USPAS Accelerator Physics 2019
(Virtually) Texas A&M University

Chapter 5: Action and Emittance

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01/29/21 L6 Ch5 Emittances and Phase Space

"One particle, many, or none?"

Recall the parameterization of PERIODIC linear motion (Tue slide 11)

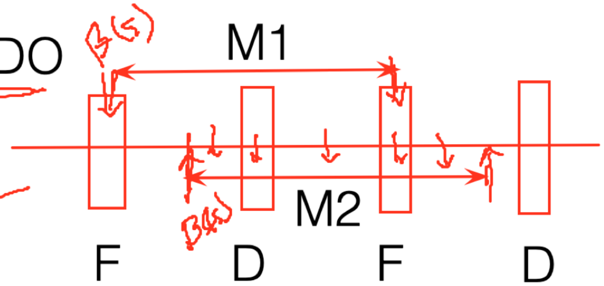
$$M = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix} \qquad \mu = 2\pi Q$$

x or y

$\det M = 1$ $\qquad \alpha(s) \equiv -\frac{1}{2}\frac{d\beta(s)}{ds} \qquad \gamma(s) \equiv \frac{1+\alpha^2(s)}{\beta(s)}$

$\mathrm{Tr}\, M = 2\cos\mu \qquad \mu$: does NOT depend on s

Example: FODO



M_1 & M_2 are both periodic of the above form

M_1 has $\beta(s), \alpha(s), \gamma(s)$ at center of F

M_2 has " " " between F, D

Both have the same μ

2

1 particle, many particles, or no particles?

No particles: Optics (Twiss parameters, or C-S parameters) are defined by the lattice (magnets)

optical properties of lattice (periodic so far)

BUT what about beamlines? Linacs? Non-periodic systems?? Are there useful parameterizations of linear motion: β, α, γ?

A: Yes (sorta) ⇒ here people usually speak more of the beam

STILL: $\begin{pmatrix} x \\ x' \end{pmatrix}_{OUT} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{IN}$ ☺

parameterize (x, x') ....

3 Single-particle coordinates in action-angle coordinates

HILLS EQN: (1D) $x'' + K(s)\,x = 0$ ....

suggests pseudo-SHO solutions

Ansatz: (A)

$$\boxed{\begin{aligned} x_n &= \sqrt{2J_x\beta_x}\,\sin(\phi_x) \\ x'_n &= \sqrt{\frac{2J_x}{\beta_x}}\left[\cos(\phi_x) - \alpha_x \sin(\phi_x)\right] \end{aligned}}$$

J_x = action, ϕ = angle } Hamiltonian

For n-turn parameterization J_x = constant

"phase advance" ⇐ ϕ_x "advances"

For ring w. tune Q_x, $\phi_{x,n} = \phi_{0x} + 2\pi Q_x n$

HW: Use (A) to find

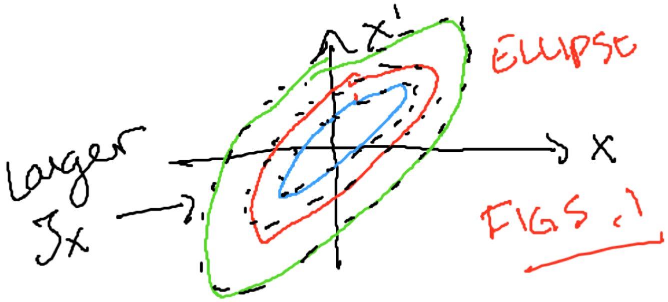
$$\boxed{2J_x = \beta_x x'^2 + 2\alpha_x x x' + \gamma x^2}$$

$\gamma = \frac{1+\alpha^2}{\beta}$

$[J_x] = [L]$

Courant-Snyder invariant → EMITTANCE

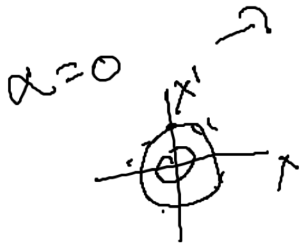
4 Ellipse and RMS displacement



$2J_x = \beta_x x'^2 + 2\alpha_x x x' + \gamma x^2$

↙ draw in (x, x')

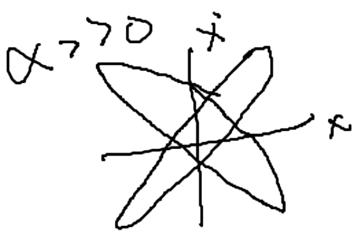
single particles



A_x = area of ellipse $= 2\pi J_x$ (do math)

