beta / C \qquad [\mathrm{s}^{-1}]$$

$$= h \; c \; C^{-1} \; B \left[B^2 [\mathrm{T}^2] + \left(\frac{E_0 [\mathrm{MeV}]}{c \rho [\mathrm{km}^2/\mathrm{s}]} \right)^2 \right]^{-1/2}$$

(see Table I.1.1 for other expressions).

See Section I.4.2.4, page 9, for differential relations.

3. SYNCHROTRON OSCILLATIONS

3.1 Equation of motion

(Above transition φ should be replaced by $\varphi + \pi - 2\varphi_s$; for transition see Section I.4.2.3, page 8.)

$$d/dt [(\beta_s^2 \; E_s / \eta_s \; \Omega_s^2 \,) \; \dot{\varphi}] = (h \; eV/2\pi)(\sin \varphi - \sin \varphi_s)$$

where E_s is in keV, $\eta = \gamma_{tr}^{-2} - \gamma^{-2}$, and Ω is the particle angular velocity on the orbit of length C. To transform to other co-ordinates one has in the absence of perturbations

$$\dot{\varphi} = h \; (\Omega_s - \Omega)$$

$$= (h \; \Omega_s \; \eta_s \; \gamma_{tr}^{-2}/R) \Delta R$$

$$= (h \; \Omega_s \; \eta_s / \beta_s \; \gamma_s)(\Delta p / m_0 c)$$

$$= (h \; \Omega_s \; \eta_s / \beta_s^2 \; E_s) \Delta E \; .$$

See page 1 for meaning of symbols

[14] 3.2 **Bucket size**

3.2.1 Without space charge effects

a) Bucket area, (half) height and coordinates

Bucket area	Bucket (half) height	Coordinates*
$(heV)^{1/2}\,\alpha(\Gamma)\,(16\gamma/h)\,(2\pi E\lvert\eta\rvert)^{-1/2}$	$(heV)^{1/2}\,Y(\Gamma)(\gamma/h)\,(\pi E\lvert\eta\rvert)^{-1/2}$	$(\Delta p/m_0c) - \varphi$
$(heV)^{1/2}\,\alpha(\Gamma)\,(16\beta/h)\,[E/(2\pi\lvert\eta\rvert)]^{1/2}$	$(heV)^{1/2}\,Y(\Gamma)(\beta/h)[E/(\pi\lvert\eta\rvert)]^{1/2}$	$(\Delta E) - \varphi$
$(heV)^{1/2}\,\alpha(\Gamma)[16\alpha_p R/(h\beta)](2\pi E\lvert\eta\rvert)^{-1/2}$	$(heV)^{1/2}\,Y(\Gamma)[R/(\gamma_{tr}^2 h\beta)](\pi E\lvert\eta\rvert)^{-1/2}$	$(\Delta R) - \varphi$
$(heV)^{1/2}\,\alpha(\Gamma)[16\beta/(h^2\Omega)][E/(2\pi\lvert\eta\rvert)]^{1/2}$	$(heV)^{1/2}\,Y(\Gamma)[\beta/(h^2\Omega)][E/(\pi\lvert\eta\rvert)]^{1/2}$	$(\Delta E/h\Omega) - \varphi$

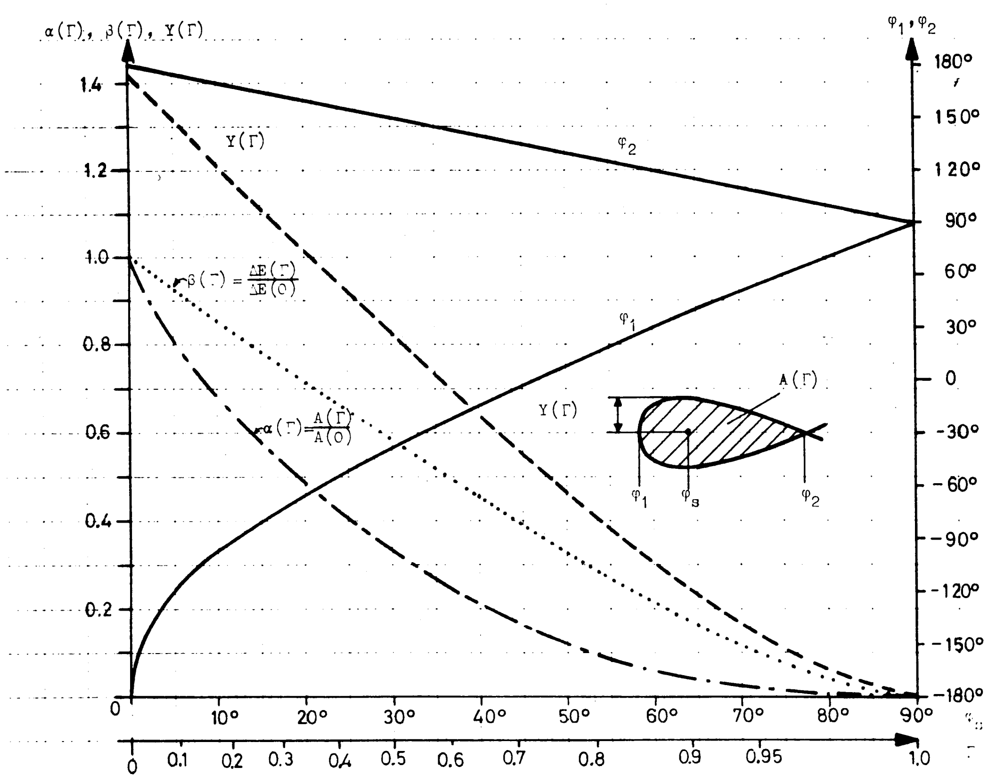
For $\alpha(\Gamma)$ see below and Appendix C; for η see Section 3.1 on preceeding page.

$$Y = Y(\Gamma) = \dot{\varphi}_{max}/(\sqrt{2}\,2\pi\,\nu_0)_{\varphi_s=0} = \mathring{\varphi}_{max}(heV)^{-1/2}(\beta/\Omega)(\pi E\lvert\eta\rvert)^{1/2}$$

*[E and eV in keV
φ in rad, ΔR in cm]

Ideal adiabatic trapping of a linac beam with $\pm\,\Delta E_L$ leads to a minimum bucket (half) height $\Delta E = (\pi/2)\Delta E_L$

b) Bucket width, normalised (half) height and area (see Appendix C for numbers)



3.2.2 Reduction of bucket area due to space charge effects (below transition)

This reduction can be obtained from Fig. III.3.2.2, where

$$\Lambda\, A_{sp.c.} = 4\pi\, h\, g_c\, E_0\, r_p\, N/(ReV\, \gamma^2)$$

with

[15,3] N = number of accelerated particles

$g_c = 1 + 2 \ln$ (vacuum chamber diameter/beam diameter)

r_p = classical proton radius

and E_0 and eV are in the same units (as are r_p and R).

