USPAS Accelerator Physics 2021
(Virtually) Texas A&M University

Lattice Examples II
Dispersion Suppressors and Achromats
(or More Stupid Lattice Tricks)

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http://www.toddsatogata.net/2021-USPAS

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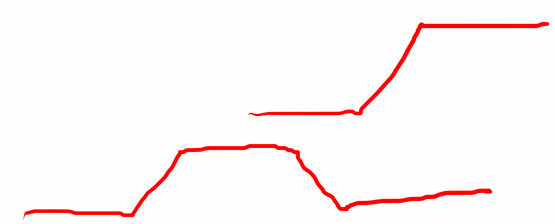
Today: 1D+ and 2-3D+

- Bending Transverse Lattices (FODO)
  - Review: FODO cell, with dipoles
  - FODO cell dispersion suppressors
- Achromats
  - Doglegs and achromatic doglegs
  - Chicanes and bunch compressors
  - Double bend achromat
  - Triple bend achromat
  - (Multi-bend achromat (HMBA))
- 4D/6D manipulation
  - Transverse/longitudinal emittance exchange
  - (Flat to round/round to flat transforms)



LINACS (e^-)

(Review: Matrices of Magnetic Elements)

- For our purposes yet again:
  - All motion is linearized
    $$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad x' \equiv \frac{p_x}{p_0}$$
  - Linear transport matrices:
    $$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$
    Book A.1.1

    DECOUPLED

    $$M_{\text{quad}} \approx \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$
    Book A.1.6
    for thin quads
  - (Sector) dipole includes constant fractional momentum offset

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_2 = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ \frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_1 \qquad \delta \equiv \frac{\Delta p}{p_0}$$

Book A.1.2
for subset of phase space

(Review: Dispersion)

- Add explicit momentum dependence to (linearized) equation of motion

SECOND ORDER IN COORDS

$$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

ignore → $(1+\delta)$

FIRST ORDER IN COORDS

Perturb our zero-dispersion solution to find

D is a matrix element here NOT dispersion

$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\delta_0$$
$$x'(s) = C'(s)x_0 + S'(s)x'_0 + D'(s)\delta_0$$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

1st order in coords

The trajectory has two parts:

linear

$$\boxed{x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}}$$

(Review: Dispersion)

- Substituting and noting dispersion is periodic, $\eta_x(s+C) = \eta_x(s)$

$$\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \\ \delta_0 \end{pmatrix} \qquad \text{achromat}: D = D' = 0$$

matrix elements

  - If we take $\delta_0 = 1$ we can solve this in a clever way

$$\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(S) \end{pmatrix} \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} + \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

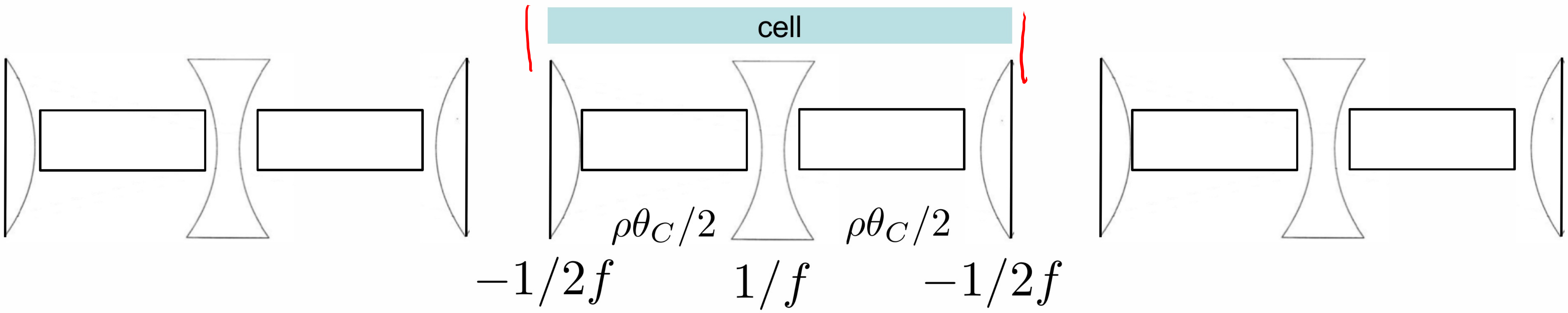
$$(I - M) \begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} \quad \Rightarrow \quad \boxed{\begin{pmatrix} \eta_x(s) \\ \eta_x'(s) \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}}$$

2 x 2 inversion explicitly

  - Solving gives

$$\boxed{\begin{aligned} \eta(s) &= \frac{[1 - S'(s)]D(s) + S(s)D'(s)}{2(1 - \cos\mu)} \\ \eta'(s) &= \frac{[1 - C(s)]D'(s) + C'(s)D(s)}{2(1 - \cos\mu)} \end{aligned}}$$

Review: FODO with dipoles



- A periodic lattice without dipoles has no **intrinsic** dispersion
- Consider FODO with long dipoles and thin quadrupoles
  - Each dipole has total length $\rho\theta_C/2$ so each cell is of length $L = \rho\theta_C$
  - Assume a large accelerator with many FODO cells so $\theta_C \ll 1$

$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\text{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta_C}{8} \\ 0 & 1 & \frac{\theta_C}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Tr M = 2cos μ

(periodic)

$$M_{\text{FODO}} = M_{-2f} M_{\text{dipole}} M_f M_{\text{dipole}} M_{-2f}$$

$$M_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_C \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_C \\ 0 & 0 & 1 \end{pmatrix}$$

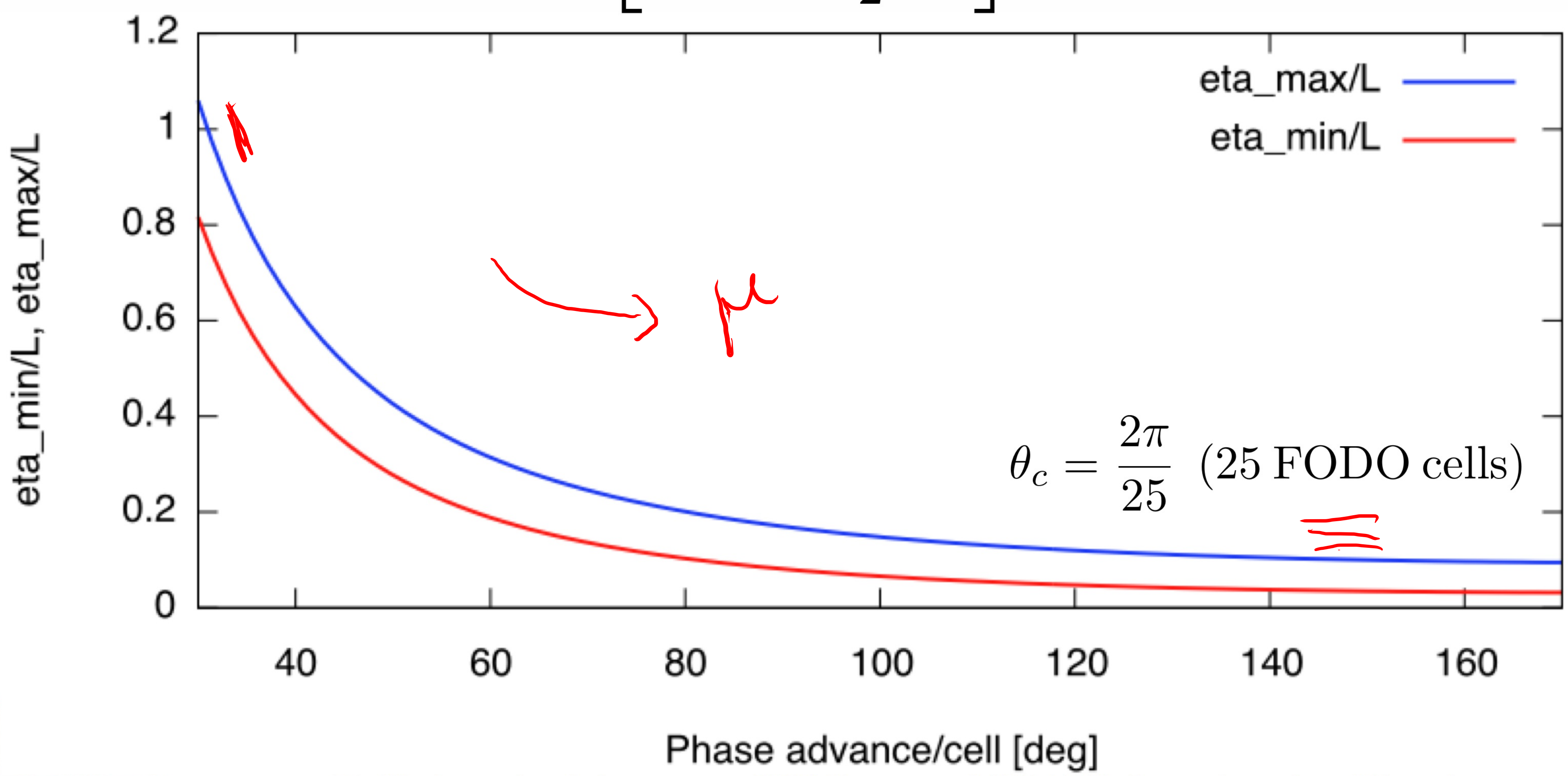
Review: FODO with dipoles

- Like $\hat{\beta}$ before, this choice of periodicity gives us $\hat{\eta}_x$

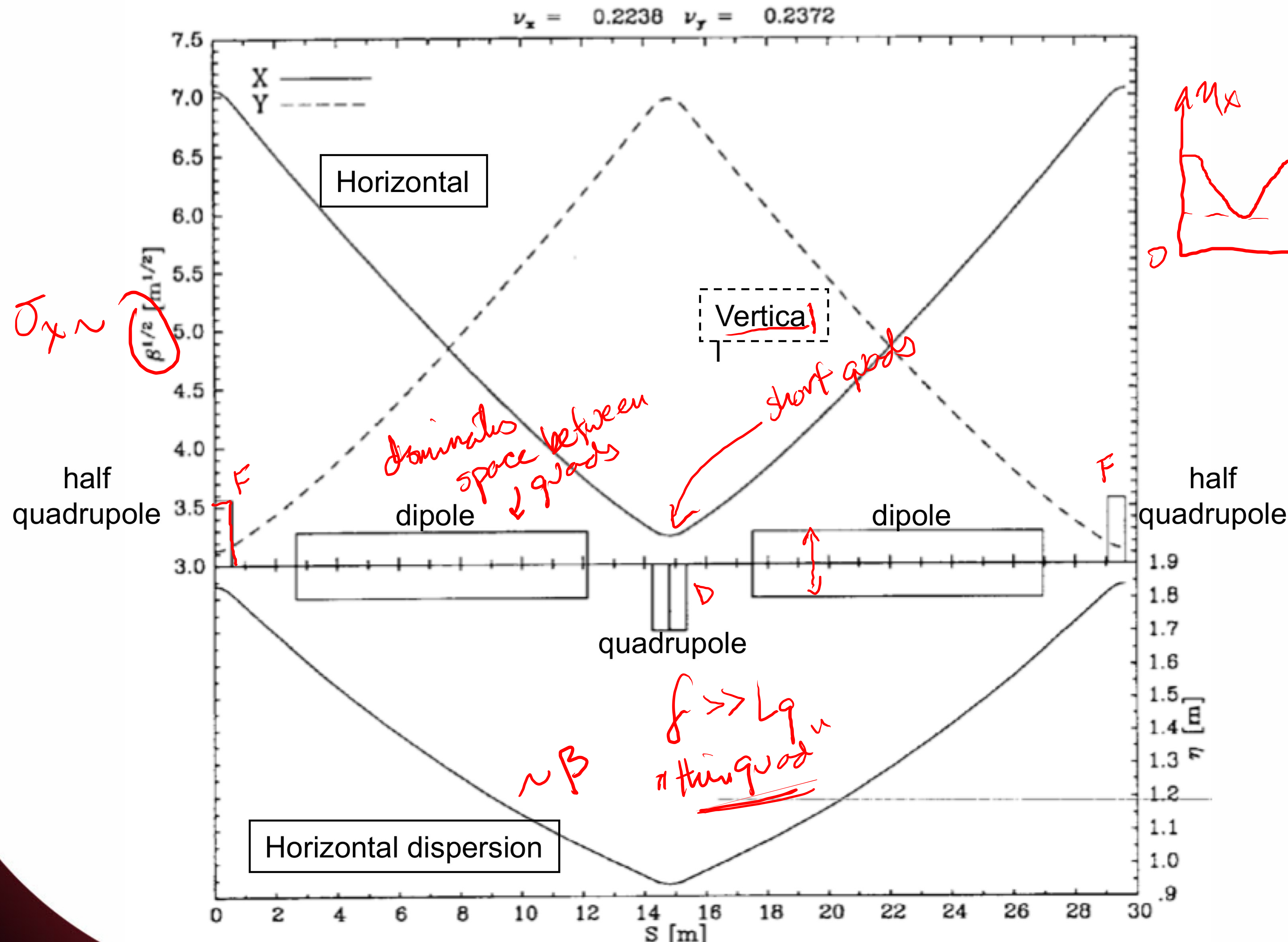
$$\hat{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] \qquad \eta'_x = 0 \text{ at max}$$

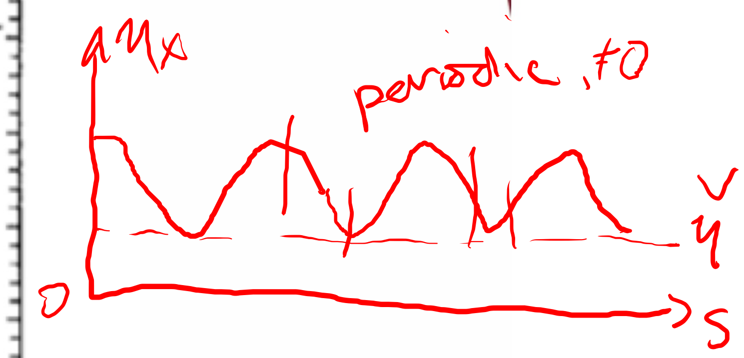
- Changing periodicity to defocusing quad centers gives $\check{\eta}_x$

$$\check{\eta}_x = \frac{L\theta_C}{4}\left[\frac{1-\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] \qquad \eta'_x = 0 \text{ at min}$$



Review: RHIC FODO Cell





How to "match" to $\eta_x = 0$, $\eta_x' = 0$
(FODO NO DIPOLES)

