Sextupoles and Chromaticity Part II

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USPAS, Accelerator Physics
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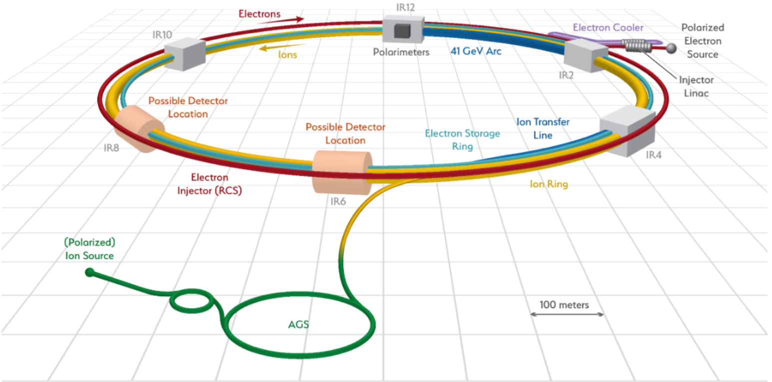
The Electron-Ion Collider (EIC)

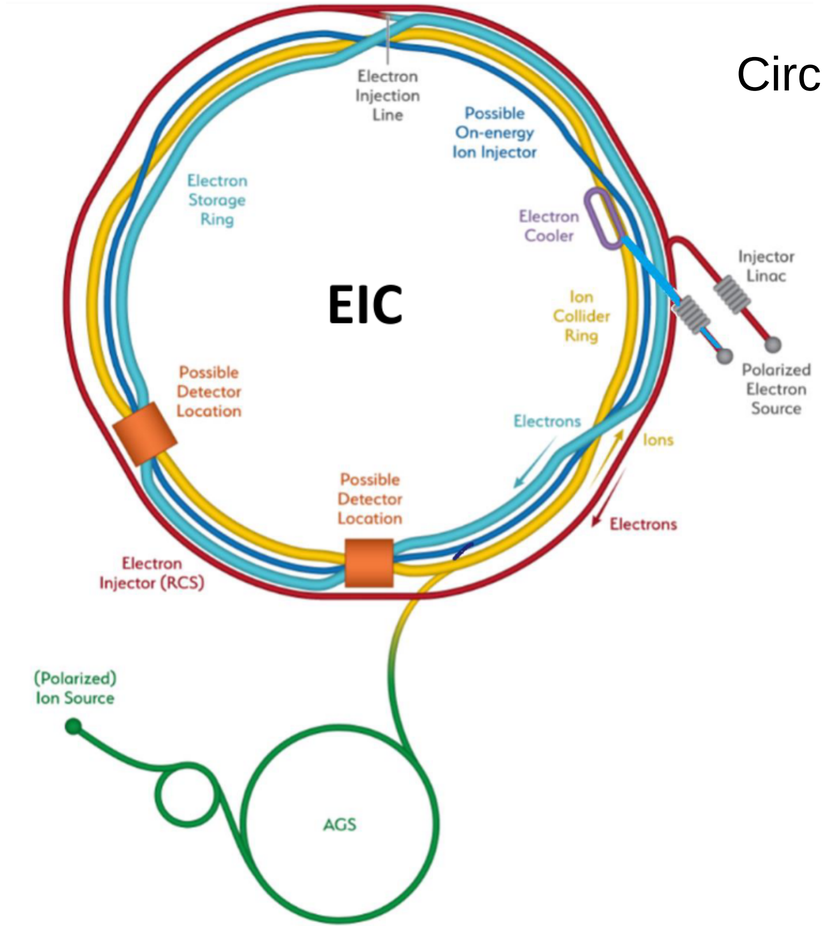
Hadron storage ring 40-275 GeV
(existing: RHIC)
Electron storage ring (ESR) 2.5–18 GeV
(new)
Electron rapid cycling synchrotron
(new)

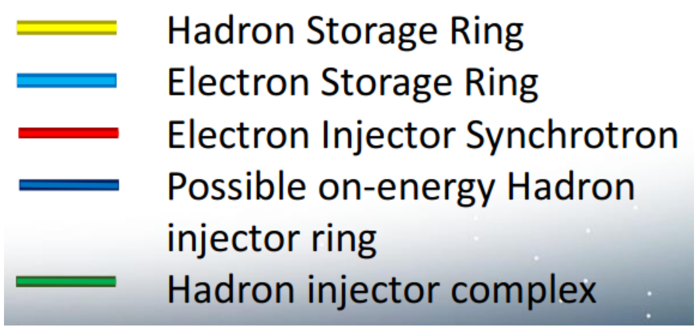
High luminosity interaction region(s)
(new)
- L = 10^{34} cm^{-2} s^{-1}
- Superconducting magnets
- 25 mrad crossing angle with crab cavities
- Spin rotators (longitudinal spin)