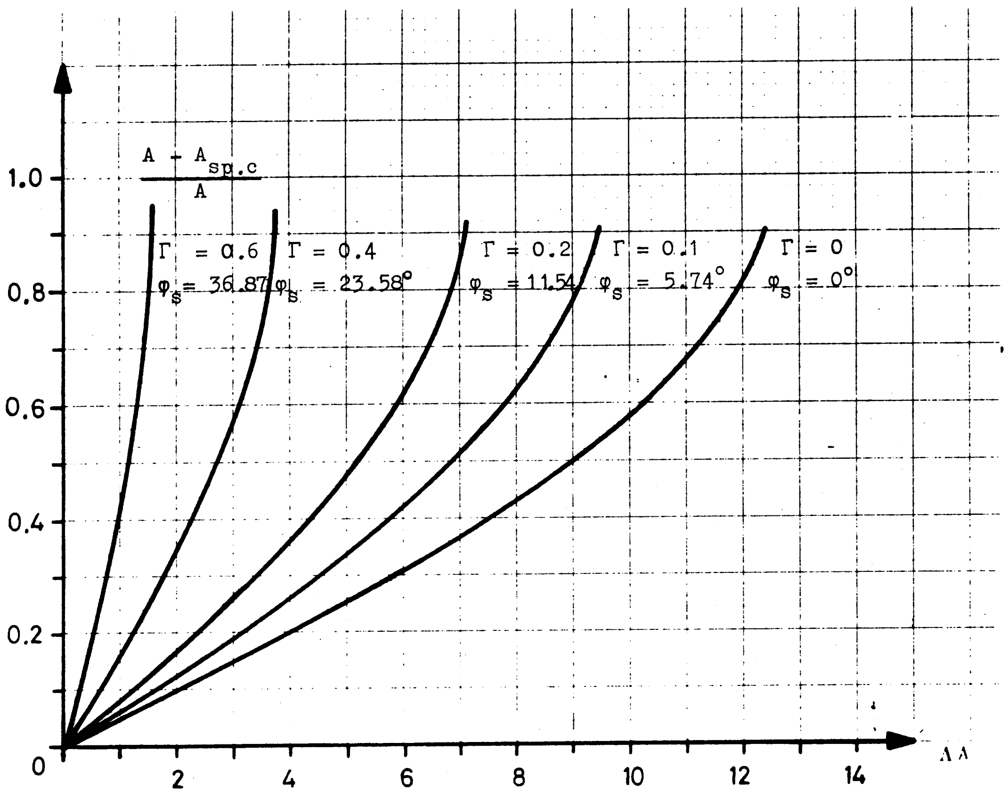


Fig. III.3.2.2 $(A - A_{sp.c})/A = f(\Lambda\, A_{sp.c.})$ (for constant density in phase space)

For $\varphi_s = 0^\circ$ (and a $\cos^2$ distribution in real space) one has

[18, Appendix IV]

$$A_{sp.c.}/A = [1 - g_c\, e\, h\, N/(4\pi\, \varepsilon_0\, \gamma^2\, R\, V)]^{1/2},$$

where V is in volts.

3.3 Frequency of synchrotron oscillations

In the case of small phase oscillation amplitudes around the stable phase φ_s, the equation of motion is

[17,10]* $$(\ddot{\Delta\varphi}) = (\delta^2 - 1)\delta^{-3}\, C'[\sin(\varphi_s + \Delta\varphi) - \sin\varphi_s] = (\delta^2 - 1)\delta^{-3}\, C' \cos\varphi_s\, \Delta\varphi$$

where $\delta^2 = \eta\,\gamma^2 + 1 = \alpha_p \gamma^2$, $C' = 2\pi\, f_\infty^2\, \gamma_{tr}^{-3}\, eV/(hE_0)$ and **eV and E_0 are in the same units.** .(See Section I.4.2.3, page 8, for transition.)

The frequency of these oscillations is [**eV and E_s in same units**]

[17,11] $$\nu_0 = [(1 - \delta^2)\delta^{-3}\, C' \cos\varphi_s]^{1/2}/(2\pi) = [f_\infty^2 |\eta| eV \cos\varphi_s/(2\pi\, E_s h)]^{1/2} .$$

In terms of the bucket area A [in $\Delta p/(m_0 c) - \varphi$ coordinates]:

$$\nu_0 = [A\, E_0\, f_\infty |\eta| /(16 E_s \alpha(\Gamma))](|\cos\varphi_s|)^{1/2}$$

or

$$\nu_0 = [Ach|\eta|/(32\pi\, R_s\, \gamma_s\, \alpha(\Gamma))](|\cos\varphi_s|)^{1/2}.$$

Alternatively

[16,4] $$\nu_0 = \{[\cos\varphi_s/(4\pi^2)][c^2/(2\pi\, R^2\, E_0)](heV)(|1 - \gamma_{tr}^{-2}\, \gamma^2|)/\gamma^3\}^{1/2} .$$

3.4 Adiabatic damping of small-amplitude oscillations

$$\gamma(t) = \{1 + [B(t)/B(t_0)]^2\}^{1/2} .$$

3.4.1 Phase amplitude [**eV and E_s in same units**]

[16,3] $$\Delta\varphi(t) = D[eV(t)\cos\varphi_s]^{-1/4}\, [|1 - \gamma_{tr}^{-2}\, \gamma^2(t)|/\gamma^3(t)]^{1/4}$$

where

$$D = \text{constant} = \Delta\varphi_i [2\pi\, R^2\, E_0/(h\, c^2)]^{1/4}$$

with i denoting the initial values.

$$\Delta\varphi(t)/\Delta\varphi_i = [Vi/V(t)]^{1/4}\, [|1 - \gamma_{tr}^{-2}\, \gamma^2(t)/\gamma^3(t)]^{1/4}\, (|1 - \gamma_{tr}^{-2}\, \gamma_i^2|/\gamma_i^3)^{-1/4}.$$

---

*) See page 43 for references

3.4.2 Energy amplitude

$$\Delta E(t) = G[eV(t)\cos\varphi_s]^{1/4}\ \beta(t)[\gamma^3(t)/|1-\gamma_{tr}^{-2}\ \gamma^2(t)|]^{1/4}$$

where

$$G = \text{constant} = \Delta\varphi_i[E_0 R/(hc)][h\ c^2/(2\pi\ R^2\ E_0)]^{1/4}$$

$$\Delta E(t)/\Delta E_i = [V(t)/V_i]^{1/4}[\beta(t)/\beta_i][|1-\gamma_{tr}^{-2}\ \gamma^2(t)|/\gamma^3(t)]^{-1/4}(|1-\gamma_{tr}^{-2}\gamma_i^2|/\gamma_i^3)^{1/4}.$$