Dispersion suppressors

MATCHING

NON-PERIODIC INSERTION

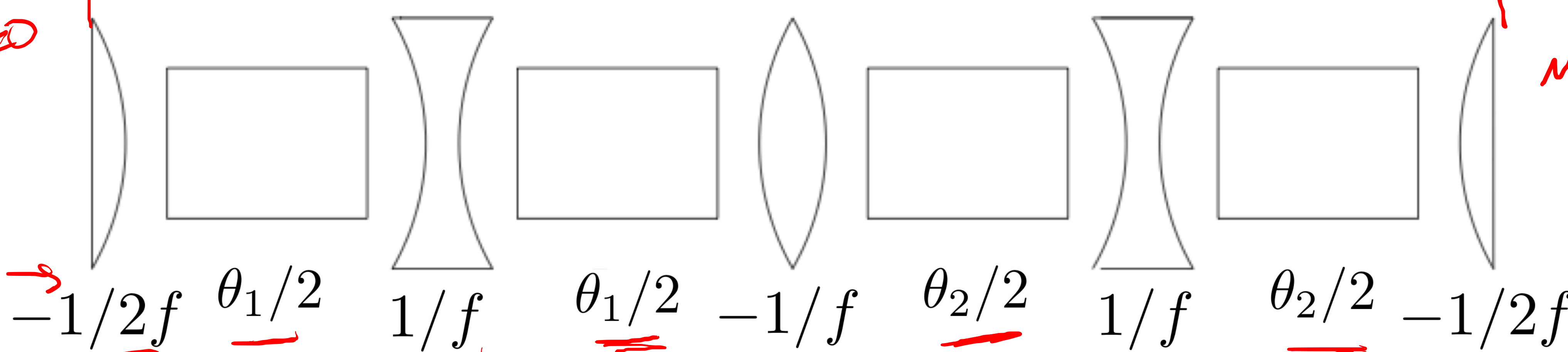
- The FODO dispersion solution is non-zero everywhere
  - But in straight sections we often want $\eta_x = \eta'_x = 0$
    - e.g. to keep beam small in wigglers/undulators in a light source
  - We can "match" between these two conditions with with a **dispersion suppressor**, a **non-periodic** set of magnets that transforms FODO (η_x, η'_x) to zero.

θ in FODO $\rightarrow$

$\alpha_x = 0$ 2 conditions $\Rightarrow$ 2 knobs

$\eta_x, \eta'_x \neq 0$

$\eta_x, \eta'_x = 0$



QUADS LOOK FODO $\rightarrow$

VARY dipoles

- Consider two FODO cells with different total bend angles θ_1, θ_2
  - Same quadrupole focusing to not disturb β_x, μ_x much
  - We want this to match $(\eta_x, \eta'_x) = (\hat{\eta}_x, 0)$ to $(\eta_x, \eta'_x) = (0, 0)$
  - $\alpha_x = 0$ at ends to simplify periodic matrix

Dispersion suppressors

Zero dispersion area

slope η' =0

$$\begin{array}{r} \eta_x = \\ \eta_x' = \\ \\ \end{array}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\mu_x & \beta_x \sin 2\mu_x & D(s) \\ -\frac{\sin 2\mu_x}{\beta_x} & \cos 2\mu_x & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_x \\ 0 \\ 1 \end{pmatrix}$$

"periodic" M ($\alpha_x = 0$)

incoming dispersion

$\eta_x' = 0$

FODO peak dispersion,

slope η' =0

$\theta_{1,2} \ll 1$

multiply matrices $\Rightarrow$

$$D(s) = \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

$$D'(s) = \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]$$

Mathematica

$$\hat{\eta}_x = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)(\theta_1 + \theta_2)$$

⇒ same μ

$$\boxed{\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \qquad \theta_2 = \left(\frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta}$$

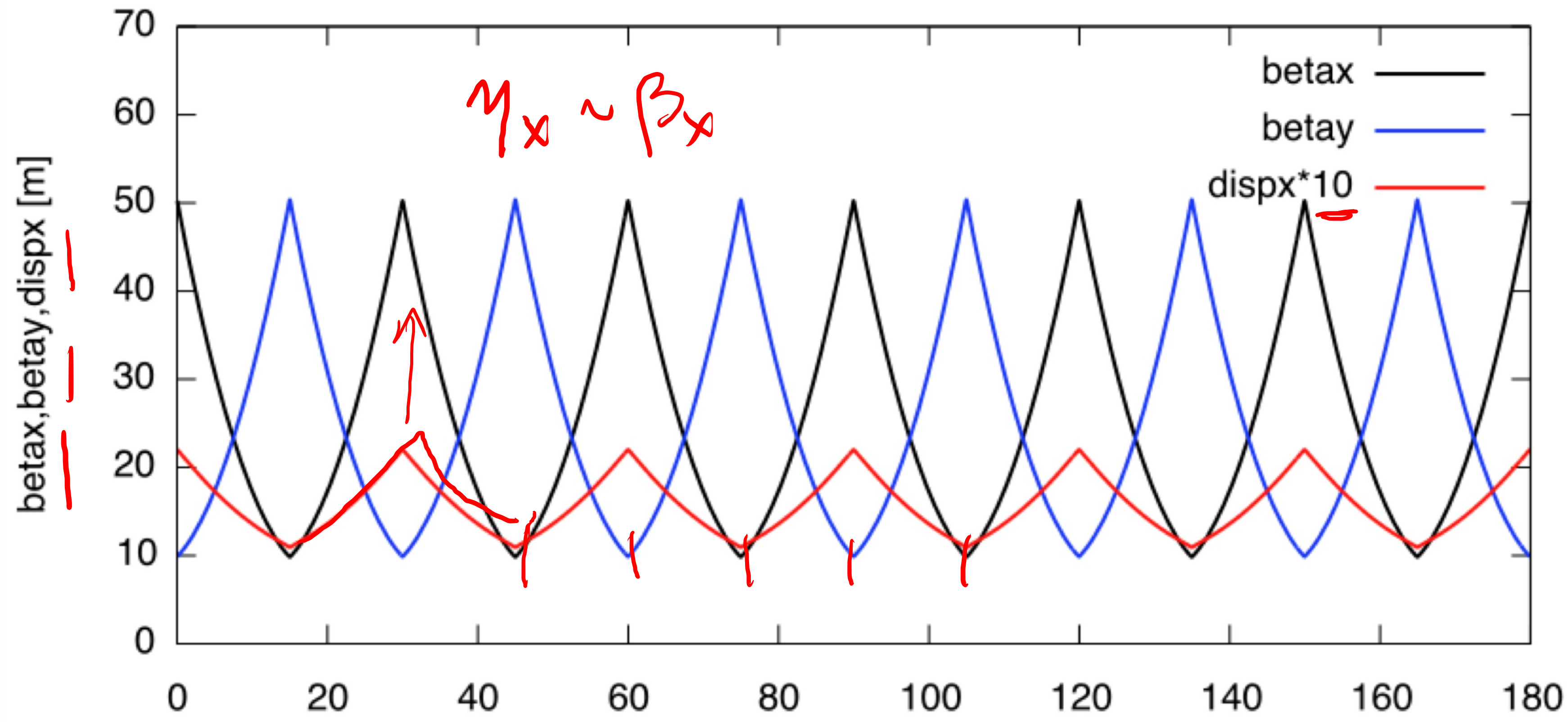
θ original FODO dipole bend angle

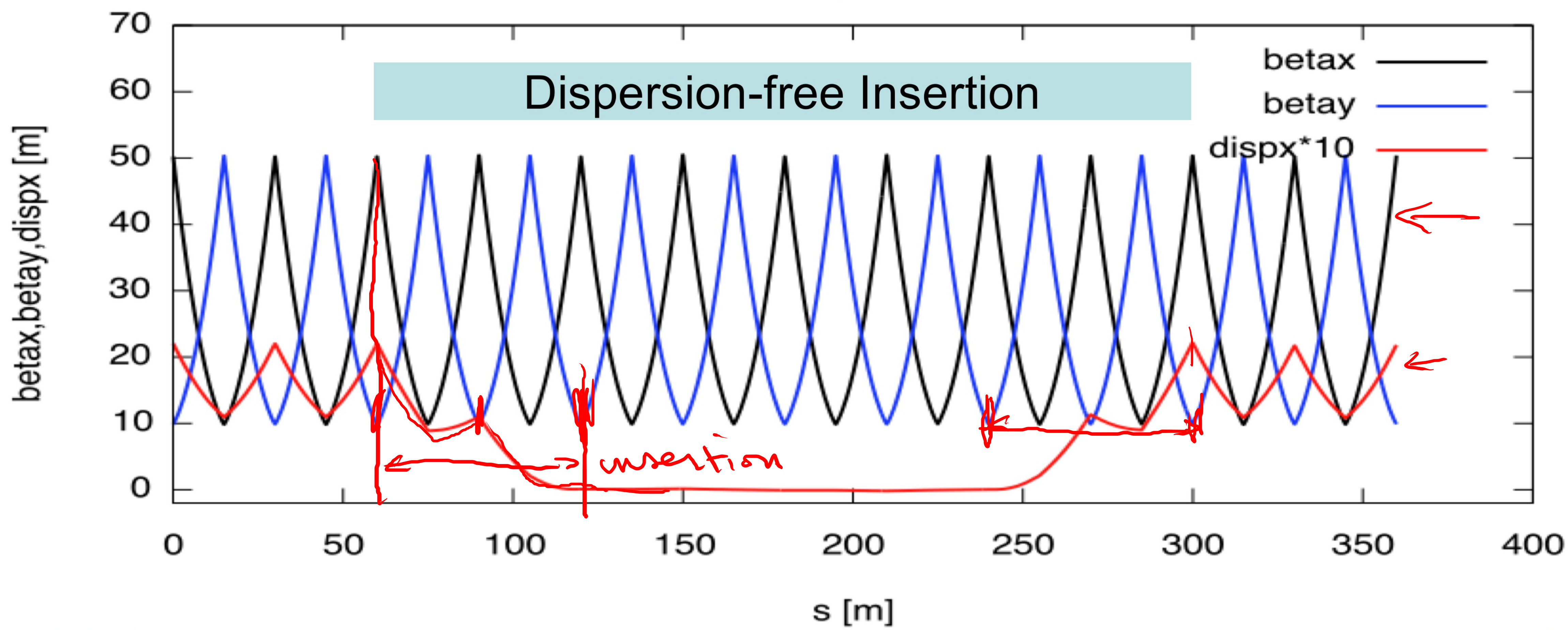
$\boxed{\theta = \theta_1 + \theta_2}$ two cells, one FODO bend angle $\rightarrow$ reduced bending