"single-particle emittance" (area in phase space)

True everywhere in system: J_x has NO s-dependence



RMS displacement of this single particle over all s (over all ϕ_x): $\sigma_x = \langle x^2 \rangle^{1/2} = 2\sqrt{J\beta_x \langle \sin^2\phi \rangle}$

$x = \sqrt{2J\beta_x}\,\sin\phi_x$

$= \boxed{\sqrt{J_x \beta_x} = \sigma_x}$

"simple" 1-particle

⇒ onward to distributions

5 Many particles: distributions

Base on 1-particle statistics

Mean-square size of "bunch": $\sigma_x^2 = \frac{1}{N}\int_0^\infty \sigma_{x_1}^2 \rho(J_x)\,dJ_x$

$\Rightarrow$ remember different particles have different J_x

$N \propto 10^6 - 10^{11}$ or more ....

$\rho(J_x)$ is distribution function

$\sigma_x^2 = \frac{\beta_x}{N}\int_0^\infty J_x\,\rho(J_x)\,dJ_x$

$\int_0^\infty \rho(J_x)\,dJ_x = \cancel{1}$ (N for bunch)

$\boxed{\sigma_x^2 = \beta_x \langle J_x \rangle}$ RMS beam size

DEFINE UNNORMALIZED EMITTANCE (RMS)

$\boxed{\epsilon_{x,} = \langle J_x \rangle}$

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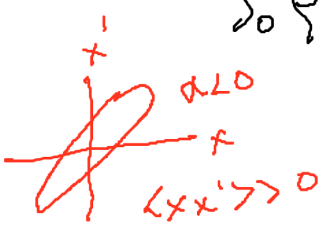
$\sigma_x^2 = \frac{\beta_x}{N}\int_0^\infty J_x\, \rho(J_x)\, dJ_x$

$\boxed{\sigma_x^2 = \beta_x \langle J_x \rangle}$ RMS beam size



DEFINE UNNORMALIZED EMITTANCE (RMS)

$\boxed{\epsilon_x = \langle J_x \rangle}$

Generalize:

RMS quantities $\boxed{\begin{pmatrix} \langle x^2 \rangle \\ \langle x x' \rangle \\ \langle x'^2 \rangle \end{pmatrix} = \epsilon_x \begin{pmatrix} \beta_x \\ -\alpha_x \\ \gamma_x \end{pmatrix}}$ ⇐ C-S or Twiss parameters for bunch or beam

Redid linear transverse dynamics: What about longitudinal? Dispersion?

6: Add dispersion to beam

$$x_{n,\mathrm{TOT}} = \underbrace{\sqrt{2J_x\beta_x}\,\sin\phi_{x,n}}_{\text{betatron}} + \underbrace{\delta_n \eta_x(s)}_{\text{dispersion}} \qquad \delta \equiv \frac{\Delta p}{p_0}$$

For each particle

$$\delta_n = a_\delta \sin(\phi_{s,n})$$

Assume $\phi_{x,n}$ & $\phi_{s,n}$ are uncorrelated

BOOK CORRECTION → 2

$$\boxed{\sigma^2_{x,\mathrm{TOT}} = \beta_x\epsilon_x + \eta_x^2\left(\frac{\sigma_p}{p}\right)^2}$$

ADDED "in quadrature"

all rms quantities

(Synchrobetatron coupling is largely pathological and not common)

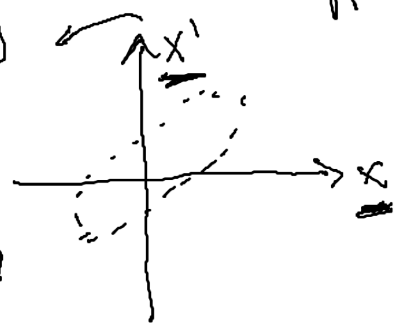
5.3: Tune spread and filamentation — website video

So far: unnormalized emittance

→ normalized emittance

7: Normalized emittance and adiabatic damping

What happens to phase space during acceleration?



$$x' = \frac{dx}{ds} = \frac{p_x}{p_0}$$

accelerate $p_0 \to p_0 + \Delta p$

adiabatic "damping"

$[\sigma_x] = [\epsilon_{x,n}] = [L]$

$$x' \to x' \Big/ \left(1 + \frac{\Delta p}{p_0}\right)$$ & shrinks during acceleration