3.4.3 Radial amplitude

$$\Delta R(t) = H[eV(t)\ \cos\ \varphi_s]^{1/4}[\beta(t)\ \gamma(t)]^{-1}[\gamma^3(t)/|1-\gamma_{tr}^{-2}\ \gamma^2(t)|]^{1/4}$$

where

$$H = \Delta\varphi_i[\gamma_{tr}^{-2}\ R^2/(hc)][h\ c^2/(2\pi R^2 E_0)]^{1/4}$$

$$\Delta R/\Delta R_i = [\beta_i\ \gamma_i/\beta(t)\gamma(t)]\ [\Delta\varphi_i/\Delta\varphi(t)]\ .$$

4. DEBUNCHING

4.1 Debunching time

In the absence of RF fields the beam "debunches"*) itself in the synchrotron (i.e. front end of one bunch reaches the tail of the next bunch ahead) after a time**)

$$t_{db} \approx (\pi - \Delta\varphi)[2\pi\ f_a|\gamma^{-2} - \gamma_{tr}^{-2}|\Delta p/p]^{-1}$$

where $2\Delta\varphi$ and $2\Delta p$ are the total phase and momentum spreads.

Special cases:

a) Low energy

$\gamma^2 \ll \gamma_{tr}^2$ $\qquad$ $\Delta\varphi \ll \pi$ (strong damping in linac)

$$t_{db} = \gamma^2(2f_a\ \Delta p/p)^{-1}$$
$$= \gamma(\gamma+1)(2f_a\Delta T/T)^{-1}.$$

---

*) This azimuthal spreading does not involve any reduction of $\Delta p/p$.

**) Strictly valid for "rectangular" bunches. For "oval" bunches more time may be required in practice.

b) Well above transition

$\gamma^2 >> \gamma_{tr}^2$ $\Delta\varphi << \pi$ (damping in synchrotron)

$$t_{db} = \gamma_{tr}^2 (2f_a \, \Delta p/p)^{-1}$$

Complete overlapping (front end reaches centre of next bunch ahead) requires $2t_{db}$.

4.2 Travelling distance required for debunching

$$s_{db} = c \, \beta \, t_{db} \; .$$

For $\gamma^2 << \gamma_{tr}^2$ and $\Delta\varphi << \pi$, this becomes with 2D = distance between centre of bunches

$$s_{db} = D/(\Delta\beta/\beta).$$

See Table I.1.1 for other expressions.

See page 1 for meaning of symbols

ADDITIONAL FORMULAE

P A R T IV

ROUGH EVALUATION OF MAJOR ACCELERATOR SYSTEMS

1. MAGNETS (non-saturated)

1.1 Bending magnet

a) Excitation current

$$N_B\, I[\text{ampere-turns}] = B\, h_B/\mu_0 [\text{mT}/(\text{H m}^{-1})]$$

where h_B is the (mean) gap height

$N_B\, I/(B\, h_B) \approx 800$ ampere-turns/(0.1T × 0.01 m gap height).

b) Inductance

$$L_B[\text{H}] \approx N_B^2\, \mu_0\, w\, \ell_B/h_B$$

$w = w_a + \frac{2}{3} w_c$ (for window frame magnet)

$w = w_p + \frac{1}{2} h_B$ (for magnet with poles)

where

w_a = aperture between coils, w_c = coil width

w_p = pole width

ℓ_B = the total magnetic length

c) Stored energy

$$W_B[\text{Ws}] \approx B^2\, h_B\, w\, \ell_B/(2\mu_0) \qquad [\text{T}^2\, \text{m}^3/(\text{H m}^{-1})]$$

where w is as in b).

1.2 Quadrupole lens

a) Excitation current per pole

$$N_Q I[\text{ampere-turns}] = g\, r_Q^2/(2\mu_0) \qquad [\text{Tm/H m}^{-1}]$$

$N_Q I/(g\, r_Q^2) \approx 400$ ampere-turns/[10 T/m)(0.01 m bore radius)^2]

b) Inductance

$$L_Q[\text{H}] \approx 8\mu_0\, N_Q^2\, y_{max}(y_{max} + 2/3\, w_c)\, \ell_Q/r_Q^2$$

where y_{max} is the distance from the lens centre to the coil face and ℓ_Q is the total magnetic length.

c) Stored energy

$$W_Q[\mathrm{Ws}] \approx g^2\, r_Q^2\, y_{max}(y_{max} + 2/3\, w_c)\mathcal{K}_Q/\mu_0 \; .$$

1.3 Bending magnet and quadrupole lens excited in series

$$B \approx N_B\, r_Q^2\, \beta\, \gamma\, K/\,(\,0.639\, h_B\, N_Q)\;,\;(\text{for protons})$$

or

$$K \approx (0.6/p)(N_Q/N_B)(h_B/r_Q^2\,)\,B$$

where r_Q and h_B in m.

See Section II.1.4.d) for other expressions.

1.4 Cooling water requirements

To cool a conductor heated by a power loss N[kW], one needs a water flow of

$$G_w[\ell/s] \approx N/(4.2\;\Delta t) = 10^{-3}\, v\, A_F = 10^{-3}\, v\, F_s\, d_h^2\;,$$

where

$\Delta t_w[°C]$	is the allowed temperature increase
$v[m/s]$	is the velocity of the cooling water
$A_F[mm^2]$	is the flow area
$F_s = A_F/d_h^2$	is the shape factor (= $\pi/4$ for round holes)
$d_h[mm] = 4A_F/\text{perimeter}$	is the hydraulic diameter.

For turbulent flow the required pressure drop may be obtained from

$$\Delta P_w[kg/cm^2] = 0.18\; L_c\; v^{1.75}/(F_s^{1.75}\; d_h^{1.25})$$

where $L_c[m]$ = length of conductor, and it is noted that $0.18(\pi/4)^{-1.75} = 0.28$.

If d_h is in metres, this becomes for round holes

[4,67]* $$\Delta P_w[kg/cm^2] = 5\;10^{-5}\;L\;v^{1.75}/d_h^{1.25}.$$

Alternatively, one has (with somewhat more pessimistic assumptions about the pressure loss)

$$G_w[\ell/s] = 1.25\;10^{-3}(1 + 0.009\;t_w)\;F_s\;d_h^{2.71}(\Delta P_w/L_c)^{0.57}$$

where $t_w[°C]$ is the water temperature.