FODO Cell Dispersion and Suppressor

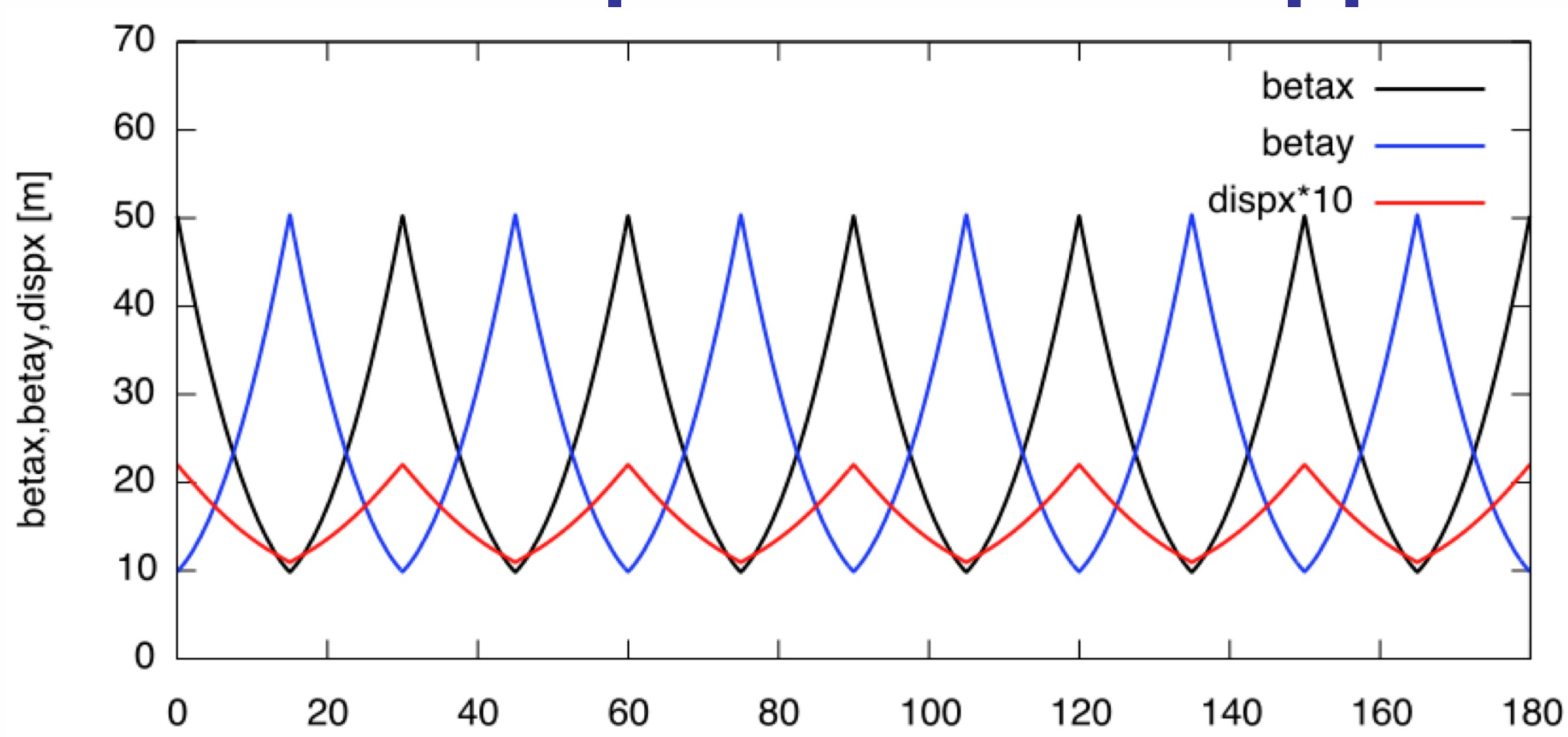
FODO

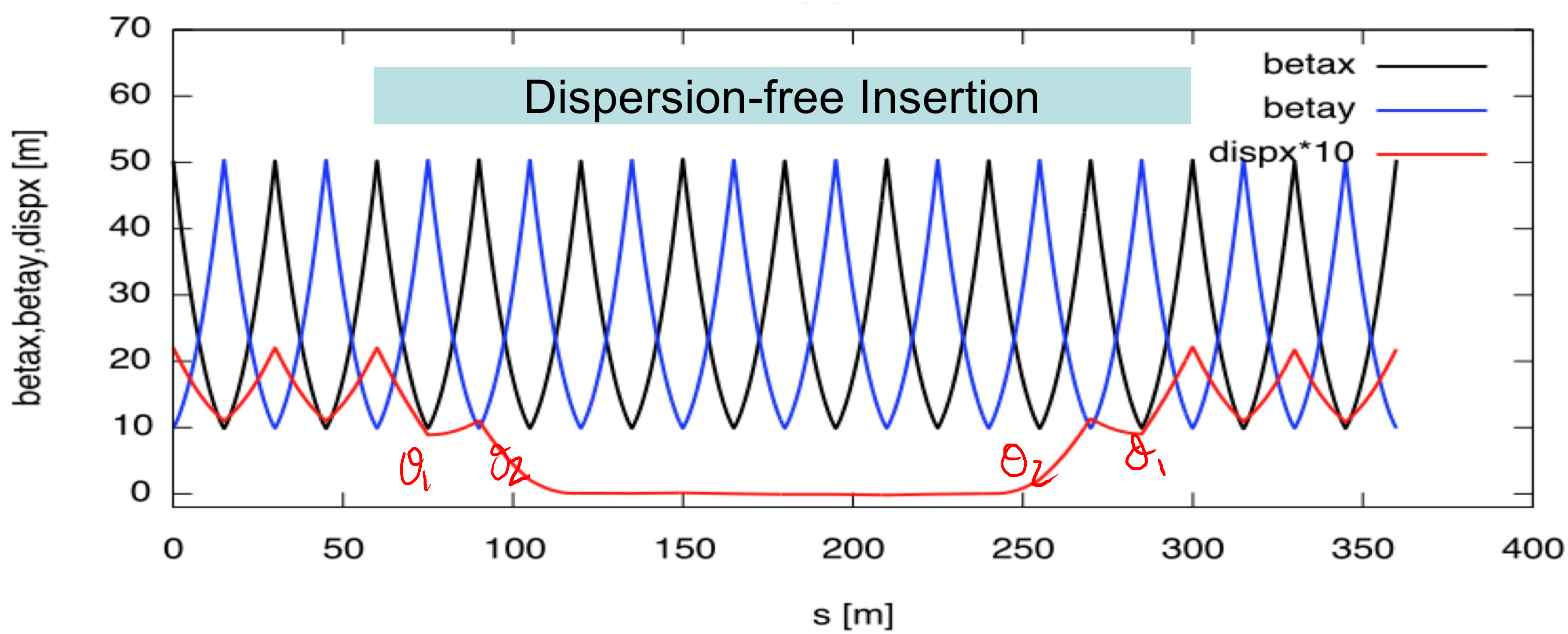
$\eta_x \sim \beta_x$





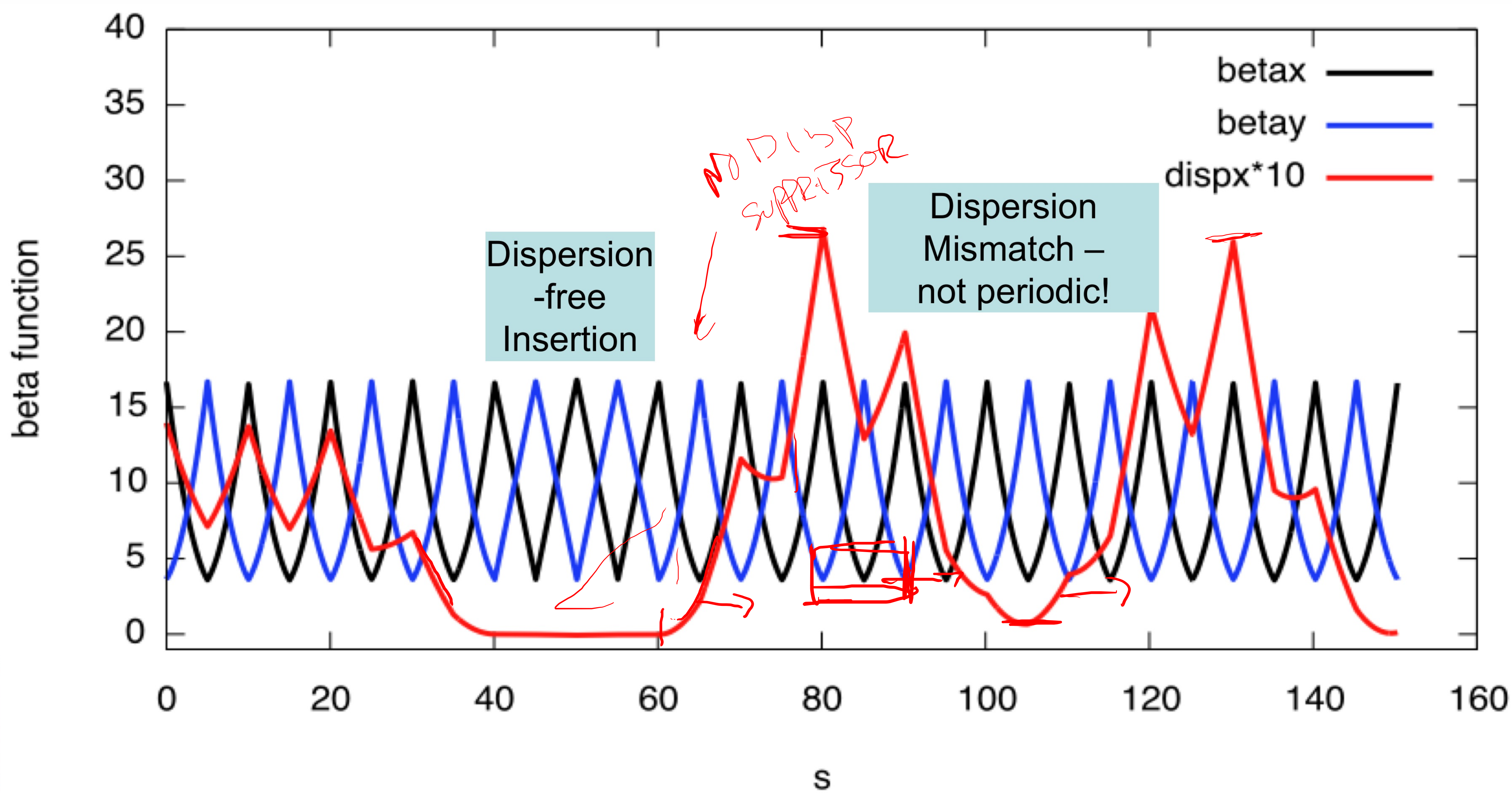
FODO Cell Dispersion and Suppressor



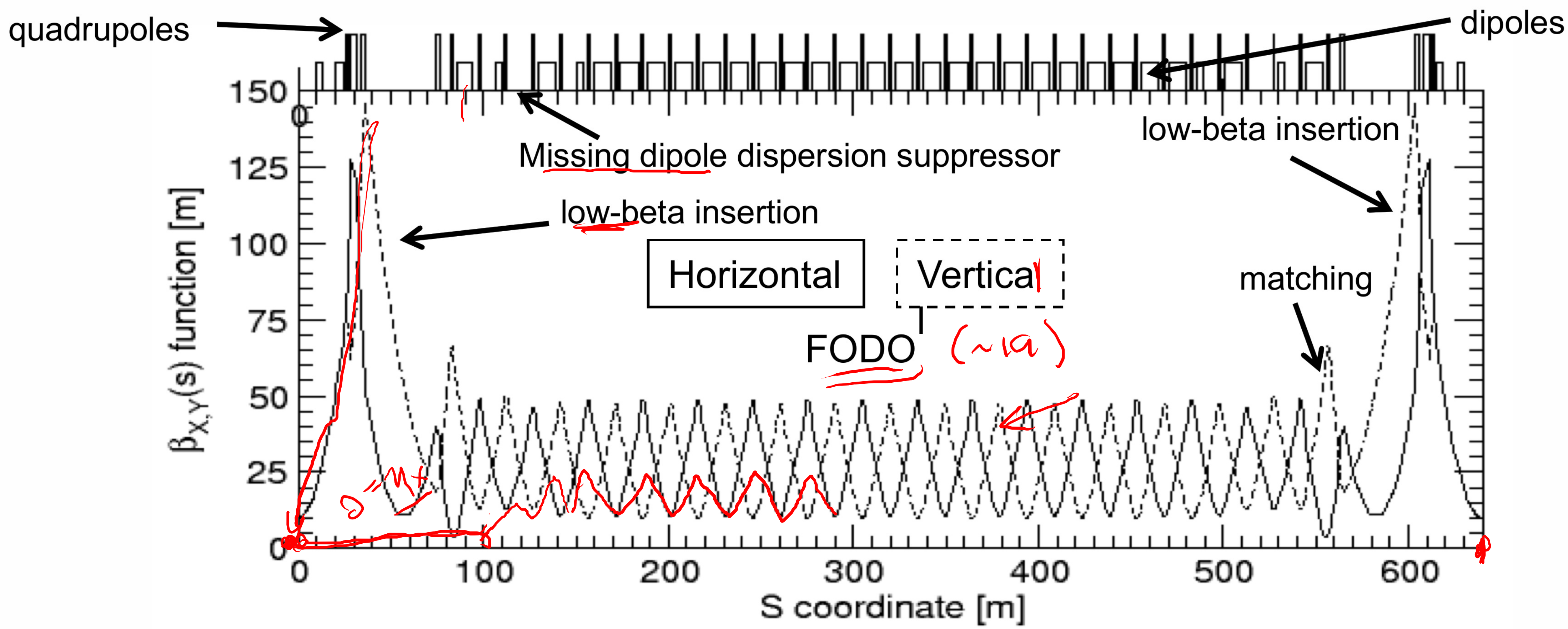


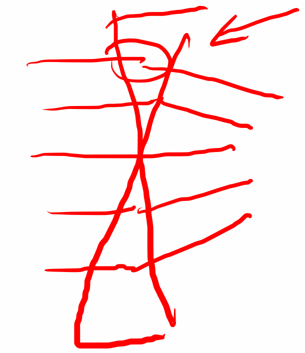
Mismatched Dispersion

- What does mismatched dispersion look like?
  - For example, this is what happens when the second dispersion suppressor is eliminated and the dipole-free FODO cells run right up against the FODO cells with dipoles



RHIC Lattice Revisited



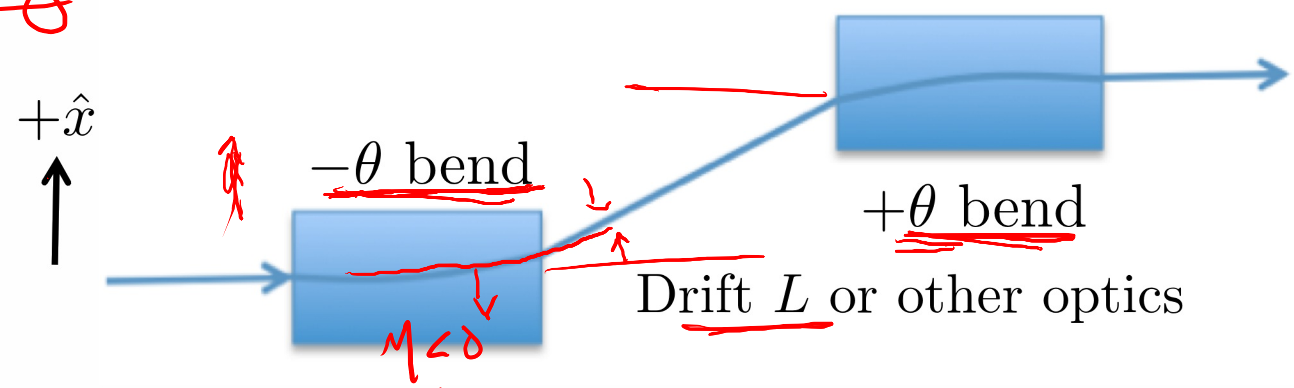


- Note modular design, including low-beta insertions
  - Used for experimental collisions
  - Minimum beam size σ (with zero dispersion)
    - **maximize luminosity**
  - Large σ, beam size in "low beta quadrupoles"
  - Other facilities also have longitudinal bunch compressors (~~this afternoon~~)

Move beam sideways

Doglegs

(Other dispersion things)



- Displaces beam transversely without changing direction
- What is effect on 6D optics?

$$\mathbf{M}_{\text{dipole}}(\rho,\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\theta & -\rho(1-\cos\theta) & 0 & 0 & 1 & \boxed{L/(\gamma^2\beta^2)} - \rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Leftarrow M_{56}\ (z,\delta)$

kinematic (electron linacs)

- Be careful about the coordinate system and signs!!
- If ρ,θ>0, positive displacement points **out** from dipole curvature
- Be careful about order of matrix multiplication!