$p = \beta\gamma mc$ ← relativistic

$$\langle J_{new} \rangle = \frac{J_0}{1 + \Delta p/p} = \frac{J_0}{(1 + \Delta(\beta\gamma)/\beta\gamma)}$$

UNNORMALIZED EMITTANCE SHRINKS! NOT CONSERVED! ☹

Commonly $\epsilon_x \sim$ mm e^-

$\epsilon_{x,n} \sim$ mm-mrad

p, ion

Define norm emit: $\epsilon_{x,n} = (\beta\gamma)\,\epsilon_x$ ← rms emit

↳ conserved even through acceleration

$$\boxed{\sigma_x^2 = \frac{\beta\,\epsilon_{x,n}}{(\beta\gamma)_{rel}}}$$

For protons/ions: synchrotrons $\epsilon_x \sim \epsilon_y$ $\sigma_x \sim \sigma_y$

ROUND BEAMS

(symmetry to transverse dynamics)

For electrons: synch radiation!

$\Rightarrow$ adds x noise (bending plane)

$\Rightarrow$ $\epsilon_x \gg \epsilon_y$ $\sigma_x \gg \sigma_y$

FLAT BEAMS

$\mathcal{L} \propto \frac{1}{\sigma_x \sigma_y}$ [$cm^{-2}\ s^{-1}$]

EIC: $\Rightarrow$ problem! p, ion beams want to be round

e^- beams want to be flat

↳ electron-ion collider

TLA (Three letter acronym)

Single-pass systems

LINACS   TRANSFER LINES

$\Rightarrow$ No periodicity so cannot use $M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ \cdots & \cdots \end{pmatrix}$

$\hookrightarrow$ This form is only for periodic focusing

! PARADIGM SHIFT: Use β, α, γ of beam distributions

electrons

$$\boxed{\begin{pmatrix} \beta \\ -\alpha \\ \gamma \end{pmatrix}_{x,A} = \frac{1}{\epsilon_{xA}} \begin{pmatrix} \langle x^2 \rangle \\ \langle x x' \rangle \\ \langle x'^2 \rangle \end{pmatrix}}$$

ϵ_{xA} ~ AREA EMITTANCE (scaling term)

USES e^-

e^- guns/sources

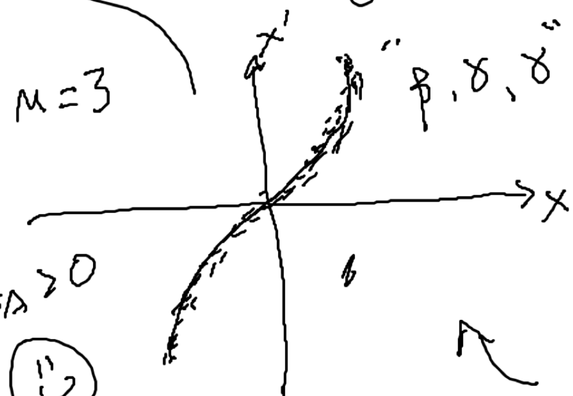
low E transport

non-equilibrium systems (ERLs, FELs...)

M=1 'Wangler distribution'

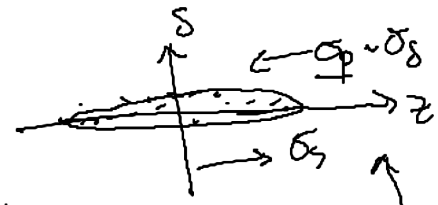
$\epsilon_{xA} = 0$!

M=3 "β, α, γ"



$\epsilon_{xA} > 0$

Are there longitudinal equivalents for this description?



longitudinally RMS bunch length $\sigma_s = \sqrt{\frac{\beta_s \epsilon_s}{(\beta\gamma)}}$

normalization

$$\sigma_p = \sqrt{\frac{\epsilon_s}{\beta_s (\beta\gamma)}}$$

ϵ_s = longitudinal emittance

units could be [L] ⇐⇒ z, δ

often see eV-s

energy spread

time

β_s : longitudinal β function parameterization

$$\beta_s [m] = \frac{\sigma_s}{\sigma_p} = \text{easily calculable}$$

$$= \frac{C}{2\pi} \frac{|\eta_s|}{Q_s}$$

circumf

$|\eta_s|$ ← slip factor $= \frac{-1}{\gamma^2} + \frac{1}{\gamma_T^2}$

Q_s ← synchrotron tune

when $\gamma = \gamma_T$

$\eta_s = 0$

→ $\beta_s \to 0$

BOOM ☺

$\sigma_p \sim \frac{1}{\sqrt{\beta_s}} \to \infty$!