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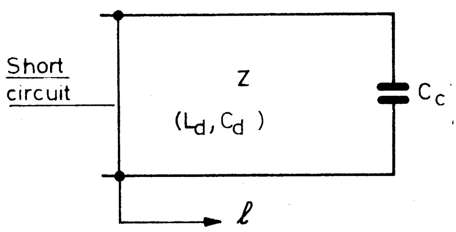
*) See page 43 for references

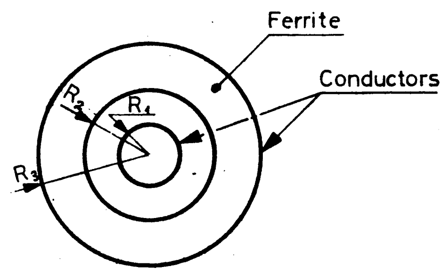
2. FERRITE-LOADED RF ACCELERATING CAVITY

2.1 Basic quantities

Equivalent transmission line



Cavity cross section



a) Effective permittivity and permeability

$$\epsilon_{eff} = \epsilon/[x + (1-x)\epsilon]$$

$$\mu_{eff} = 1 + x(\mu - 1) \approx \mu x$$

where $x = \ln(R_3/R_2)/\ln(R_3/R_1)$.

b) Inductance of ferrite cylindre of length ℓ_c

$$L_d = [\mu_{eff}\ \mu_0\ \ell_c\ \ln(R_3/R_1)]/2\pi$$

c) Capacitance

$$C_d = 2\pi\ \epsilon_{eff}\ \epsilon_0\ \ell_c/\ln(R_3/R_1)$$

d) Characteristic impedance

$$Z = (L_d/C_d)^{1/2} = 60(\mu_{eff}/\epsilon_{eff})^{1/2}\ \ln(R_3/R_1)$$

e) Wavelength

$$\lambda = v/f_a = c/[f_a(\mu_{eff}\ \epsilon_{eff})^{1/2}] = \ell_c/[f_a(L_d\ C_d)^{1/2}]$$

2.2 Length of cavity

For resonance (assuming negligeable losses)

$$1/(w_0\ C_c) = Z\ \tan(2\pi\ \ell_c/\lambda) \qquad \text{i.e.}$$

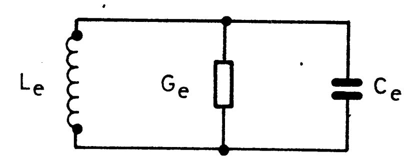
$$\tan(2\pi\ \ell_c/\lambda) = (\omega_0\ C_c\ Z)^{-1},$$

which becomes for small arguments with $\omega_0 \approx (L_d\ C_c)^{-1/2}$

$$\ell_c \approx \lambda (C_d/C_c)^{1/2}/(2\pi).$$

2.3 Equivalent resonant circuit

If one knows the complex cavity admittance $Y = G + jB = (G^2 + B^2)^{1/2}\ e^{j\Phi}$ as a function of ω, one can find C_e from



$$C_e = (G_e \tan \Phi)/(2\Delta\omega)$$

where the phase angle Φ pertains to the frequency $\omega = \omega_0 \pm \Delta\omega$, and hence

$$L_e = (\omega_0^2\ C_e)^{-1} .$$

The relation between the equivalent and the "real" quantities is as follows (on the basis of $B = B_e = 0$; $dB/d\omega = dB_e/d\omega$ for $\omega = \omega_0$)

$$C_e = 0.5\ C_c + 0.5\ C_d[\sin^2(2\pi \ell_c/\lambda)]^{-1}$$

$$L_e = 2L_d\left((2\pi \ell_c/\lambda)^2\ \{(C_c/C_d) + [\sin^2(2\pi\ \ell_c/\lambda)]^{-1}\}\right)^{-1} .$$

2.4 Longitudinal variation of power loss

$$P(\ell) = P_{max} \cos^2(2\pi\ \ell/\lambda)$$

$$\bar{P} = P_{max}\{\ 0.5 + [0.25\ \sin(4\pi\ell_c/\lambda)]/(2\pi\ell_c/\lambda)\}$$

where P_{max} is the maximum loss (occuring at the short circuit) and the power loss per unit volume is assumed to be constant.

3. **VACUUM PRESSURE REQUIRED**

The natural growth of the beam emittance in either transverse plane is given by

[19,8]* $$\Delta(\varepsilon\beta\gamma)\,[\text{rad m}] = 0.32\ \lambda\!\!\!^{-} P\,|\ln(1-\eta)|\int_{t_0}^{t_1}\beta^{-2}\gamma^{-1}\,dt\ [\text{m Torr s}]$$

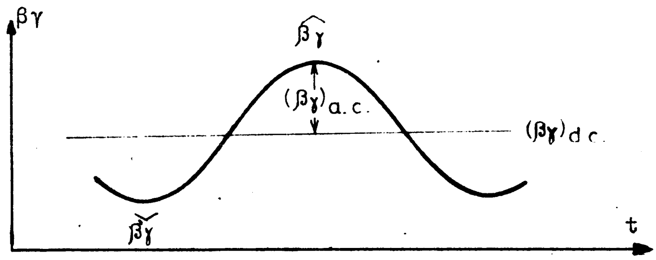
where $\lambda\!\!\!^{-} = R/Q$, P = nitrogen equivalent pressure and η is the fraction of the particles contained in the emittance

η	0.5	0.8	0.9	0.95	0.97
$\|\ln(1-\eta)\|$	0.694	1.61	2.3	3.0	3.5

a) For $(\dot{\beta\gamma})$ = const, one has

$$\int_{t_0}^{t_1}\beta^{-2}\gamma^{-1}dt = (\dot{\beta\gamma})^{-1}\Big(1/\beta(t_0) - 1/\beta(t_1) + \ln\{\gamma(t_1)[1+\beta(t_1)]/\gamma(t_0)[1+\beta(t_0)]\}\ .$$

b) In the case of sinusoidal excitation of the magnet field



$$\beta(t)\gamma(t) = (\beta\gamma)_{d.c.} - (\beta\gamma)_{a.c.}\cos\omega_m t$$

$$= 0.5\ \{(\widehat{\beta\gamma}) + (\check{\beta\gamma}) - [(\widehat{\beta\gamma}) - (\check{\beta\gamma})]\cos\omega_m t\ \}$$

one has to good approximation (for $\hat{\beta} \approx 1$ and $\check{\beta} \ll 1$)

$$\int_{t_0}^{t_1}\beta^{-2}\,\gamma^{-1}\,dt \approx \pi(1/\check{\beta} + 1/\hat{\beta})/\{2\omega_m\,[(\widehat{\beta\gamma})(\check{\beta\gamma})]^{1/2}\ \}\ .$$