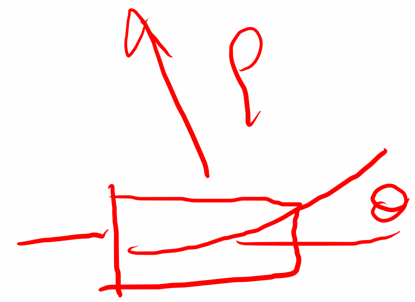
Reverse Bend Dipole Transport

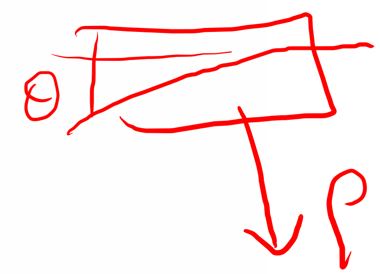
- What is the correct 6x6 transport matrix of a reverse bend dipole?
- It turns out to be achieved by reversing **both** ρ **and** θ
  - $\rho\theta = L$ (which stays positive) so both must change sign

$L = \rho\theta > 0$

$$\mathbf{M}_{\text{dipole}}(-\rho, -\theta) = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \boxed{-\rho(1-\cos\theta)} \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 & 0 & \boxed{-\sin\theta} \\ 0 & 0 & 1 & \rho\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \boxed{\sin\theta} & \boxed{\rho(1-\cos\theta)} & 0 & 0 & 1 & \boxed{L/(\gamma^2\beta^2)} - \rho(\theta - \sin\theta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

relativistic

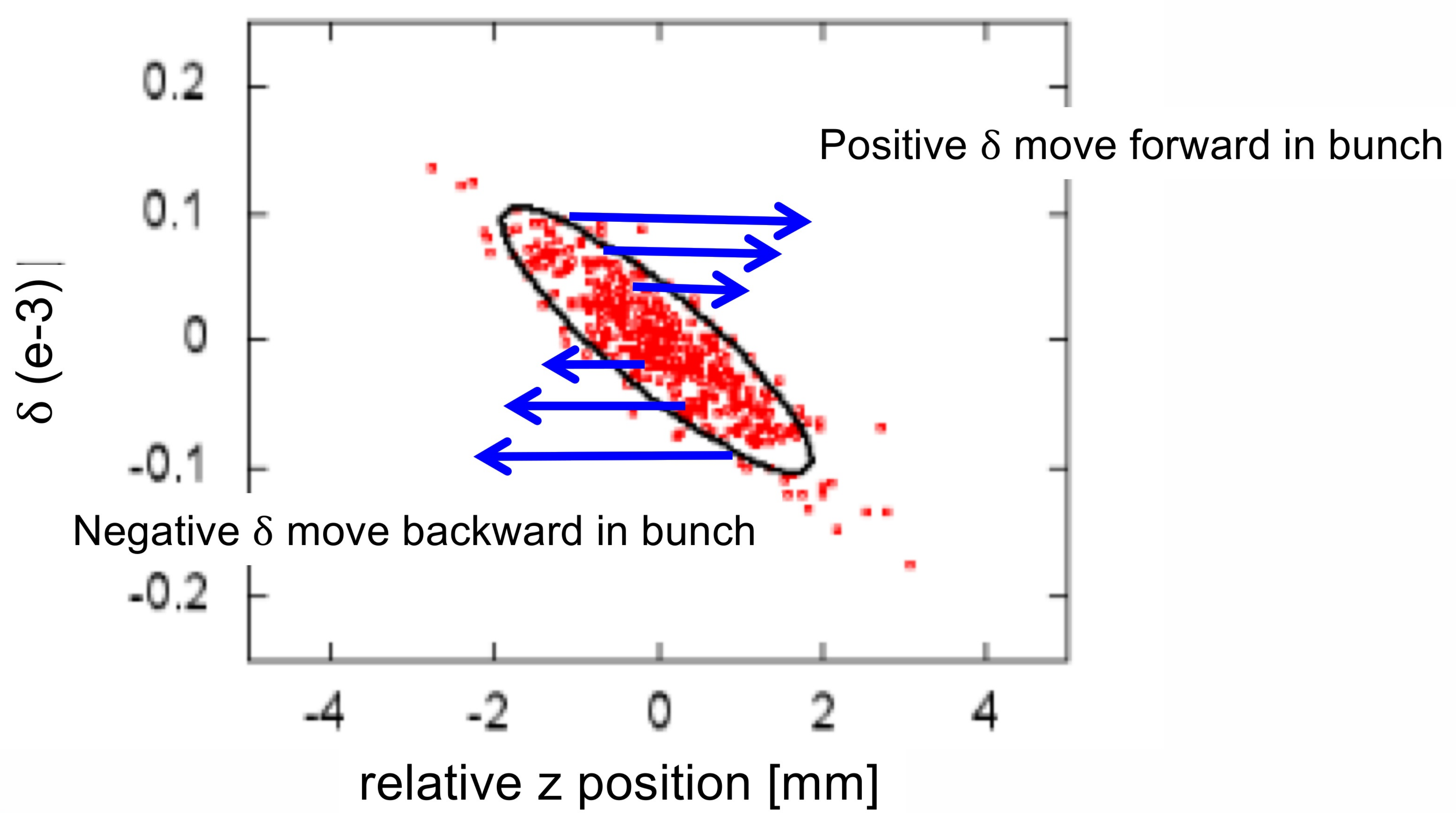




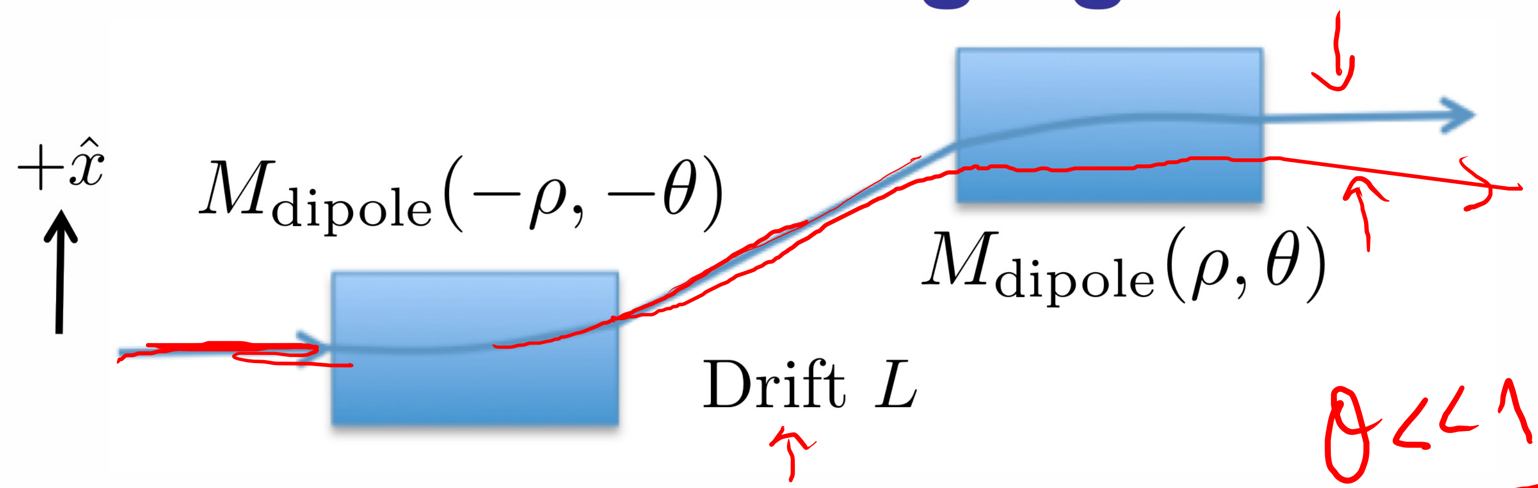
$$M_{\text{drift}} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \boxed{L/(\gamma^2\beta^2)} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Aside: Longitudinal Phase Space Drift

- Wait, what was that M_{56} term with the relativistic effects?
  - Recall longitudinal coordinates are (z, δ)
  - This extra term is called "ballistic drift": not in all codes!
    - Important at low to modest energies and for bunch compression
    - Relativistic terms enter converting momentum p to velocity v



Weak Dogleg



$\theta \ll 1$

$$\mathbf{M}_{\text{dogleg}} = \mathbf{M}_{\text{dipole}}(\rho, \theta)\mathbf{M}_{\text{drift}}\mathbf{M}_{\text{dipole}}(-\rho, -\theta)$$

3×3 $\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}$

$$\mathbf{M}_{\text{weak dogleg}} = \begin{pmatrix} 1 & \rho\theta & \frac{\rho\theta^2}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho\theta & \frac{\rho\theta^2}{2} \\ 0 & 1 & -\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Check sign

$$= \begin{pmatrix} 1 & L + 2\rho\theta & \boxed{-L\theta} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} (\eta, \eta')_{\text{in}} = 0 \\ (\eta, \eta')_{\text{out}} = (-L\theta, 0) \end{array}$$

$\eta'_x(\text{out}) = 0$

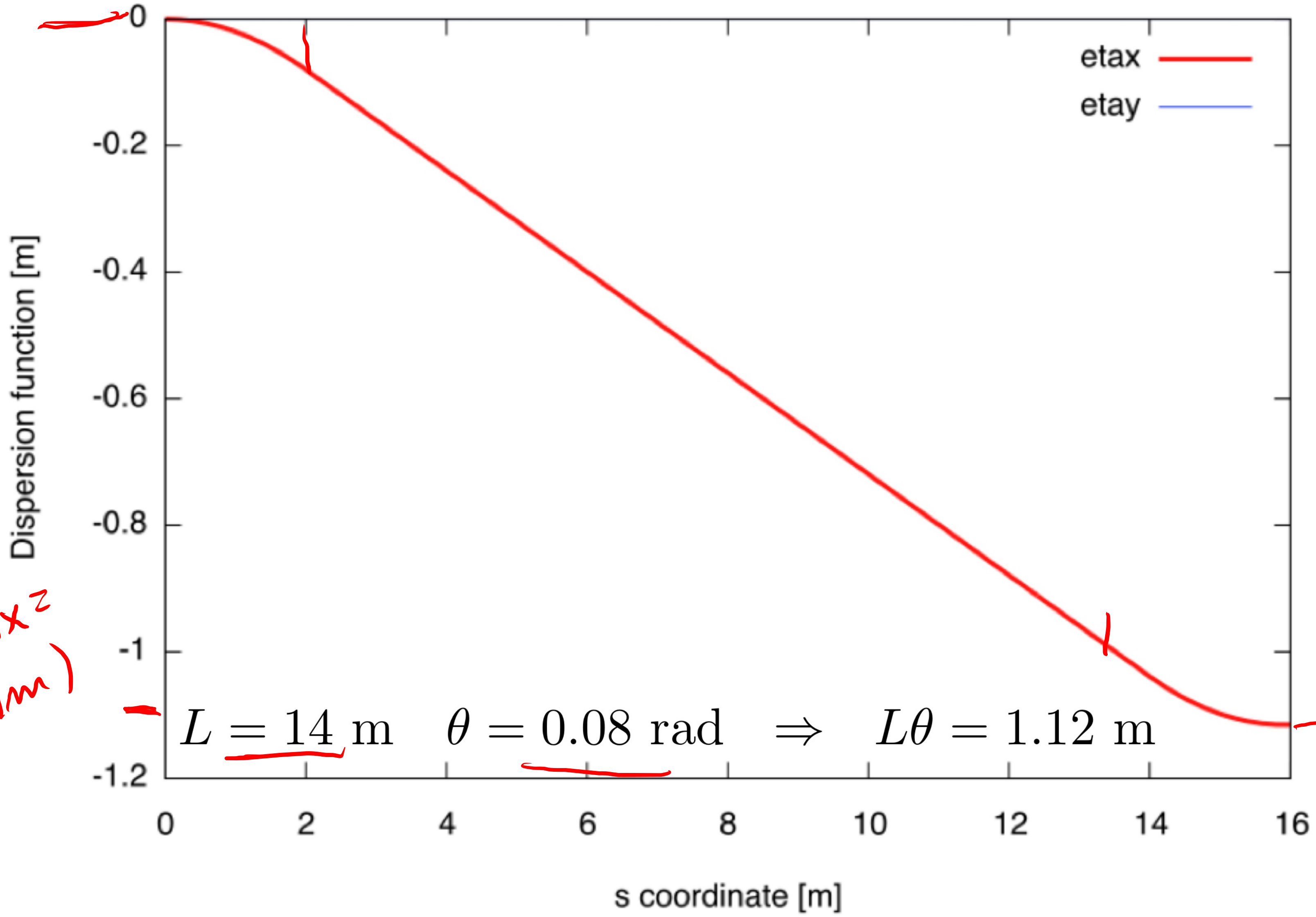
Strong dogleg can also be derived:

$$D = -L\cos\theta\sin\theta \qquad D' = \frac{L\sin^2\theta}{\rho}$$

Dogleg Dispersion

$\eta_x = 0$ m

$\eta_x' = 0$



$\eta_x = \delta(1\text{m})$

For weak dogleg

$$(\eta, \eta')_{\text{in}} = 0$$

$$\Rightarrow \quad (\eta, \eta')_{\text{out}} = (-L\theta, 0)$$

Does this make sense?

$$\Delta x'(\delta) = \frac{BL}{(B\rho)} = \frac{q}{p(1+\delta)}[BL] \approx \frac{q}{p}[BL](1-\delta) = (1-\delta)\Delta x'(\delta = 0)$$

A small momentum offset of $+\delta$ reduces the dipole kick by a factor of delta, and this is magnified to a transverse offset from design at the end of the dogleg by $-\delta L\theta$.

Achromatic Dogleg

- How can we make an achromatic dogleg?

IF $(\eta, \eta')_{\text{in}} = (0 \text{ m}, 0)$ THEN $\Rightarrow$ $(\eta, \eta')_{\text{out}} = (0 \text{ m}, 0)$ ACHROMATIC

- Use an I insertion (e.g. four consecutive π/2 insertions)

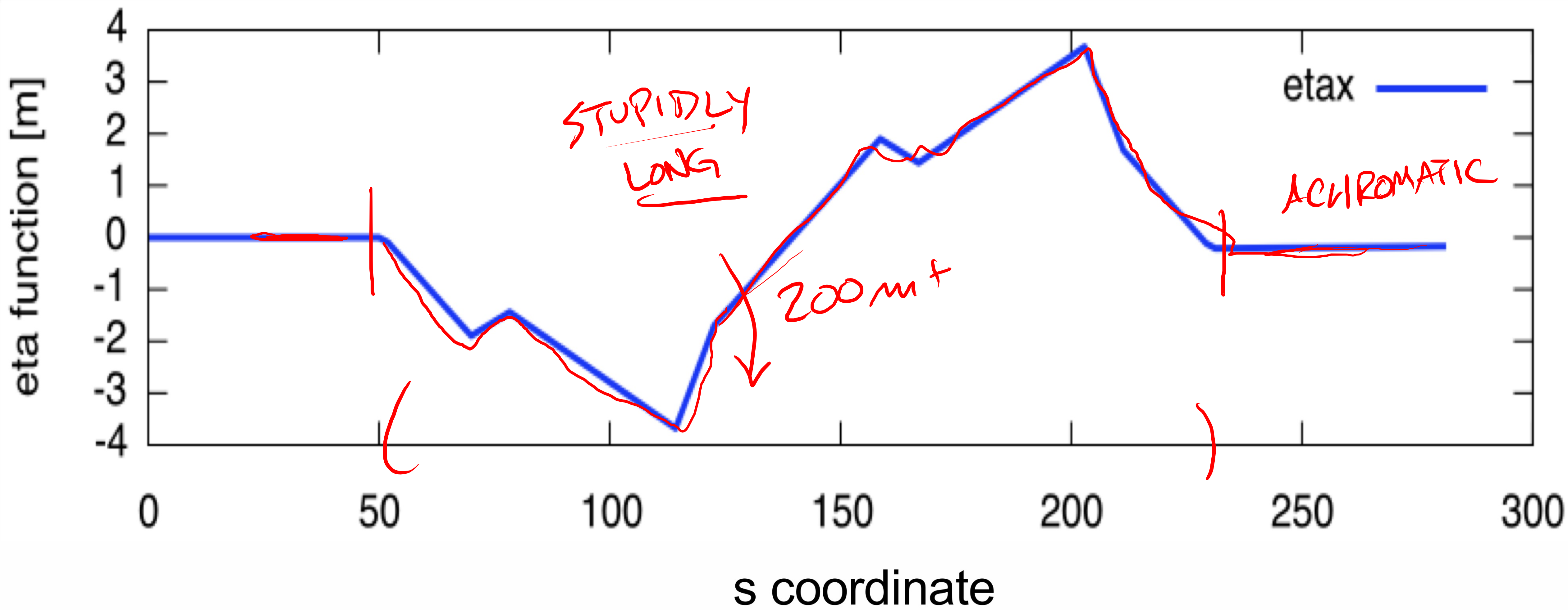
(NOT SAME AS D, D' = 0)

$$\mathbf{M}_{\pi/2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J} \qquad (\text{Recall } \mathbf{J}^4 = \mathbf{I})$$

$$\mathbf{M}_{\text{achromatic dogleg}} = \begin{pmatrix} \cos(2\theta) & \rho \sin(2\theta) & 0 \\ -\frac{\sin(2\theta)}{\rho} & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{achromatic!}$$

- **Any** transport with net phase advance of 2nπ will be achromatic (nπ if all dipoles bend in same direction)
  - common trick for matching dispersive bending arcs to non-dispersive straight sections.

