USPAS Graduate Accelerator Physics Homework 8

Due date: Thursday February 4, 2021

1 Low energy proton acceleration

(5 points) We want to accelerate protons with **kinetic energy** of 40 MeV using superconducting rf cavities. During the lectures, we have mentioned how the pillbox design is practical only for relatively high values of $\beta = v/c$. To demonstrate this, let us design a pillbox cavity for these low energy protons. Assuming that the velocity of the protons remain constant inside the cavity, calculate the optimal ratio of length L to the radius R of a pillbox cavity which could be used to accelerate the low energy protons.
(You can assume that the rest mass of a proton is 938.27 MeV/c^2 and the first root of the zero order Bessel function $J_0(x)$ is $u_{01} = 2.405$.)

2 Cavity Scaling

a) (5 points) The geometry factor is one of the figures of merit used to characterize a cavity and is defined as,

$$G \equiv Q_0 R_s = \omega\mu_0 \frac{\int_V |\vec{H}(\vec{r})|^2 \, \mathrm{d}V}{\int_S |\vec{H}(\vec{r})|^2 \, \mathrm{d}S} ,$$

where $\vec{H}(\vec{r})$ is the time independent field profile of a resonant mode. Prove that the geometry factor G does not change on scaling the size of the cavity.
Hint: You can start from the Helmholtz equation $\vec{\nabla}^2 \vec{H}(\vec{r}) + k^2 \vec{H}(\vec{r}) = 0$ where $k \equiv \omega/c$ is the wave number for the resonant mode to determine how ω and the integrals scale.

b) (5 points) The ratio of shunt impedance to the quality factor (in circuit definition) is another figure of merit defined as,

$$\frac{R}{Q} \equiv \frac{V^2}{2\omega U} ,$$

where V and U are the accelerating voltage and the stored energy respectively of the resonant mode. How does R/Q scale with size?

c) (5 points) A single cell Niobium cavity of the TESLA shape with a fundamental mode frequency of 1.3 GHz (typical for $e^{\pm}$ linacs) has an intrinsic quality factor $Q_0 = 3 \times 10^8$ at an accelerating gradient of 10 MV/m when operating at a temperature of 4.2 K. Assuming that the surface resistance is dominated by the BCS resistance, what would be the intrinsic quality factor of a cavity of the same shape but scaled to the resonance frequency of 500 MHz (typical for $e^{\pm}$ rings) when operating at the same temperature and field?