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APPENDIX A

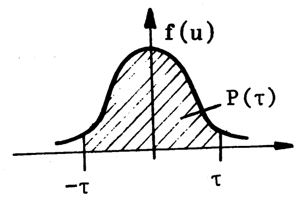
TABLE OF CONSTANTS

Symbols	Meaning	Value	Units
c	velocity of light	$2.997925\ 10^{8}$	ms^{-1}
e	electronic charge	$1.6021\ 10^{-19}$	C
e/m_e	charge to mass ratio for an electron	$1.75880\ 10^{8}$	Cg^{-1}
e/m_p	charge to mass ratio for a proton	$9.57896\ 10^{4}$	Cg^{-1}
h	Planck's constant	$6.6256\ 10^{-34}$	Js
$\hbar$	Planck's constant/2π	$1.0545\ 10^{-34}$	Js
h/e	quantum charge ratio	$4.1355\ 10^{-15}$	$Js\ C^{-1}$
k	Boltzmann's constant ((	$1.3805\ 10^{-23}$ $8.6171\ 10^{-14}$	J/(°K) GeV/(°K)
m_d	rest mass of deuteron ((	$3.3433\ 10^{-27}$ 1.87558	kg GeV/c^2
m_e	rest mass of electron ((	$9.1091\ 10^{-31}$ $5.11006\ 10^{-4}$	kg GeV/c^2
m_p	rest mass of proton ((	$1.6725\ 10^{-27}$ 0.93826	kg GeV/c^2
m_p/m_e	ratio of proton and electron masses	$1.83610\ 10^{3}$	
r_e	classical electron radius	$2.8178\ 10^{-15}$	m
r_d	classical deuteron radius	$0.76774\ 10^{-18}$	m
r_p	classical proton radius	$1.5347\ 10^{-18}$	m
μ_o	permeability of free space ((	$= 4\pi \times 10^{-7}$ $= 1.25664\times10^{-6}$	Hm^{-1}
ϵ_o	permittivity of free space ((	$= (\mu_o c^2)^{-1}$ $= 8.8542\times 10^{-12}$	Fm^{-1}

These values are taken from the 48th edition of the CRC Handbook of Chemistry and Physics. (Still within the limits of errors of the values given in the 50th edition.)

APPENDIX B

GAUSSIAN DISTRIBUTIONS



Definitions: $f(u) = (2\pi)^{-1/2}\exp(-u^2/2)$; $\overline{u^2} = 1$

$$P(\tau) = \int_{-\tau}^{\tau} f(u)du'$$

One-dimensional density distribution:

$$\rho(x) = (2\pi)^{-1/2}\,\sigma^{-1}\,\exp[-x^2/(2\sigma^2)] = \sigma^{-1}f(x/\sigma)$$

Normalization: $\int_{-\infty}^{\infty} \rho dx = 1$; $\int_{-x}^{x} \rho(x)dx = P(x/\sigma)$

Variance: $\overline{x^2} = \int_{-\infty}^{\infty} x^2 \rho dx = \sigma^2$

Standard deviation: $\overline{x^2}^{1/2} = \langle x \rangle = \sigma$

Two-dimensional density distribution (x,z uncorrelated):

$$\rho(x,z) = (2\pi\sigma_1\sigma_2)^{-1}\exp[-x^2/(2\sigma_1^2) - z^2/(2\sigma_2^2)] = (\sigma_1\sigma_2)^{-1}f(x/\sigma_1)f(z/\sigma_2)$$

Normalization: over the whole plane $\iint \rho dxdz = 1$

Variances: $\overline{x^2} = \sigma_1^2$ for any z or for all z; $\overline{z^2} = \sigma_2^2$ similarly

The ellipses $x^2/\sigma_1^2 + z^2/\sigma_2^2 = v^2$ are lines of constant ρ with

$$\rho = (2\pi\sigma_1\sigma_2)^{-1}\exp(-v^2/2); \quad \overline{v^2} = 2.$$

Over this ellipse $\iint \rho dxdz = 1 - \exp(-v^2/2)$

u,τ,v	f(u)	P(τ)	$1-\exp(-v^2/2)$	$v/\sqrt{2}$	$v^2/2$
0	0.399	0	0	0	0
0.5	0.352	0.383	0.118	0.354	0.125
0.674	0.318	0.500	0.203	0.477	0.227
0.707	0.311	0.520	0.221	0.500	0.250
1	0.242	0.683	0.393	0.707	0.500
1.414	0.147	0.843	0.632	1.000	1.000
1.5	0.130	0.866	0.675	1.061	1.125
2	0.0540	0.954	0.865	1.414	2.000
2.5	0.0175	0.9876	0.9561	1.768	3.125
2.797	0.0080	0.9948	0.980	1.978	3.912
3	0.0044	0.9973	0.9889	2.121	4.500
∞	0	1.000	1.000	∞	∞

PROTON KINEMATICAL TABLE *)