Achromatic Dogleg



Achromatic Dogleg: Steffen CERN School Notes

Example of nondispersive translating system

↳ ACHROMATIC DOGLEG

Φ = sector magnet bend. angle

$\varphi = \ell\sqrt{k}$ = quadrupole magnet phase angle

d, λ = drift space lengths.

The system is nondispersive if the sinelike trajectory (with respect to the central symmetry point) goes through the mid-point of the bending magnets, i.e. if

$$\rho \tan\frac{\Phi}{2} + \lambda = \frac{1}{\sqrt{k}} \, \frac{d\sqrt{k}\cos\varphi + 2\sin\varphi}{d\sqrt{k}\sin\varphi - 2\cos\varphi} .$$

Focusing also in the other plane may be obtained by adding a third quadrupole of opposite polarity at the symmetry point.

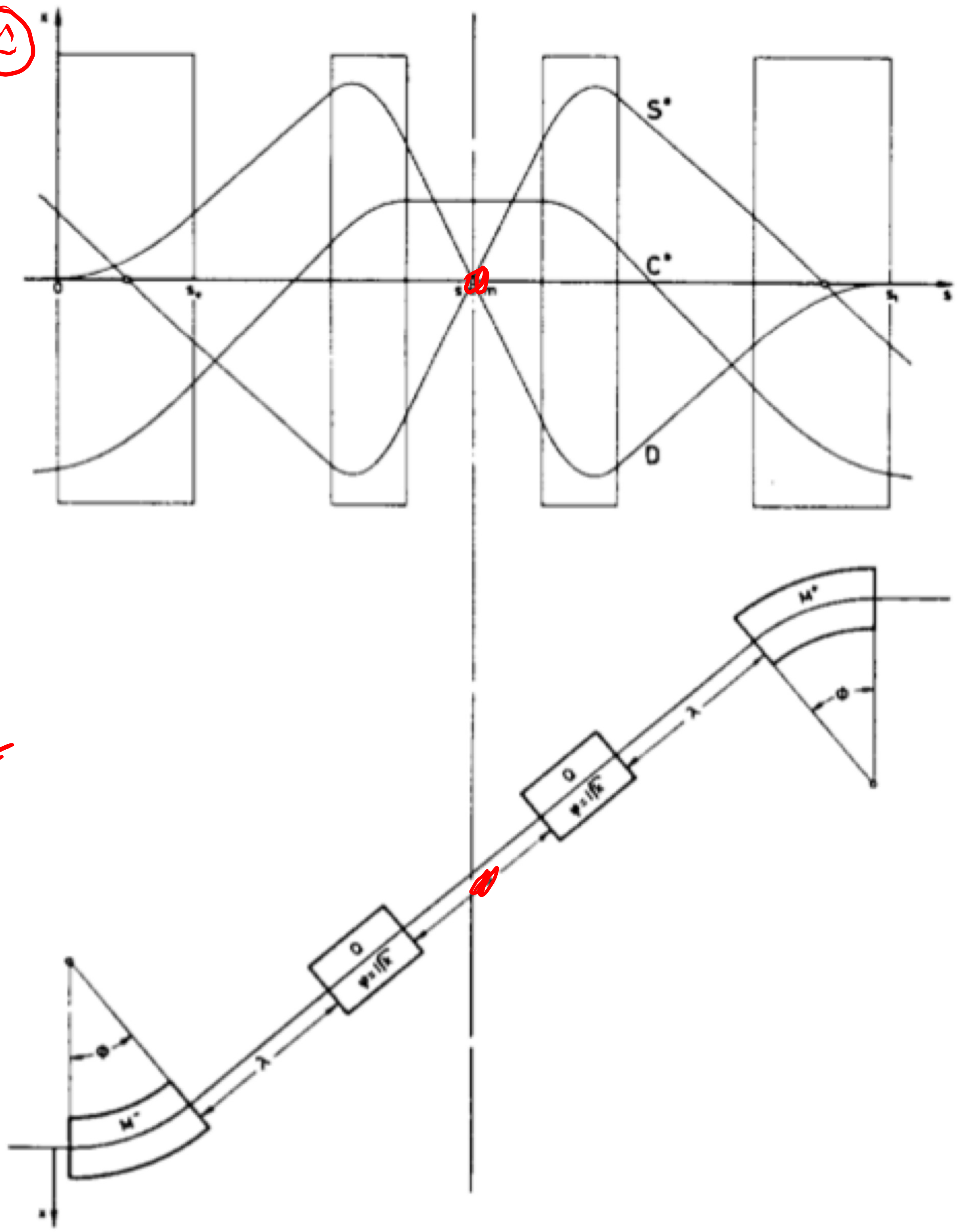


Fig. 15: Nondispersive translating system.

K. Steffen, CERN-85-19-V-1, 1985, p. 55

First-Order Achromat Theorem

- A lattice of n repetitive cells is achromatic (to first order, or in the linear approximation) iff $\mathbf{M}^n = I$ or each cell is achromatic
- Proof:

$$\text{Consider } \mathbf{R} \equiv \begin{pmatrix} \mathbf{M} & \bar{d} \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_2 = \mathbf{R} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_1 \qquad \bar{d} = \begin{pmatrix} M_{16} \\ M_{26} \end{pmatrix}$$

$$\textbf{For } n \text{ cells}: \quad \mathbf{R}^n = \begin{pmatrix} \mathbf{M}^n & (\mathbf{M}^{n-1} + \mathbf{M}^{n-2} + \ldots + \mathbf{I})\bar{d} \\ 0 & 1 \end{pmatrix}$$

$$\text{but} \quad (\mathbf{M}^{n-1} + \mathbf{M}^{n-2} + \ldots + \mathbf{I}) = (\mathbf{M}^n - I)(\mathbf{M} - I)^{-1}$$

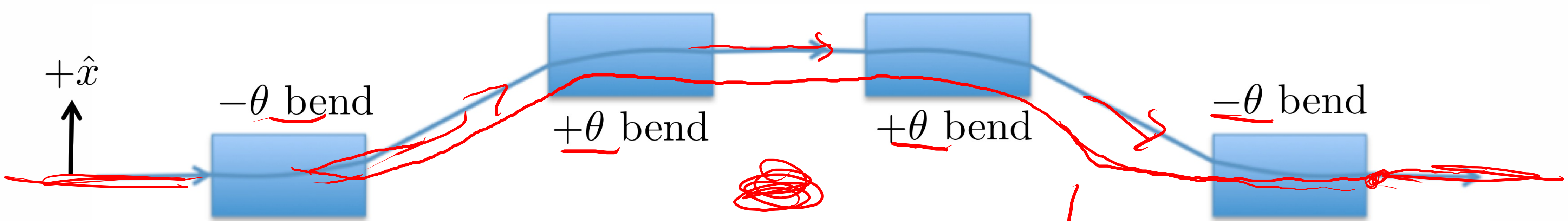
$$\text{So for } n \text{ cells}: \quad \mathbf{R}^n = \begin{pmatrix} \mathbf{M}^n & (\mathbf{M}^n - \mathbf{I})(\mathbf{M} - \mathbf{I})^{-1}\bar{d} \\ 0 & 1 \end{pmatrix}$$

- So the lattice is achromatic only if $\bar{d} = 0$ or $\mathbf{M}^n = \mathbf{I}$

$$\mathbf{M}^n = \mathbf{I} \cos \mu_{\text{tot}} + \mathbf{J} \sin \mu_{\text{tot}} \quad \Rightarrow \quad \mu_{\text{tot}} = 2\pi k$$

S.Y. Lee, "Accelerator Physics"

Chicane (DOUBLE DOGLEG)

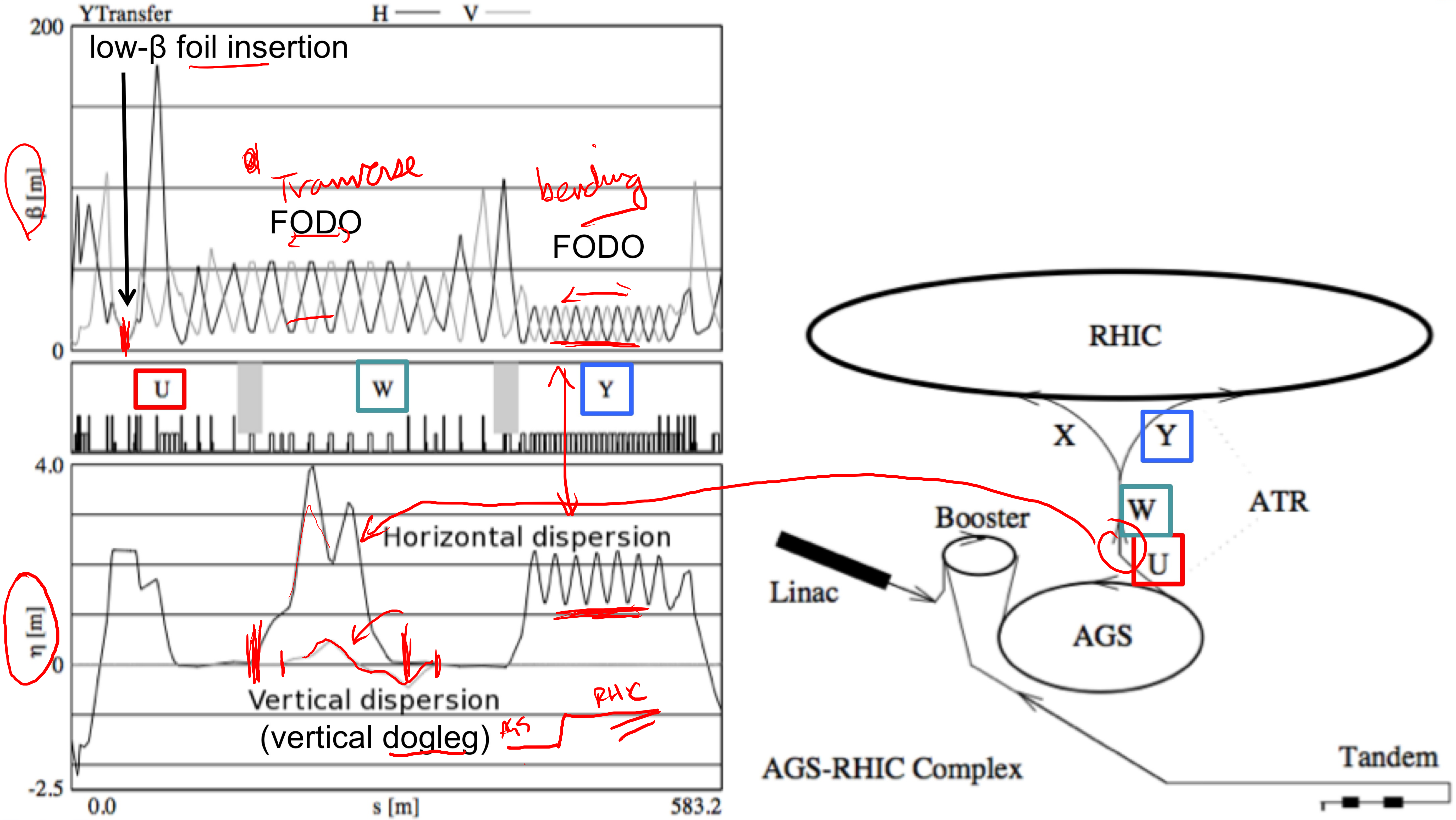


$\delta > 0 \Rightarrow$ length traversed $<$ L design

- Divert beam around an obstruction
  - e.g. vertical bypass chicane in Fermilab Main Ring
  - e.g. horizontal injection chicane in CEBAF recirculating linac
  - Essentially a design orbit "4-bump" (4 dipoles)
- Usually need some focusing, optics between dipoles
- Usually design optics to be achromatic
  - Operationally null orbit motion at end of chicane vs changes in input beam energy
- Naively expect $M_{56}<0$ (bunch lengthening or decompression)
  - Higher energy particles ($+\delta$) have shorter path lengths
  - But can compress bunches with introduction of longitudinal correlation