The Electron-Ion Collider (EIC)

Circumference = 3.8 km

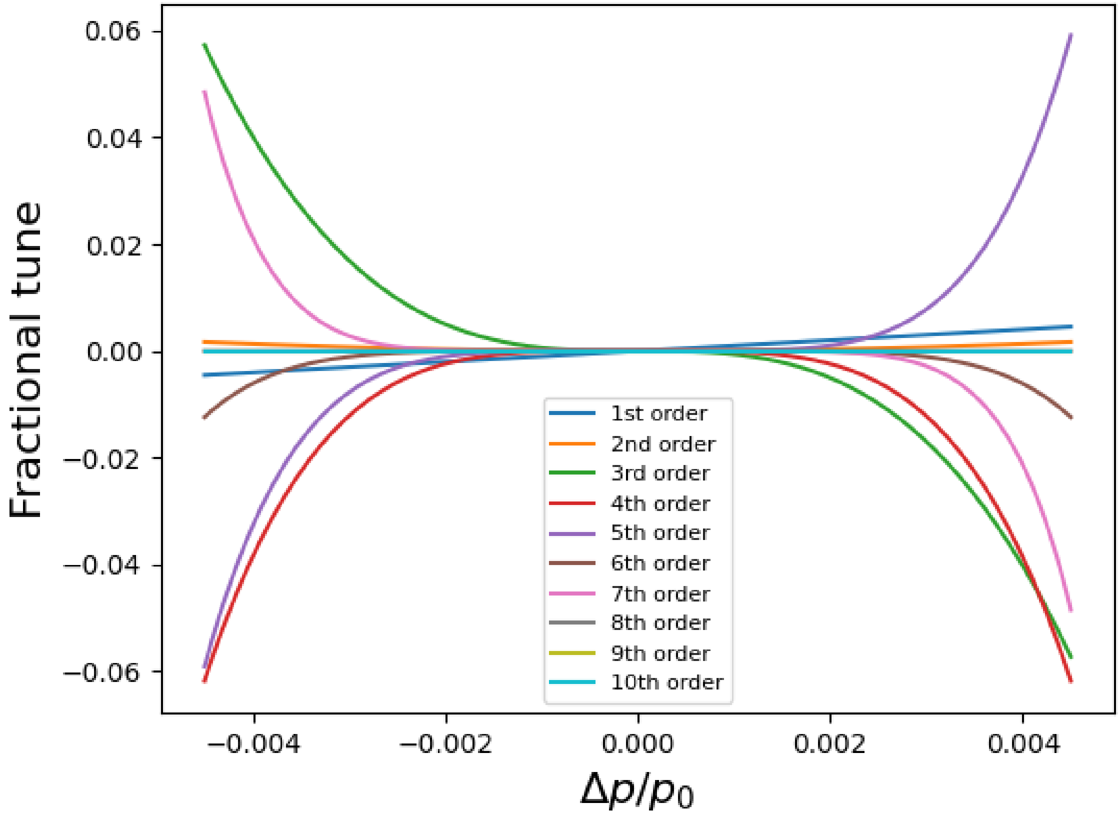




Outline

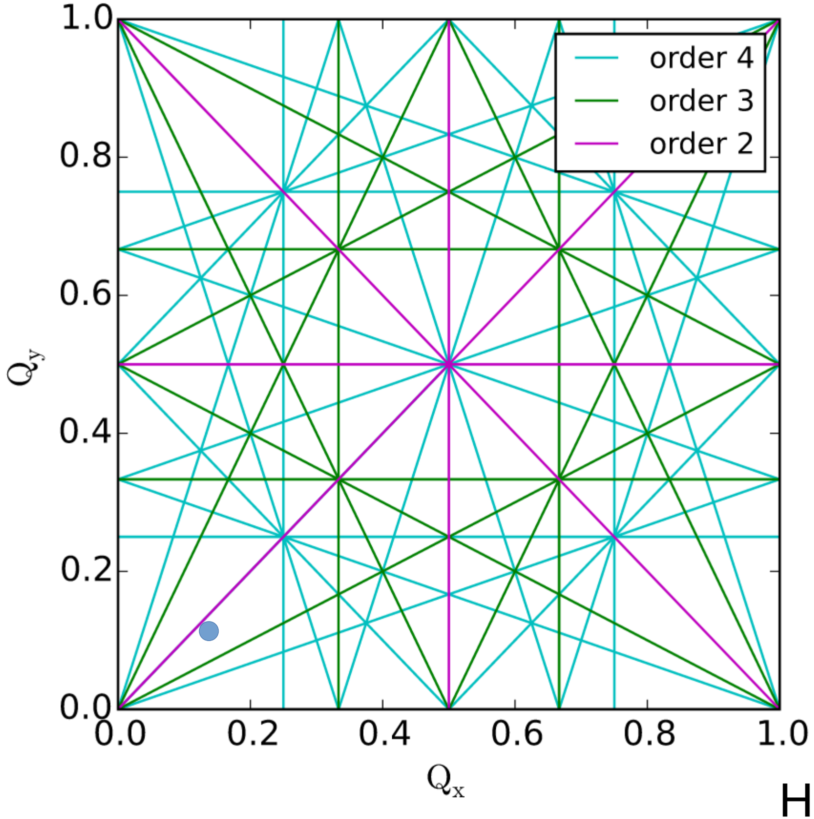
- Recap of yesterday’s lecture
- Derivation of 1st & 2nd order chromaticity
- W-vector (Montague) formalism
- Chromaticity correction in a collider
- Chromaticity correction and dynamic aperture in the EIC ESR

Chromaticity is the variation of tune with momentum – and higher orders do matter!



Example of EIC ESR lattice with linear chromaticity corrected

We need to minimize tune variation with momentum to increase momentum acceptance.



Homework 9

Chromaticity: $\chi \equiv \frac{\mathrm{d}Q}{\mathrm{d}\delta}$

The natural chromaticity is always negative. Why?

$$\chi_{\mathrm{nat}} = \frac{N}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}\delta} = -\frac{N}{2\pi}\tan(\phi/2) = -Q\frac{\tan(\phi/2)}{\phi/2}$$

This equation applies to an ideal lattice of FODO cells. In a collider, the interaction points (IPs) also contribute strongly.

To which value do we want to set linear chromaticity?