$E_o = 938.26$ MeV

T [MeV]	cp [MeV]	Bρ [Tm]	β	β^2	γ	γ^2	βγ	$\beta^2\gamma$	$\beta\gamma^2$	$\beta^2\gamma^3$
20.00	194.7573	6.4964E-01	2.0324E-01	4.1307E-02	1.0213E+00	1.0431E+00	2.0757E-01	4.2187E-02	2.1200E-01	4.4005E-02
50.00	310.3643	1.0353E+00	3.1405E-01	9.8628E-02	1.0533E+00	1.1094E+00	3.3079E-01	1.0388E-01	3.4841E-01	1.1525E-01
200.00	644.4408	2.1496E+00	5.6616E-01	3.2054E-01	1.2132E+00	1.4718E+00	6.8685E-01	3.8887E-01	8.3326E-01	5.7232E-01
400.00	954.2578	3.1831E+00	7.1306E-01	5.0845E-01	1.4263E+00	2.0344E+00	1.0171E+00	7.2522E-01	1.4506E+00	1.4754E+00
500.00	1090.0734	3.6361E+00	7.5791E-01	5.7443E-01	1.5329E+00	2.3498E+00	1.1618E+00	8.8054E-01	1.7809E+00	2.0691E+00
600.00	1218.9799	4.0661E+00	7.9244E-01	6.2796E-01	1.6395E+00	2.6879E+00	1.2992E+00	1.0295E+00	2.1300E+00	2.7673E+00
700.00	1342.9684	4.4797E+00	8.1975E-01	6.7199E-01	1.7461E+00	3.0487E+00	1.4313E+00	1.1733E+00	2.4992E+00	3.5772E+00
750.00	1403.5277	4.6817E+00	8.3135E-01	6.9114E-01	1.7994E+00	3.2377E+00	1.4959E+00	1.2436E+00	2.6916E+00	4.0264E+00
800.00	1463.2894	4.8810E+00	8.4181E-01	7.0865E-01	1.8526E+00	3.4323E+00	1.5596E+00	1.3129E+00	2.8893E+00	4.5061E+00
850.00	1522.3475	5.0780E+00	8.5130E-01	7.2471E-01	1.9059E+00	3.6326E+00	1.6225E+00	1.3813E+00	3.0924E+00	5.0175E+00
900.00	1580.7808	5.2729E+00	8.5993E-01	7.3949E-01	1.9592E+00	3.8386E+00	1.6848E+00	1.4488E+00	3.3009E+00	5.5614E+00
950.00	1638.6562	5.4660E+00	8.6781E-01	7.5310E-01	2.0125E+00	4.0502E+00	1.7465E+00	1.5156E+00	3.5148E+00	6.1386E+00
[GeV]	[GeV]									
1.00	1.6960	5.6574E+00	8.7503E-01	7.6567E-01	2.0658E+00	4.2675E+00	1.8076E+00	1.5817E+00	3.7342E+00	6.7501E+00
3.00	3.8249	1.2758E+01	9.7121E-01	9.4324E-01	4.1974E+00	1.7618E+01	4.0765E+00	3.9592E+00	1.7111E+01	6.9754E+01
6.00	6.8745	2.2931E+01	9.9081E-01	9.8171E-01	7.3948E+00	5.4683E+01	7.3269E+00	7.2596E+00	5.4181E+01	3.9698E+02
8.00	8.8889	2.9650E+01	9.9448E-01	9.8898E-01	9.5264E+00	9.0753E+01	9.4738E+00	9.4215E+00	9.0251E+01	8.5502E+02
10.00	10.8979	3.6352E+01	9.9631E-01	9.9264E-01	1.1658E+01	1.3591E+02	1.1615E+01	1.1572E+01	1.3541E+02	1.5728E+03
12.00	12.9042	4.3044E+01	9.9737E-01	9.9474E-01	1.3790E+01	1.9015E+02	1.3753E+01	1.3717E+01	1.8965E+02	2.6084E+03
18.00	18.9150	6.3094E+01	9.9877E-01	9.9755E-01	2.0184E+01	4.0741E+02	2.0160E+01	2.0135E+01	4.0691E+02	8.2032E+03
21.00	21.9182	7.3111E+01	9.9909E-01	9.9817E-01	2.3382E+01	5.4671E+02	2.3360E+01	2.3339E+01	5.4621E+02	1.2760E+04
24.00	24.9206	8.3126E+01	9.9929E-01	9.9858E-01	2.6579E+01	7.0646E+02	2.6560E+01	2.6542E+01	7.0596E+02	1.8751E+04
27.00	27.9225	9.3139E+01	9.9944E-01	9.9887E-01	2.9777E+01	8.8665E+02	2.9760E+01	2.9743E+01	8.8615E+02	2.6372E+04
33.00	33.9253	1.1316E+02	9.9962E-01	9.9924E-01	3.6171E+01	1.3084E+03	3.6158E+01	3.6144E+01	1.3079E+03	4.7290E+04
76.00	76.9325	2.5662E+02	9.9993E-01	9.9985E-01	8.2001E+01	6.7242E+03	8.1995E+01	8.1989E+01	6.7237E+03	5.5131E+05
200.00	200.9361	6.7025E+02	9.9999E-01	9.9998E-01	2.1416E+02	4.5865E+04	2.1416E+02	2.1416E+02	4.5864E+04	9.8222E+06
300.00	300.9368	1.0038E+03	1.0000E+00	9.9999E-01	3.2074E+02	1.0287E+05	3.2074E+02	3.2074E+02	1.0287E+05	3.2996E+07
400.00	400.9372	1.3374E+03	1.0000E+00	9.9999E-01	4.2732E+02	1.8260E+05	4.2732E+02	4.2732E+02	1.8260E+05	7.8030E+07

*) See page 48 for values useful for PSB and CPS operation.

RF "BUCKET" WIDTH, NORMALISED (HALF) HEIGHT AND AREA

(From Ref. 14; note that in this reference $Y(0) = \dot{\varphi}_{max}(0)[\sqrt{2}\,2\pi\ \nu_o(0)] = \sqrt{2}$ - rather than 2 - for computational convenience. See page 31 for other definitions and a graphical representation.)

