USPAS Accelerator Physics 2019
(Virtually) Texas A&M University

Longitudinal Motion and Dispersion

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1/28/21 L5 Longitudinal and Off-Momentum Motion

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{TRANSVERSE} \quad x' \equiv \frac{dx}{ds} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{LONGITUDINAL} \end{matrix}$$

$\underline{z}$: longitudinal position RELATIVE TO DESIGN "PARTICLE"

$\underline{\delta}$: FRACTIONAL MOMENTUM OFFSET

$$\delta \equiv \frac{p - p_0}{p_0} = \frac{\Delta p}{p_0} \qquad p_0 = \text{DESIGN PARTICLE MOMENTUM}$$

HERE WE WILL START INVESTIGATING $\delta \neq 0$.

$\Rightarrow$ Dispersion - dependence of x on δ (or y)

Chromaticity - dependence of tune or focusing on δ

Q: WHY DO WE NEED (NONLINEAR) SEXTUPOLES? MAGNETS

CONSTANT MOMENTUM OFFSETS δ : $\delta = \frac{p - p_0}{p_0}$ $\begin{pmatrix} \vdots \\ z \\ \delta \end{pmatrix}$ phase space coords

RECALL HILL'S EQN: FOR BETATRON TRANSVERSE OSCILLATIONS

Where does Hill's EQN come from?

$$\left\{ \begin{array}{l} x'' + K(s)\, x = 0 \\ y'' - K(s)\, y = 0 \end{array} \right\} \text{ DECOUPLED}$$

$K(s)$ is PERIODIC IN s

$K(s) = K(s + C)$

$0 < s \leq C$

QUAD: $B_x \propto y$

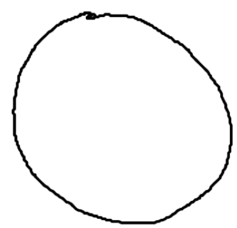
$B_y \propto x$

$F = q\vec{v} \times \vec{B}$ $\quad F_x \propto x$

$F_y \propto y$

DRIFT

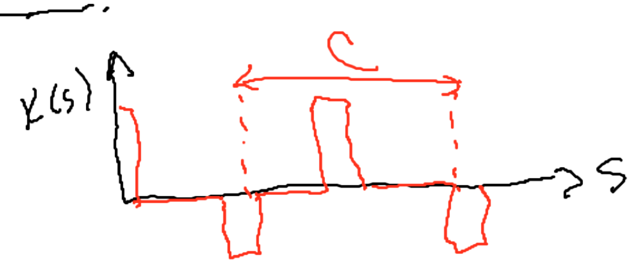
$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$ $\quad F_x = F_y = 0 \propto x$



C = CIRCUMF

K = affected by $\delta \neq 0$.

→ PERIODIC IN C



WORRY RE DIPOLES

BENDING RADIUS R

IN DIPOLE FOR $\delta \neq 0$ BECOMES

$$\delta \equiv \frac{\Delta p}{p_0} = \frac{p - p_0}{p_0} = \frac{p}{p_0} - 1$$

$$\Rightarrow 1 + \delta = p/p_0$$

$$\frac{1}{R} = \frac{1}{\rho}\frac{1}{(1+\delta)} = G\frac{1}{(1+\delta)}$$

ρ = DESIGN BEND RADIUS

PASS THROUGH ΔS INSIDE DIPOLE:

$$\boxed{\Delta x'_{TOT} = G\left(1 - \frac{1}{1+\delta}\right)\Delta s}$$

$\Rightarrow$ HILL'S EQN

(G NOT INSIDE DIPOLE $= 0$)

TOT: $x_{TOT} = x(\delta) + x_\beta$ $\leftarrow$ x_β BETATRON OSCILLATIONS

$x(\delta)$: "CLOSED ORBIT"

(TRAJECTORY FOR PARTICLE W/ $x=0,\ y=0,\ x'=0,\ y'=0,\ \delta \neq 0$.)

BACK TO HILLS EQN:

$\delta \equiv \frac{P-P_0}{P_0}$

$1+\delta = P/P_0$

$$\boxed{\begin{aligned} x''_{TOT} + \frac{K(s)}{1+\delta}\, x_{TOT} &= G(s)\left(1 - \frac{1}{1+\delta}\right) \\ y''_{TOT} - \frac{K(s)}{1+\delta}\, y_{TOT} &= 0 \end{aligned}}$$

$\Rightarrow$ WORKS FOR
DRIFTS $K, G = 0$
QUADS $K \neq 0$, $G = 0$
DIPOLES $K = 0$, $G \neq 0$

NO VERT BENDS

EXACT TO ALL ORDERS IN δ (SO FAR) — SOON $\delta \ll 1$ & LINEARIZE

REMEMBER PERIODIC $K(s)$ & $G(s)$

LINEARIZE:

$$\boxed{\begin{aligned} x''_{TOT} + K(s)(1-\delta)\,x_{TOT} &= G(s)\,\delta \\ y''_{TOT} - K(s)(1-\delta)\,y_{TOT} &= 0 \end{aligned}}\quad \text{(A)}$$