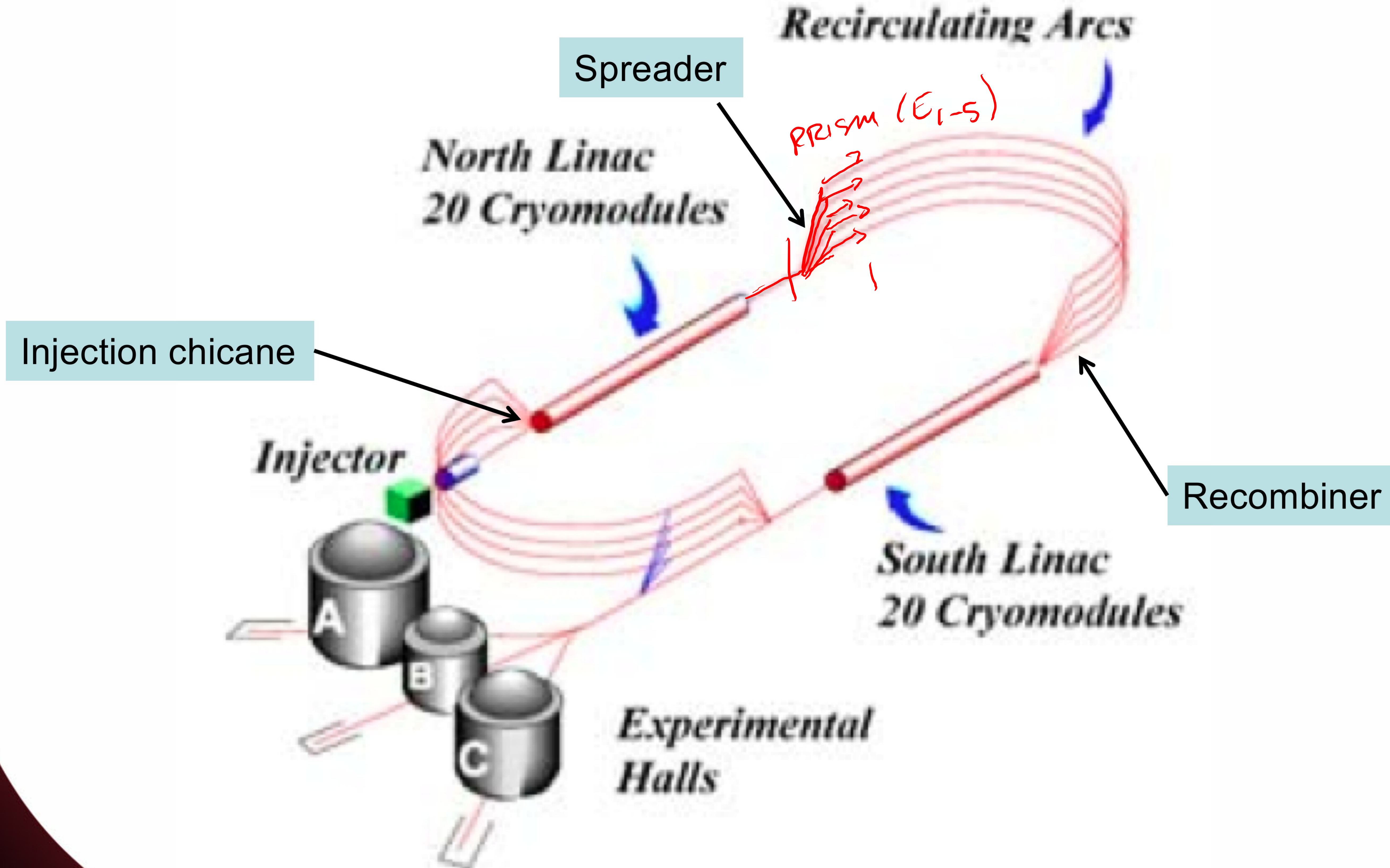
watch sign

AGS to RHIC (ATR) Transfer Line



The ATR vertical dogleg is not strictly a dogleg since the planes of the AGS and RHIC accelerators are not parallel

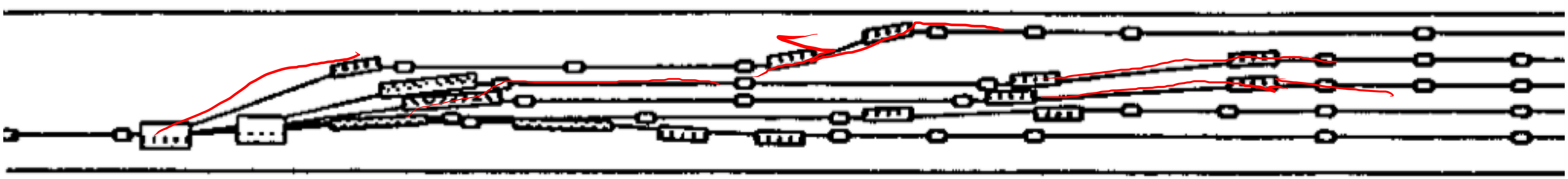
CEBAF



Recirculating Arcs
Spreader
PRISM (E1-5)
North Linac
20 Cryomodules
Injection chicane
Injector
Recombiner
South Linac
20 Cryomodules
A
B
C
Experimental
Halls

CEBAF Spreaders/Recombiners

- Problem: Separate different energy beams for transport into arcs of CEBAF, and recombine before next linac
  - Achromats: arcs are FODO-like, linacs are dispersion-free
  - "I" insertion: 1 betatron wavelength between dipoles
  - Single dogleg: unacceptably high beta functions
  - Two consecutive "staircase" doglegs with same total phase advance was solution

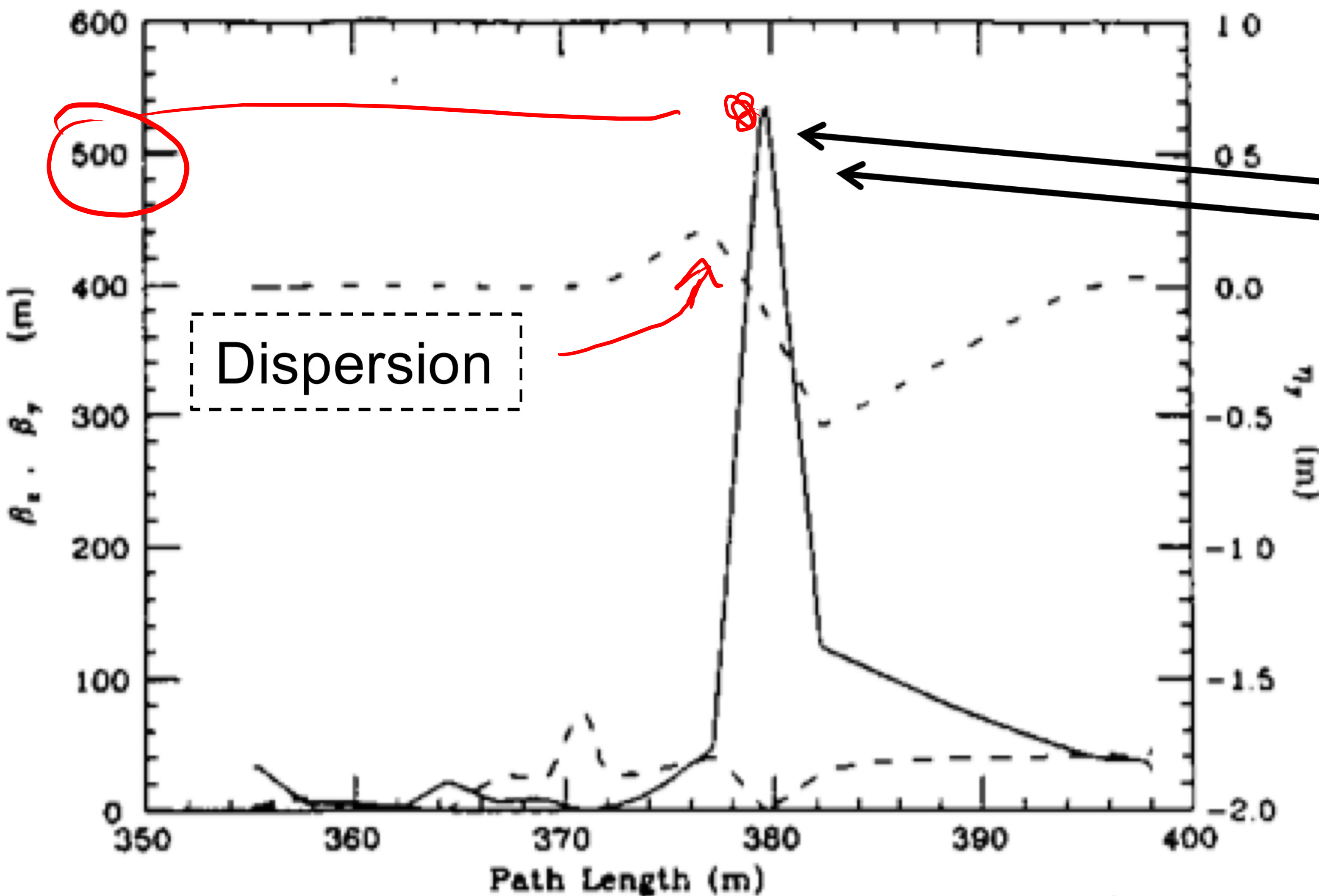


EAST ARC ELEVATION

  - Still quite a challenge in physical layout of real magnets!

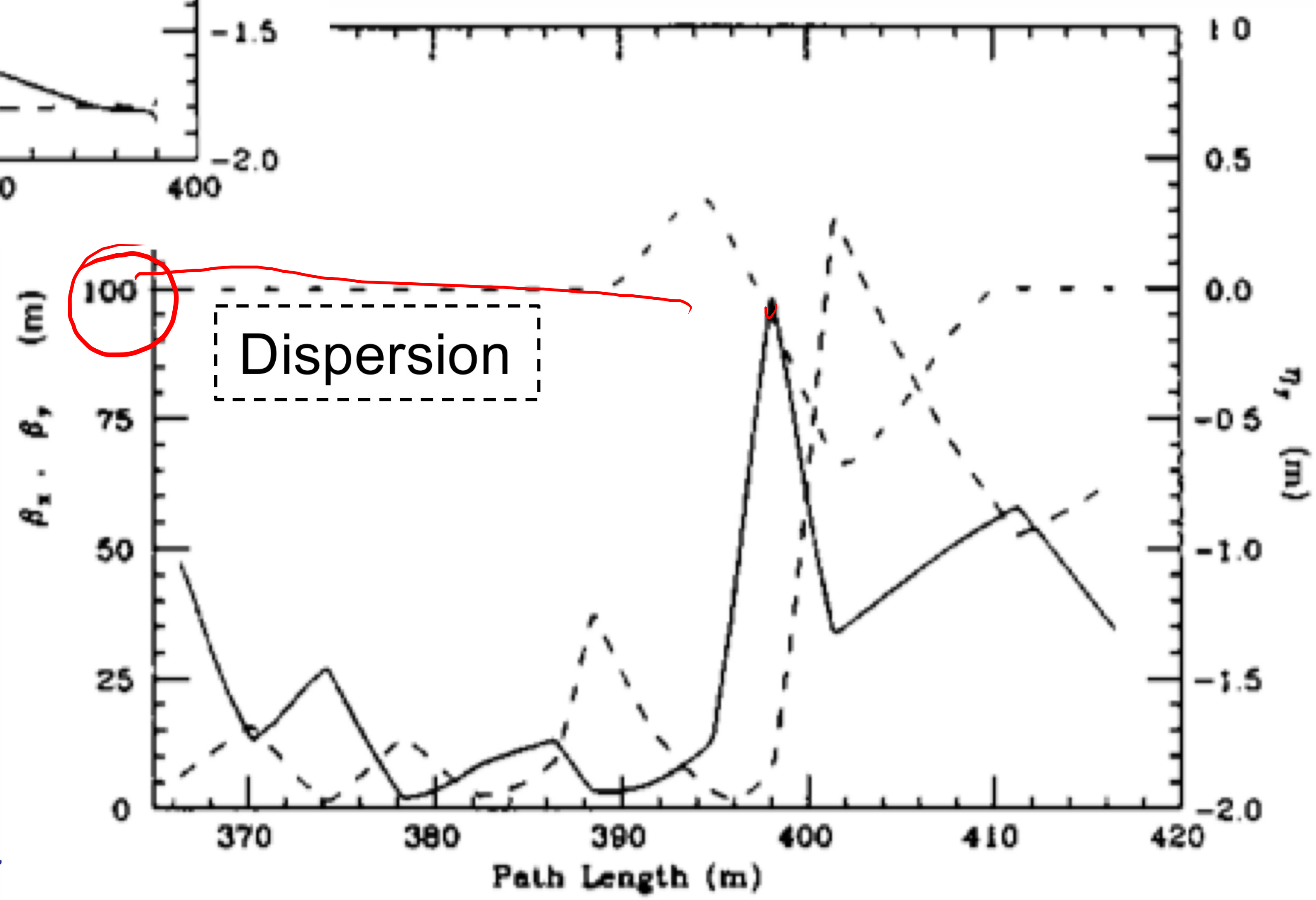
D. Douglas, R.C. York, J. Kewisch, "Optical Design of the CEBAF Beam Transport System", 1989