All values of φ_s, φ_1, φ_2, $\Delta\varphi$ in degrees

Stable phase		"Bucket" width			Half height		Area
φ_s	Γ	φ_1	φ_2	$\Delta\varphi$	$Y(\Gamma)$	$\beta = \frac{\Delta E(\Gamma)}{\Delta E(0)}$	$\alpha = \frac{A(\Gamma)}{A(0)}$
0	0.000000	-180.0	180	360	1.414214	1.000000	1.000000
1	.017452	-154.0	179	333	1.394803	.986275	.954105
2	.034899	-143.5	178	321	1.375347	.972517	.917558
3	.052336	-135.5	177	312	1.355847	.958729	.884511
4	.069756	-128.8	176	305	1.336309	.944913	.853741
5	.087156	-122.9	175	298	1.316736	.931073	.824676
6	.104528	-117.6	174	292	1.297132	.917211	.796983
7	.121869	-112.6	173	286	1.277500	.903329	.770443
8	.139173	-108.1	172	280	1.257846	.889431	.744906
9	.156434	-103.7	171	275	1.238171	.875519	.720257
10	.173648	-99.6	170	270	1.218482	.861597	.696413
11	.190809	-95.7	169	265	1.198781	.847666	.673303
12	.207912	-92.0	168	260	1.179072	.833730	.650875
13	.224951	-88.4	167	255	1.159360	.819791	.629082
14	.241922	-84.9	166	251	1.139648	.805853	.607888
15	.258819	-81.5	165	247	1.119940	.791917	.587261
16	.275637	-78.2	164	242	1.100240	.777987	.567174
17	.292372	-75.0	163	238	1.080552	.764066	.547603
18	.309017	-71.9	162	234	1.060881	.750156	.528529
19	.325568	-68.9	161	230	1.041230	.736261	.509933
20	.342020	-65.9	160	226	1.021603	.722382	.491799
21	.358368	-63.0	159	222	1.002004	.708524	.474114
22	.374607	-60.1	158	218	.982438	.694688	.456865
23	.390731	-57.3	157	214	.962907	.680878	.440040
24	.406737	-54.5	156	210	.943418	.667097	.423630
25	.422618	-51.8	155	207	.923972	.653347	.407624
26	.438371	-49.1	154	203	.904576	.639632	.392015
27	.453990	-46.4	153	199	.885232	.625954	.376794
28	.469472	-43.8	152	196	.865945	.612316	.361955
29	.484810	-41.2	151	192	.846719	.598721	.347489
30	.500000	-38.7	150	189	.827559	.585172	.333392
31	.515038	-36.2	149	185	.808467	.571673	.319657
32	.529919	-33.7	148	182	.789450	.558225	.306279
33	.544639	-31.2	147	178	.770510	.544833	.293252
34	.559193	-28.8	146	175	.751653	.531499	.280571
35	.573576	-26.3	145	171	.732882	.518226	.268231
36	.587785	-23.9	144	168	.714202	.505017	.256229
37	.601815	-21.6	143	165	.695618	.491876	.244560
38	.615661	-19.2	142	161	.677132	.478805	.233218
39	.629320	-16.9	141	158	.658751	.465807	.222202
40	.642788	-14.6	140	155	.640479	.452887	.211505
41	.656059	-12.3	139	151	.622319	.440046	.201125
42	.669131	-10.0	138	148	.604277	.427288	.191058
43	.681998	-7.7	137	145	.586357	.414617	.181301
44	.694658	-5.4	136	141	.568564	.402035	.171848
45	.707107	-3.2	135	138	.550902	.389546	.162698
46	.719340	-1.0	134	135	.533376	.377154	.153845
47	.731354	1.2	133	132	.515991	.364861	.145288
48	.743145	3.5	132	129	.498752	.352671	.137022
49	.754710	5.6	131	125	.481664	.340588	.129044
50	.766044	7.8	130	122	.464732	.328615	.121349
51	.777146	10.0	129	119	.447960	.316755	.113936
52	.788011	12.2	128	116	.431354	.305013	.106800
53	.798636	14.3	127	113	.414919	.293392	.099938
54	.809017	16.4	126	110	.398660	.281895	.093346
55	.819152	18.6	125	106	.382583	.270527	.087021
56	.829038	20.7	124	103	.366694	.259292	.080959
57	.838671	22.8	123	100	.350997	.248192	.075156
58	.848048	24.9	122	97	.335499	.237234	.069609
59	.857167	27.0	121	94	.320206	.226420	.064314
60	.866025	29.1	120	91	.305123	.215755	.059267
61	.874620	31.2	119	88	.290258	.205243	.054464
62	.882948	33.3	118	85	.275616	.194890	.049902
63	.891007	35.4	117	82	.261203	.184699	.045576
64	.898794	37.4	116	79	.247028	.174675	.041482
65	.906308	39.5	115	75	.233096	.164824	.037616
66	.913545	41.6	114	72	.219416	.155150	.033974
67	.920505	43.6	113	69	.205994	.145660	.030551
68	.927184	45.7	112	66	.192840	.136358	.027344
69	.933580	47.7	111	63	.179960	.127251	.024346
70	.939693	49.7	110	60	.167365	.118345	.021555
71	.945519	51.8	109	57	.155063	.109646	.018964
72	.951057	53.8	108	54	.143065	.101162	.016569
73	.956305	55.8	107	51	.131380	.092900	.014365
74	.961262	57.9	106	48	.120020	.084867	.012347
75	.965926	59.9	105	45	.108998	.077073	.010509
76	.970296	61.9	104	42	.098325	.069526	.008845
77	.974370	63.9	103	39	.088017	.062237	.007350
78	.978148	65.9	102	36	.078089	.055217	.006018
79	.981627	68.0	101	33	.068558	.048478	.004842
80	.984808	70.0	100	30	.059444	.042033	.003816
81	.987688	72.0	99	27	.050769	.035899	.002933
82	.990268	74.0	98	24	.042558	.030093	.002185
83	.992546	76.0	97	21	.034841	.024636	.001565
84	.994522	78.0	96	18	.027654	.019554	.001064
85	.996195	80.0	95	15	.021041	.014878	.000675
86	.997564	82.0	94	12	.015058	.010647	.000386
87	.998630	84.0	93	9	.009781	.006916	.000188
88	.999391	86.0	92	6	.005325	.003765	.000068
89	.999848	88.0	91	3	.001883	.001331	.000012
90	1.000000	90.0	90	0	0.000000	0.000000	0.000000