LINEARIZED OFF-MOMENTUM HILL'S EQNS

RECALL:

$$\boxed{\begin{aligned} x_{TOT} &= x_\delta(s) + x_\beta \\ &= \delta\,\eta_x(s) + x_\beta \end{aligned}}\quad \text{(B)}$$

x_β does NOT depend on δ

$\Leftarrow$ FIRST ORDER EXPANSION in δ

$\delta \neq 0$

$\leftarrow$ closed orbit obeys C periodicity

η_x: dispersion depends on periodic in s

A+B: x_β is a solution of homogeneous (A) $\left(x_\beta'' + \frac{K}{(1+\delta)} x_\beta = 0\right)$

↳ CHROMATICITY

$$\cancel{\delta}\,\eta_x''(s) + K(s)\,\cancel{\delta}\,\eta = G\,\cancel{\delta}$$

$$\boxed{\eta_x''(s) + K(s)\,\eta_x(s) = G(s)}$$

$\Leftarrow$ solve using matrices

SOLVE USING MATRICES

$$\begin{pmatrix} x \\ x' \\ \vdots \\ \delta \end{pmatrix}_6 \Rightarrow \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_3$$ FOR SIMPLICITY

VERT: NO DIPOLES

z: IRRELEVENT

Inside long dipole $L = \rho\theta$ $(K=0)$

$$\eta_x'' = G = \frac{1}{\rho}$$

$$\delta = 1 \rightarrow \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_{OUT} = \begin{pmatrix} 1 & L & \rho(1-\cos\theta) \\ 0 & 1 & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_{IN}$$

MATRIX WAY OF EXPRESSING N.L. HILL'S EQN IN η_x

η ! PERIODIC!

REF

M = ONE TURN MATRIX

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_{REF} = M \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_{REF} = \begin{pmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_{REF}$$

SOLVE FOR

η_{REF}, η'_{REF}

HALF-FODO

$$\begin{pmatrix}\eta\\ \eta'\\ 1\end{pmatrix}_F = \underbrace{\begin{pmatrix}1&0&0\\-q&1&0\\0&0&1\end{pmatrix}}_{\text{quad}}\underbrace{\begin{pmatrix}1&L&\frac{1}{2}L\theta\\0&1&\theta\\0&0&1\end{pmatrix}}_{\text{dipole (linearized, }\theta\ll 1)}\underbrace{\begin{pmatrix}1&0&0\\q&1&0\\0&0&1\end{pmatrix}}_{\text{quad}}\begin{pmatrix}\eta\\ \eta'\\ 1\end{pmatrix}_D$$

$q \equiv 1/f$

$L = \rho\theta$

Periodic solution: $\eta'_F = \eta'_D = 0$ (extrema of motion)

2 eqns 2 unknowns: η_F, η_D

$$\Rightarrow \left\{ \boxed{\begin{aligned} \eta_F &= \eta_{max} = \hat{\eta} = L\theta\left(\frac{2+S}{2S^2}\right) \\ \eta_D &= \eta_{min} = \check{\eta} = L\theta\left(\frac{2-S}{2S^2}\right) \end{aligned}} \right.$$

$\propto L^2/\rho$

$S \equiv \sin\left(\frac{\Delta\phi}{2}\right) = |qL|$

$[\eta] = [L]$

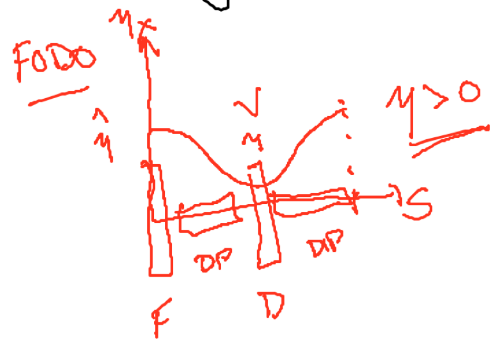
$x_{TOT} = \delta\eta + x_\beta$

$\delta \equiv \frac{p - p_0}{p_0}$

Have been assuming $\delta \neq 0$ but constant.

We keep assuming that ☺

$\delta > 0$



$\eta < 0$ is possible!

Q: $\delta > 0$ in circ accel

1) goes faster $p > p_0$
2) has further to go around

WHICH WINS? IT DEPENDS

PATH LENGTH IN 1 TURN:

$$\Delta C_{PATH} = \delta \oint \eta \, d\theta \quad \text{since} \quad \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$\boxed{\Delta C_{PATH} = \delta \oint \frac{\eta(s)}{\rho(s)} ds}$$

only contributions inside dipoles

Consider 2 cases

$$\boxed{\Delta C_{PATH} = \delta \oint \frac{\eta(s)}{\rho(s)}\, ds}$$

Higher speed moves forward (gains pathlength)

$$\Delta C_{SPEED} = C\,\frac{\Delta\beta}{\beta} = C\,\frac{\delta}{\gamma^2} \qquad \left(\frac{\Delta\beta}{\beta} \Rightarrow \frac{\Delta p}{p}\right)$$

$\beta = \frac{v}{c}$

add effects together for z:

one turn: $z_{n+1} = z_n - \eta_s\, \delta_n$ $\quad n = \text{turn \#}$

prop factor: slip factor or momentum compaction

$$\boxed{\eta_s = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}}$$

$$\frac{1}{\gamma_T^2} = \frac{1}{C}\oint \frac{\eta(s)}{\rho(s)}\, ds \leftarrow \text{lattice}$$

depends on design or particle momentum