CEBAF Spreaders/Recombiners



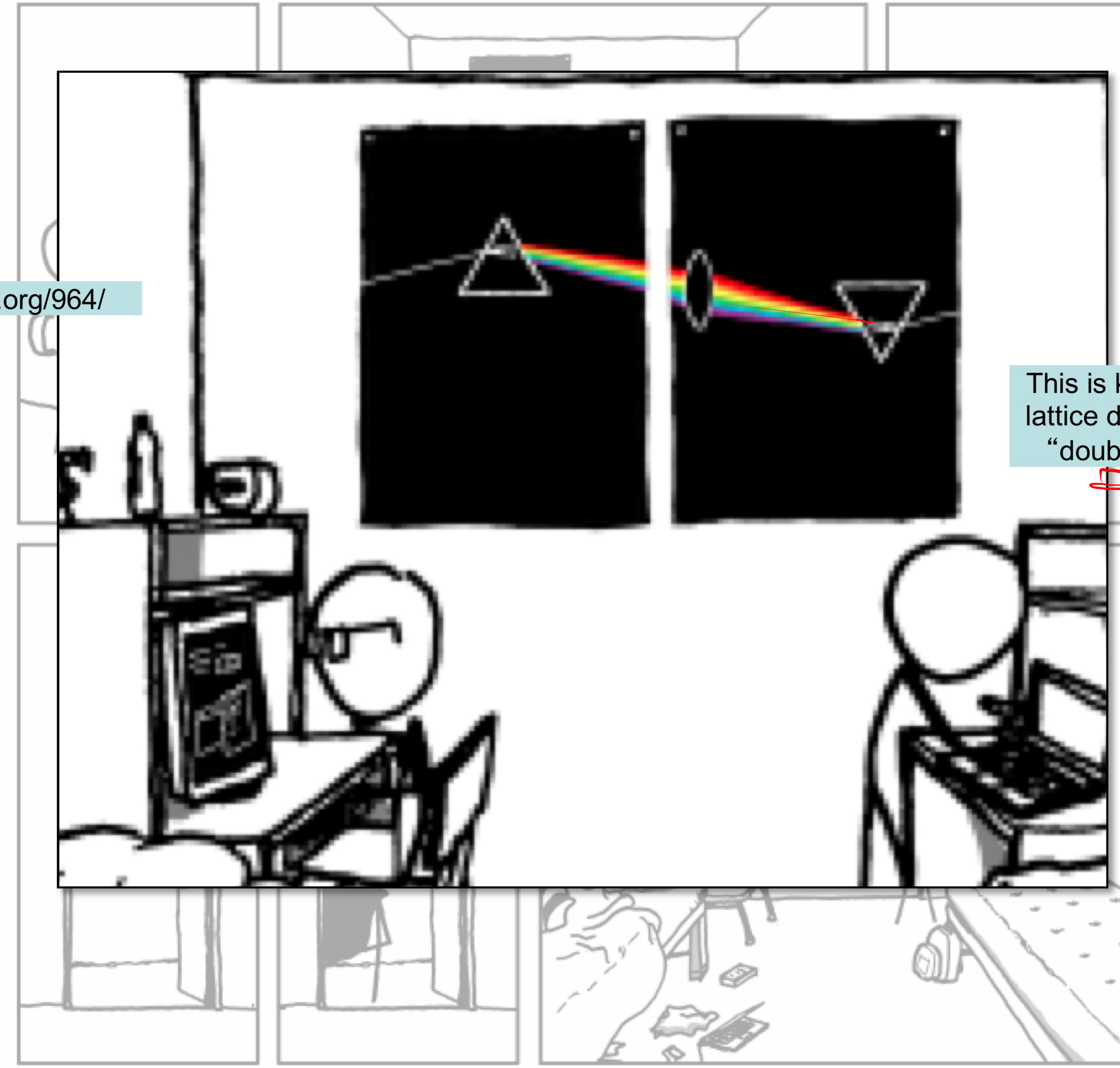
"One step" recombiner
Unacceptably large vertical beta function/beam size (550 m)

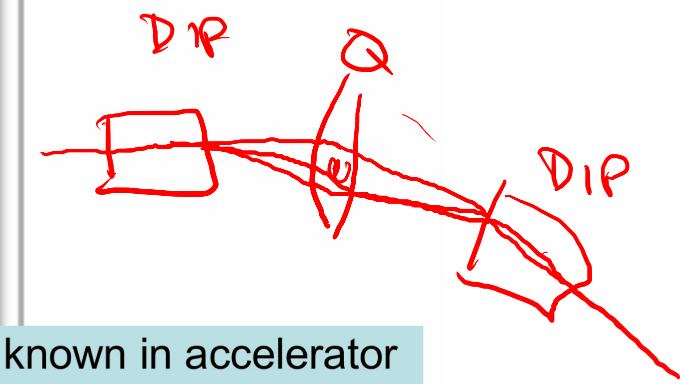
"Staircase" two-step recombiner
Acceptable beta functions and beam sizes in both planes (100 m)



(xkcd interlude)

http://www.xkcd.org/964/

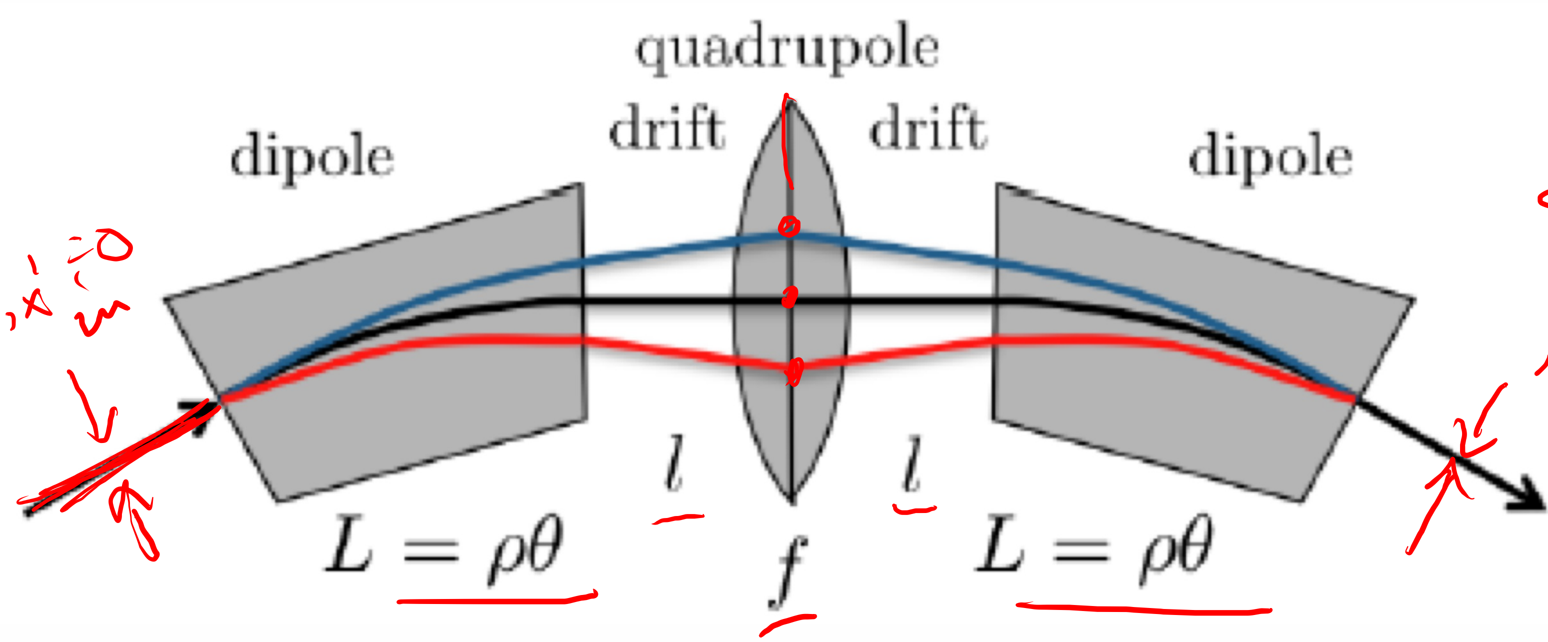




This is known in accelerator lattice design language as a "double bend achromat"

Double Bend Achromat (approximate)

$M_{x,x'}{}_{in} = 0$



$M_{x,x'}{}_{out} = 0$

symmetric

$\begin{pmatrix} M_x \\ M_x' \end{pmatrix}_{in} = 0 \Rightarrow \begin{pmatrix} M_x \\ M_x' \end{pmatrix}_{out} = 0$

- Let's calculate constraints for the double bend achromat

$$M_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho[1-\cos\theta] \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Keep lowest-order terms in θ, including θ^2 in upper right term since $\rho\theta = L$

$$M_{\text{dipole}} = \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

$\theta \ll 1$

Double Bend Achromat (approximate)

$$\mathbf{M}_{\rm DBA} = \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

MATHEMATICA ☺

$$\mathbf{M}_{\rm DBA} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}$$

PERIODIC?

NOT NECESSARILY

$$C = 1 - \frac{(L+l)}{f}$$

$$S = \frac{(L+l)(2f-L-l)}{f}$$

$$\boxed{D = \theta\frac{(L+l)(4f-L-2l)}{2f}}$$

$$C' = -\frac{1}{f}$$

$$S' = 1 - \frac{(L+l)}{f} = C$$

$$D' = \theta\frac{(4f-L-2l)}{2f} = \frac{D}{L+l}$$

Double Bend Achromat (approximate)

- The periodic solutions for dispersion for the general M matrix were shown in class earlier today

$$\eta(\text{periodic}) = \frac{[1-S']D + SD'}{2(1-\cos\mu)} = 0$$

$$\eta'(\text{periodic}) = \frac{[1-C]D' + C'D}{2(1-\cos\mu)} = 0$$

- It turns out that the η' equation is satisfied automatically!
    - This is a consequence of the mirror symmetry of the system
- The η equation is satisfied if D=0:

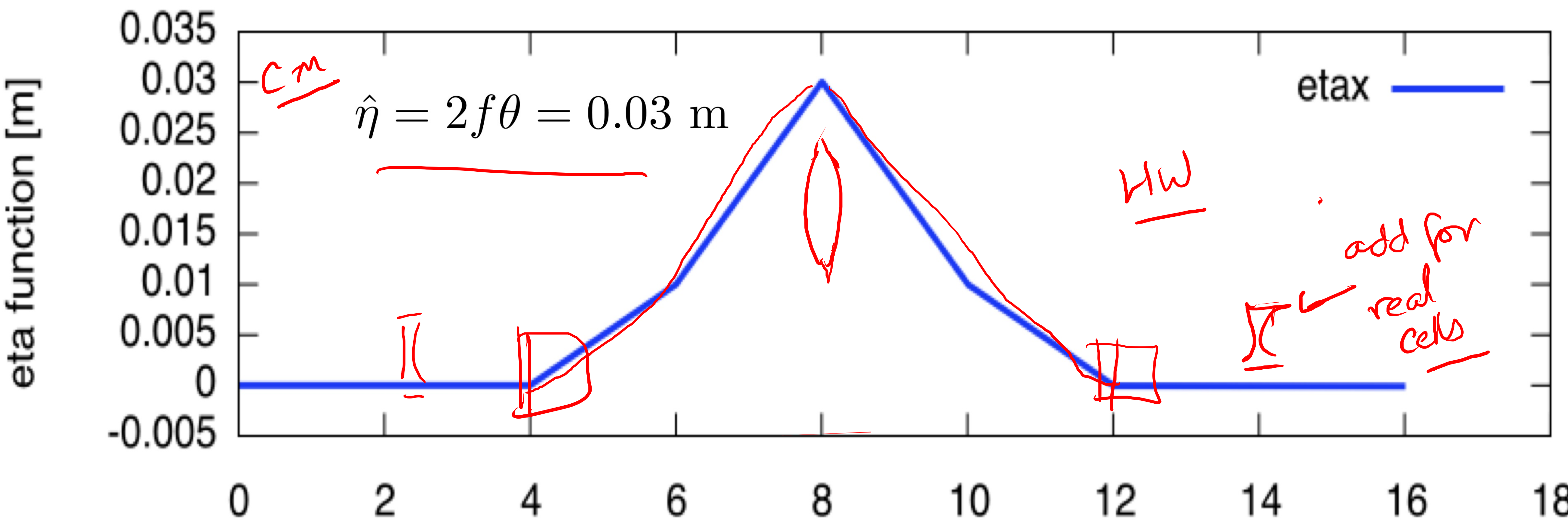
$$D = \theta\frac{(L+l)(4f-L-2l)}{2f} = 0$$

independent of θ

DBA constraint

$$\Rightarrow 4f - L - 2l = 0 \quad \Rightarrow \quad f = \frac{L+2l}{4} \qquad \hat{\eta} = \frac{(L+2l)\theta}{2} = 2f\theta$$

Double Bend Achromat



Best of FODO:
periodic b.c with bends

⇓ BUT CAN'T BE BUILDING BLOCK

$$L = l = 2 \text{ m} \quad \theta = 0.01 \text{ rad} \quad f = \frac{L+2l}{4} = 1.5 \text{ m} \quad (KL_{\text{quad}}) = 0.667 \text{ m}^{-1}$$

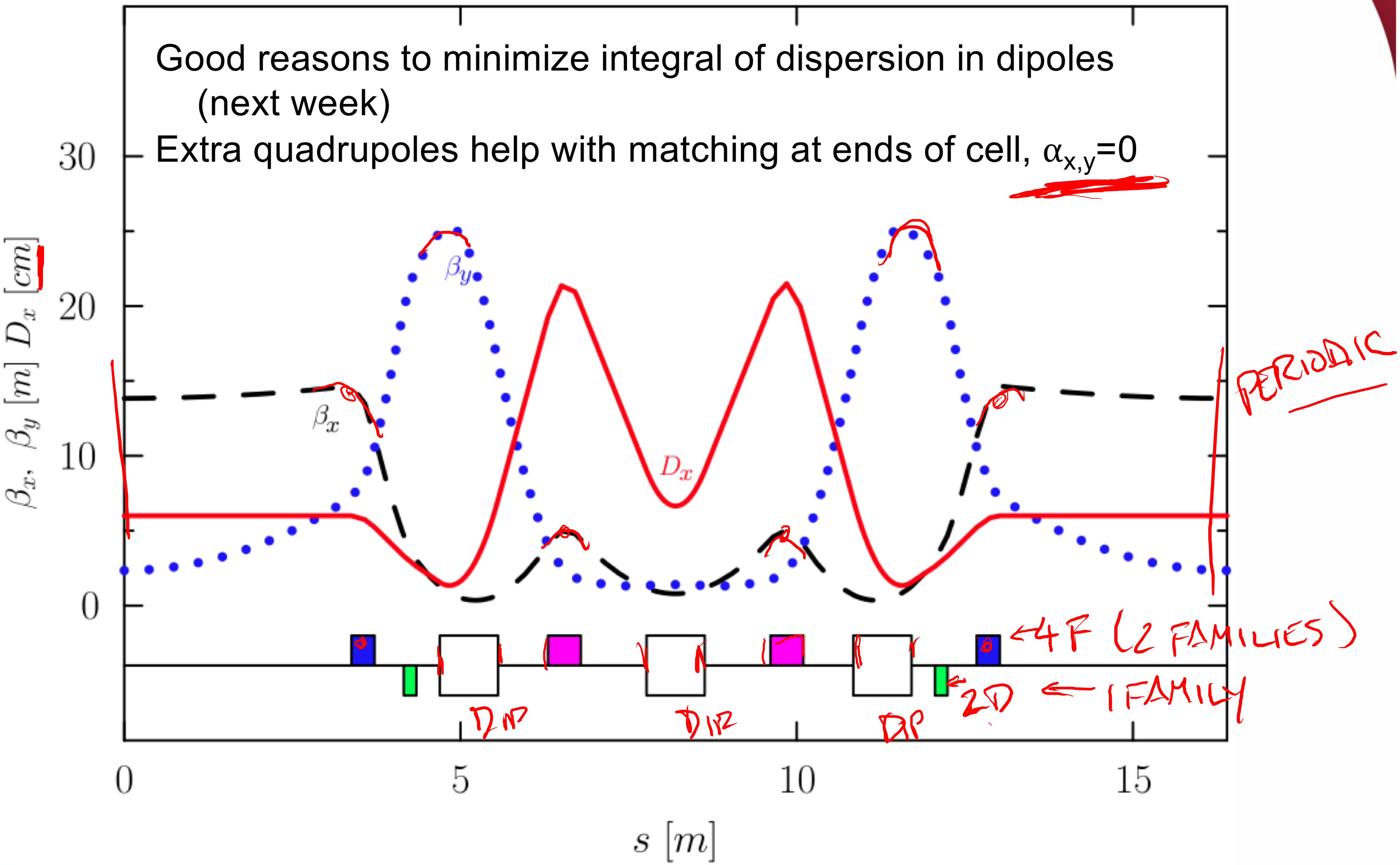
$$\text{Exact DBA}: \quad f = \frac{l}{2} + \frac{\rho}{2}\tan(\theta/2) \qquad \hat{\eta} = \rho(1-\cos\theta) + l\sin\theta$$