Chromaticity can be corrected with sextupoles.

$$\Delta x'_{TOT} = \frac{-q}{(1+\delta)}\, x_{TOT} \qquad \Delta x'_{TOT} = \frac{-S}{(1+\delta)}\, x^2_{TOT}$$

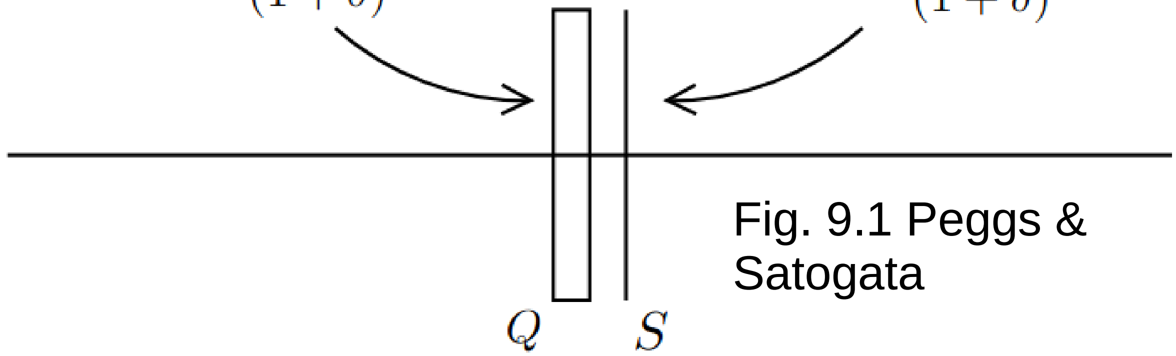


Fig. 9.1 Peggs & Satogata

Net linear kick is: $\Delta x' = -q.x + (q - 2\eta S)x\delta$

So can correct chromaticity by setting sextupole strengths to: $S = \dfrac{q}{2\eta}$

Sextupoles are grouped into families (all with the same strength).

A sext family that increases the **horizontal** chromaticity necessarily decreases the **vertical** chromaticity.

At least 2 families of sextupoles are needed to set both horizontal and vertical chromaticities to small values.

Sextupoles can only correct the chromaticity if placed in a **dispersive section**. The closer they are placed to quads, the better the correction.

There are 4 kinds of phase-space trajectories that are observed in simulations:

- Regular non-resonant trajectories
- Regular resonant trajectories
- Rapidly divergent regular trajectories
- Chaotic trajectories

Which ones will result in particle loss?