Stable phase		"Bucket" width			Half height		Area
Γ	φ_s	φ_1	φ_2	$\Delta\varphi$	$Y(\Gamma)$	$\beta = \frac{\Delta E(\Gamma)}{\Delta E(0)}$	$\alpha = \frac{A(\Gamma)}{A(0)}$
0.000	0.000	-180.0	180.0	360.0	1.414214	1.000000	1.000000
.010	.573	-160.2	179.4	339.6	1.403098	.992140	.971542
.020	1.146	-152.2	178.9	331.0	1.391966	.984268	.948459
.030	1.719	-146.1	178.3	324.4	1.380816	.976384	.927384
.040	2.292	-141.0	177.7	318.7	1.369648	.968488	.907606
.050	2.866	-136.5	177.1	313.6	1.358463	.960578	.888790
.060	3.440	-132.4	176.6	309.0	1.347259	.952656	.870739
.070	4.014	-128.7	176.0	304.7	1.336035	.944720	.853324
.080	4.589	-125.2	175.4	300.7	1.324793	.936770	.836453
.090	5.164	-122.0	174.8	296.8	1.313530	.928806	.820058
.100	5.739	-118.9	174.3	293.2	1.302248	.920828	.804087
.110	6.315	-116.0	173.7	289.7	1.290944	.912835	.788498
.120	6.892	-113.2	173.1	286.3	1.279620	.904828	.773256
.130	7.470	-110.5	172.5	283.0	1.268273	.896805	.758334
.140	8.048	-107.8	172.0	279.8	1.256905	.888766	.743707
.150	8.627	-105.3	171.4	276.7	1.245513	.880711	.729355
.160	9.207	-102.9	170.8	273.7	1.234099	.872640	.715261
.170	9.788	-100.5	170.2	270.7	1.222661	.864552	.701408
.180	10.370	-98.2	169.6	267.8	1.211198	.856447	.687785
.190	10.953	-95.9	169.0	265.0	1.199711	.848324	.674379
.200	11.537	-93.7	168.5	262.2	1.188199	.840183	.661178
.210	12.122	-91.5	167.9	259.4	1.176660	.832025	.648175
.220	12.709	-89.4	167.3	256.7	1.165096	.823847	.635360
.230	13.297	-87.3	166.7	254.0	1.153504	.815650	.622725
.240	13.887	-85.3	166.1	251.4	1.141884	.807434	.610264
.250	14.478	-83.3	165.5	248.8	1.130236	.799198	.597969
.260	15.070	-81.3	164.9	246.2	1.118559	.790941	.585836
.270	15.664	-79.3	164.3	243.7	1.106853	.782663	.573859
.280	16.260	-77.4	163.7	241.1	1.095116	.774364	.562033
.290	16.858	-75.5	163.1	238.6	1.083348	.766043	.550352
.300	17.458	-73.6	162.5	236.1	1.071549	.757699	.538814
.310	18.059	-71.7	161.9	233.7	1.059717	.749333	.527414
.320	18.663	-69.9	161.3	231.2	1.047851	.740943	.516149
.330	19.269	-68.1	160.7	228.8	1.035952	.732529	.505014
.340	19.877	-66.2	160.1	226.4	1.024018	.724090	.494007
.350	20.487	-64.4	159.5	224.0	1.012048	.715626	.483126
.360	21.100	-62.7	158.9	221.6	1.000042	.707137	.472366
.370	21.716	-60.9	158.3	219.2	.987998	.698620	.461727
.380	22.334	-59.1	157.7	216.8	.975916	.690077	.451204
.390	22.954	-57.4	157.0	214.4	.963795	.681506	.440797
.400	23.578	-55.7	156.4	212.1	.951634	.672907	.430502
.410	24.205	-53.9	155.8	209.7	.939431	.664278	.420319
.420	24.835	-52.2	155.2	207.4	.927186	.655619	.410244
.430	25.468	-50.5	154.5	205.0	.914897	.646930	.400277
.440	26.104	-48.8	153.9	202.7	.902564	.638209	.390416
.450	26.744	-47.1	153.3	200.4	.890185	.629456	.380659
.460	27.387	-45.4	152.6	198.0	.877759	.620669	.371005
.470	28.034	-43.7	152.0	195.7	.865285	.611849	.361452
.480	28.685	-42.0	151.3	193.4	.852761	.602993	.352000
.490	29.341	-40.4	150.7	191.0	.840186	.594101	.342647

Stable phase		"Bucket" width			Half height		Area
Γ	φ_s	φ_1	φ_2	$\Delta\varphi$	$Y(\Gamma)$	$\beta = \frac{\Delta E(\Gamma)}{\Delta E(0)}$	$\alpha = \frac{A(\Gamma)}{A(0)}$
.500	30.000	-38.7	150.0	188.7	.827559	.585172	.333392
.510	30.664	-37.0	149.3	186.3	.814877	.576205	.324234
.520	31.332	-35.3	148.7	184.0	.802140	.567199	.315173
.530	32.005	-33.7	148.0	181.7	.789346	.558152	.306207
.540	32.684	-32.0	147.3	179.3	.776493	.549064	.297335
.550	33.367	-30.3	146.6	176.9	.763580	.539932	.288558
.560	34.056	-28.6	145.9	174.6	.750603	.530757	.279873
.570	34.750	-26.9	145.2	172.2	.737562	.521535	.271282
.580	35.451	-25.3	144.5	169.8	.724455	.512267	.262782
.590	36.157	-23.6	143.8	167.4	.711278	.502949	.254375
.600	36.870	-21.9	143.1	165.0	.698030	.493582	.246059
.610	37.590	-20.2	142.4	162.6	.684708	.484162	.237834
.620	38.316	-18.5	141.7	160.2	.671310	.474688	.229701
.630	39.050	-16.8	140.9	157.7	.657833	.465158	.221658
.640	39.792	-15.0	140.2	155.2	.644273	.455570	.213706
.650	40.542	-13.3	139.5	152.8	.630629	.445922	.205845
.660	41.300	-11.6	138.7	150.3	.616896	.436211	.198074
.670	42.067	-9.8	137.9	147.7	.603071	.426436	.190394
.680	42.844	-8.1	137.2	145.2	.589151	.416592	.182806
.690	43.630	-6.3	136.4	142.6	.575130	.406678	.175309
.700	44.427	-4.5	135.6	140.1	.561006	.396691	.167904
.710	45.235	-2.7	134.8	137.4	.546772	.386626	.160591
.720	46.054	-.8	133.9	134.8	.532425	.376482	.153372
.730	46.886	1.0	133.1	132.1	.517959	.366252	.146245
.740	47.731	2.9	132.3	129.4	.503368	.355935	.139214
.750	48.590	4.7	131.4	126.7	.488645	.345524	.132277
.760	49.464	6.7	130.5	123.9	.473784	.335016	.125437
.770	50.354	8.6	129.6	121.1	.458778	.324405	.118694
.780	51.261	10.6	128.7	118.2	.443616	.313684	.112050
.790	52.186	12.6	127.8	115.3	.428292	.302848	.105507
.800	53.130	14.6	126.9	112.3	.412793	.291889	.099065
.810	54.096	16.7	125.9	109.2	.397110	.280799	.092728
.820	55.085	18.8	124.9	106.2	.381228	.269569	.086497
.830	56.099	20.9	123.9	103.0	.365135	.258189	.080375
.840	57.140	23.1	122.9	99.7	.348813	.246648	.074364
.850	58.212	25.4	121.8	96.4	.332245	.234933	.068467
.860	59.317	27.7	120.7	93.0	.315408	.223027	.062690
.870	60.459	30.1	119.5	89.5	.298278	.210915	.057034
.880	61.642	32.6	118.4	85.8	.280826	.198574	.051506
.890	62.873	35.1	117.1	82.0	.263017	.185981	.046111
.900	64.158	37.8	115.8	78.1	.244809	.173106	.040856
.910	65.505	40.5	114.5	73.9	.226151	.159913	.035748
.920	66.926	43.5	113.1	69.6	.206977	.146355	.030797
.930	68.435	46.6	111.6	65.0	.187205	.132374	.026015
.940	70.052	49.9	109.9	60.1	.166724	.117891	.021416
.950	71.805	53.4	108.2	54.8	.145379	.102798	.017021
.960	73.740	57.3	106.3	48.9	.122944	.086935	.012855
.970	75.930	61.8	104.1	42.3	.099059	.070045	.008956
.980	78.522	67.0	101.5	34.5	.073066	.051666	.005385
.990	81.890	73.8	98.1	24.3	.043435	.030713	.002260
1.000	90.000	90.0	90.0	0.0	0.000000	0.000000	0.000000