- DBA is also known as a Chasman-Green lattice
  - Used in early third-generation light sources (e.g. NSLS at BNL)
  - More after we discuss synchrotron radiation, $\mathcal{H}$ functions

Triple Bend Achromat Cell (ALS at LBL)

Good reasons to minimize integral of dispersion in dipoles (next week)

Extra quadrupoles help with matching at ends of cell, $\alpha_{x,y}=0$



L. Yang et al, Global Optimization of an Accelerator Lattice Using Multiobjective Genetic Algorithms, 2009

MOBIUS WAS x.y exchange, purely transverse

Transverse/Longitudinal Emittance Exchange

MODERN

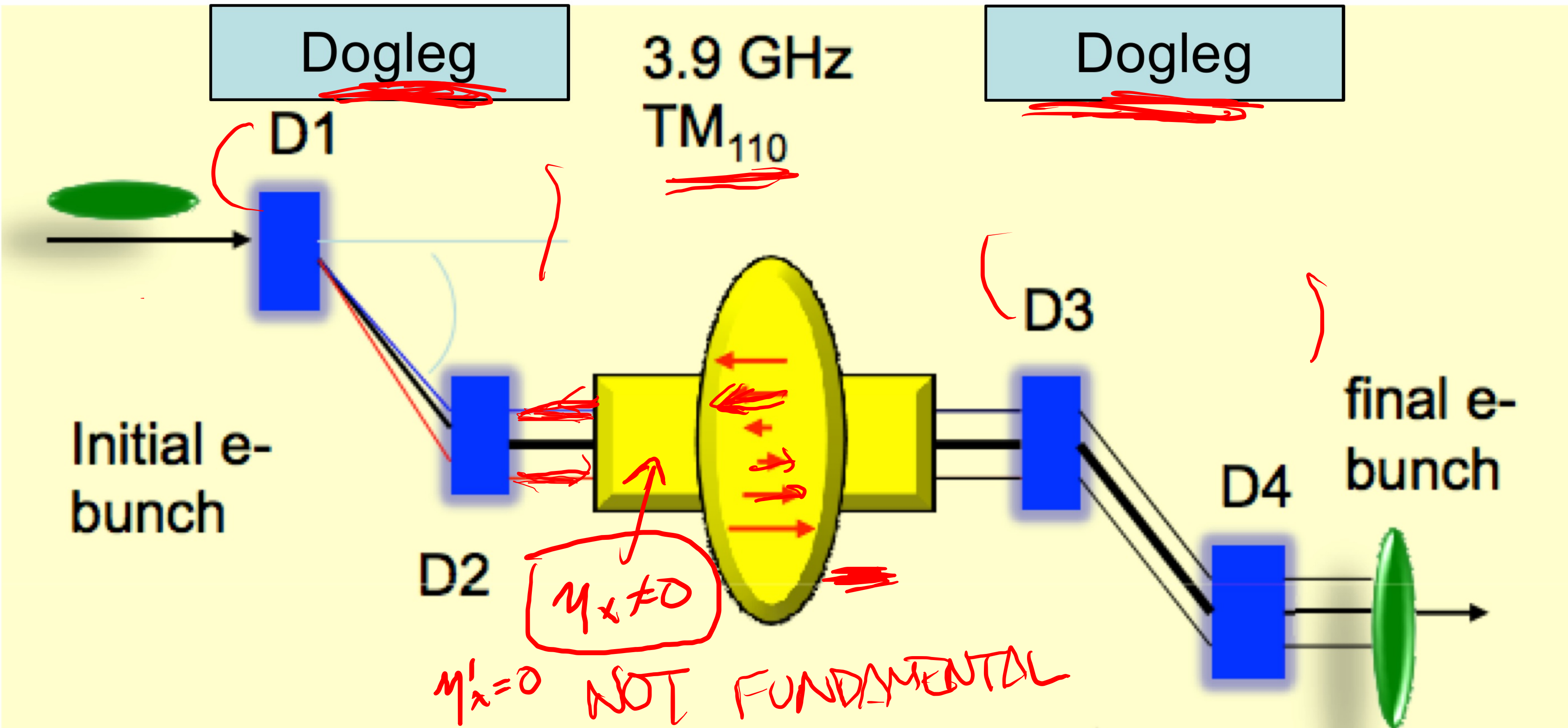
- X-ray FELs demand ultra-low transverse emittance beam*

e^- source: small transeverse size, smaller longitudinal (laser)

- State-of-the art photo-injectors can generate low 6-D emittance. Typically asymmetric emittances. Emittance exchange can swap transverse with the longitudinal emittance.
- Allows one to convert transverse modulations to longitudinal modulations : Beam shaping application
- Can also be used to suppress microbunching instability**

J.C.T. Thangaraj, Experimental Studies on an Emittance Exchange Beamline at the A0 Photoinjector, 2012

Fermilab A0 Emittance Exchanger



NOT TM_{010}

θ : Bending angle

η : dogleg dispersion

L : dogleg length

L_c : RF cell length ???

$$\mathbf{M} = \begin{pmatrix} 1 & \frac{L_c}{4} & -\frac{(4L+L_c)}{4\eta} & \eta - \frac{\theta(4L+L_c)}{4} \\ 0 & 1 & -\frac{1}{\eta} & -\theta \\ -\theta & \eta - \frac{\theta(4L+L_c)}{4} & 1 + \frac{\theta L_c}{4\eta} & \frac{\theta^2 L_c}{4} \\ -\frac{1}{\eta} & -\frac{4L+L_c}{4\eta} & \frac{\theta L_c}{4\eta^2} & 1 + \frac{\theta L_c}{4\eta} \end{pmatrix} \begin{pmatrix} x \\ x' \\ z \\ \delta \end{pmatrix}$$

??

J.C.T. Thangaraj, Experimental Studies on an Emittance Exchange Beamline at the A0 Photoinjector, 2012

TM110 RF Cavity Mode

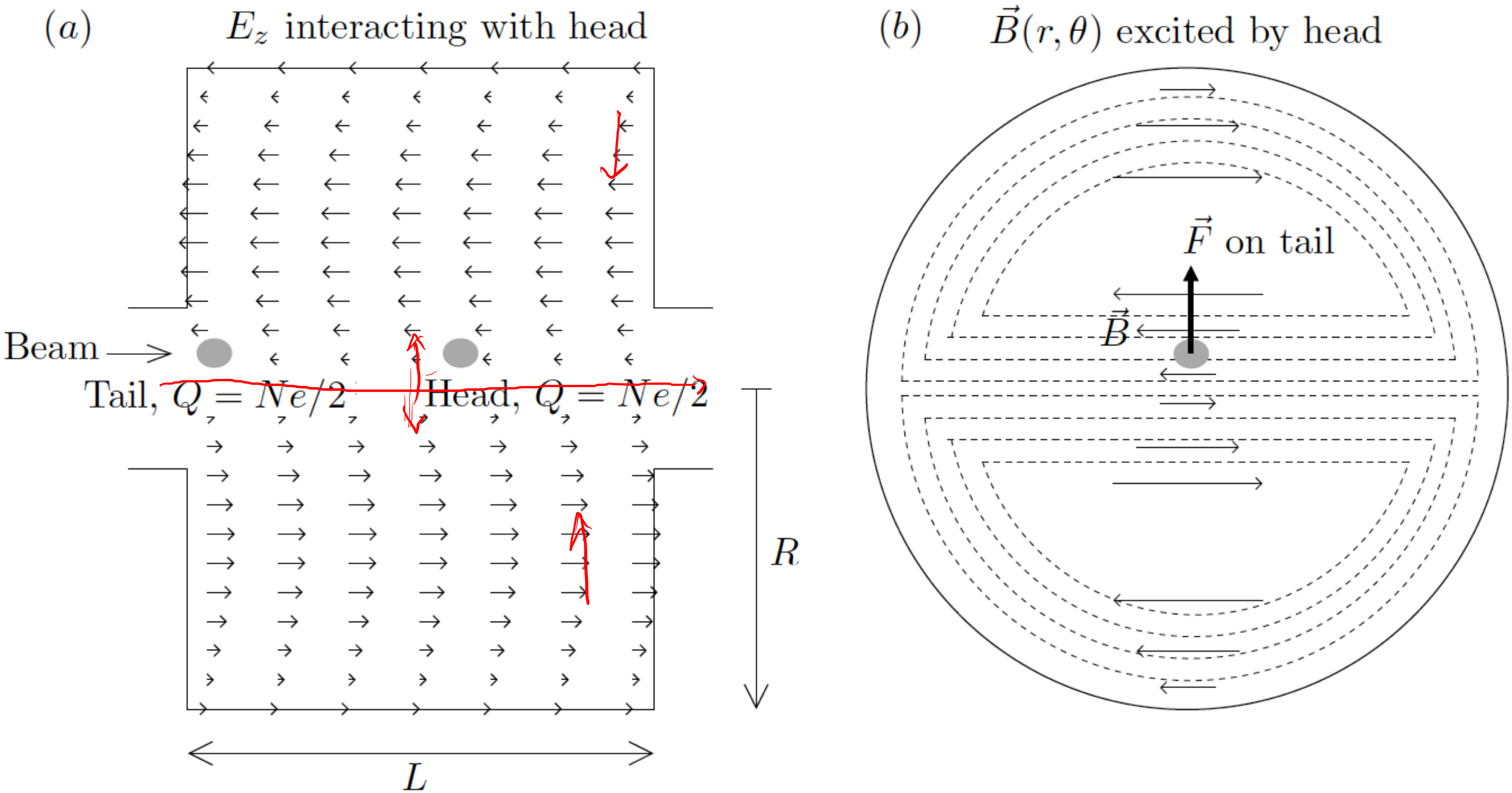


Figure 14.6 in textbook

Chicane Style Emittance Exchange

DAO XIANG *et al.* Phys. Rev. ST Accel. Beams **14**, 114001 (2011)

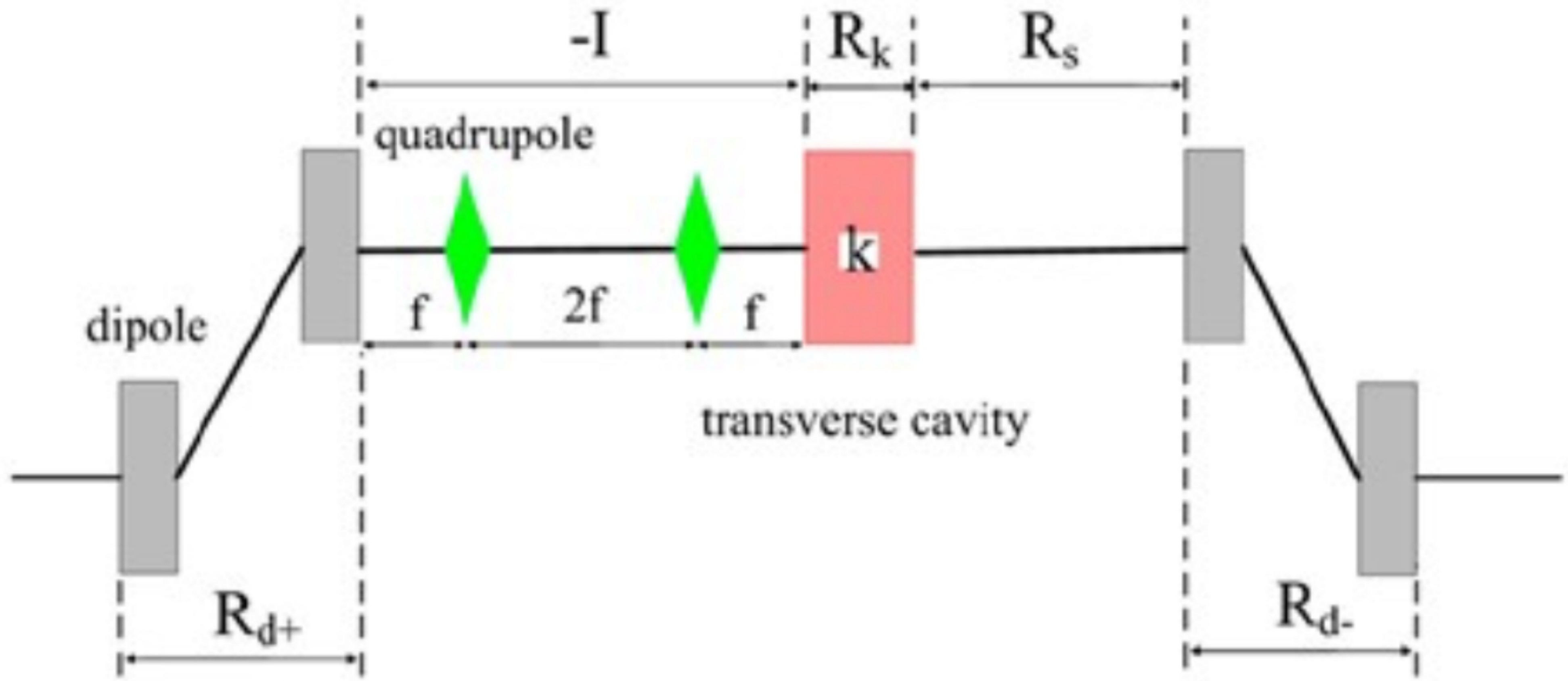


FIG. 2. A chicane-type exact EEX beam line. Two quadrupoles (green diamonds) are put upstream of the transverse cavity to reverse the dispersion.

- Reversing dispersion before the TM cavity allows you to flip the second dogleg to make a chicane
  - More transversely compact emittance exchange