What we ultimately want is a large stable region in phase space in which particles survive – dynamic aperture.

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Derivation of 1st & 2nd order chromaticity

Hill's equation:

$$z'' + Kz = 0 \qquad z = x \text{ or } y$$

Solution:

$$\vec{Z}(s) = M(s_0; s)\vec{Z}(s_0) \qquad \vec{Z}(s) \equiv \begin{pmatrix} z(s) \\ z'(s) \end{pmatrix}$$

where

$$\begin{cases} M(s_0; s_0) & = I \\ \frac{\mathrm{d}M}{\mathrm{d}s} & = AM = \left[A^{(0)} + \sum_{n=1}^{\infty} A^{(n)}\delta^n\right]\left[M^{(0)} + \sum_{n=1}^{\infty} M^{(n)}\delta^n\right] \end{cases}$$

$$A^{(0)} \equiv \begin{pmatrix} 0 & 1 \\ -K^{(0)} & 0 \end{pmatrix} \; ; \; A^{(1)} \equiv \begin{pmatrix} 0 & 0 \\ -K^{(1)} & 0 \end{pmatrix} \; ; \; A^{(2)} \equiv \begin{pmatrix} 0 & 0 \\ -K^{(2)} & 0 \end{pmatrix} \; ; \; \ldots$$

ΔK represents quadrupolar field errors

$\mathrm{M}(\mathrm{s}_1; s_2)$ is fractional-turn matrix from s_1 to s_2

$M^{(n)}$ is n$^{\text{th}}$ order expansion in δ

From previous slide:
$$\begin{cases} M(s_0; s_0) & = I \\ \frac{\mathrm{d}M}{\mathrm{d}s} & = AM = \left[A^{(0)} + \sum_{n=1}^{\infty} A^{(n)} \delta^n\right] \left[M^{(0)} + \sum_{n=1}^{\infty} M^{(n)} \delta^n\right] \end{cases}$$

So:
$$\begin{cases} M^{(0)}(s_0; s_0) & = I \\ \frac{\mathrm{d}M^{(0)}}{\mathrm{d}s} & = A^{(0)} M^{(0)} \end{cases}$$

$$\begin{cases} M^{(n)}(s_0; s_0) & = \mathbf{0} \\ \frac{\mathrm{d}M^{(n)}}{\mathrm{d}s} & = A^{(0)} M^{(n)} + \sum_{j=0}^{n-1} A^{(n-j)} M^{(j)} \end{cases} , n \geq 1$$

e.g.
$$\frac{\mathrm{d}M^{(1)}}{\mathrm{d}s} = A^{(0)} M^{(1)} + A^{(1)} M^{(0)}$$
$$\frac{\mathrm{d}M^{(2)}}{\mathrm{d}s} = A^{(0)} M^{(2)} + A^{(1)} M^{(1)} + A^{(2)} M^{(0)}$$

The following relation holds:

$$\boxed{M^{(n)}(s_0; s) = M^{(0)}(s_0; s) \int_{s_0}^{s} \mathrm{d}s' M^{(0)}(s'; s_0) \sum_{j=0}^{n-1} A^{(n-j)}(s') M^{(j)}(s_0; s')}$$

e.g.
$$M^{(1)}(s_0; s) = M^{(0)}(s_0; s) \int_{s_0}^{s} M^{(0)}(s'; s_0) A^{(1)}(s') M^{(0)}(s_0; s') \mathrm{d}s'$$
$$M^{(2)}(s_0; s) = M^{(0)}(s_0; s) \left[\int_{s_0}^{s} M^{(0)}(s'; s_0) A^{(2)}(s') M^{(0)}(s_0; s') \mathrm{d}s' + \int_{s_0}^{s} M^{(0)}(s'; s_0) A^{(1)}(s') M^{(1)}(s_0; s') \mathrm{d}s'\right]$$

$$\cos(\mu) = \frac{1}{2}\mathrm{Tr}[M(0;C)]$$

Taylor series: $$\cos(\mu) = \cos(\mu^{(0)}) - \sin(\mu^{(0)})\Delta\mu - \cos(\mu^{(0)})\frac{(\Delta\mu)^2}{2} + \ldots$$

Select 1st order:

$$\mu = \mu^{(0)} + \Delta\mu = \mu^{(0)} + \mu^{(1)}\delta + \mu^{(2)}\delta^2 + \ldots$$

$$\begin{aligned}
-\mu^{(1)}\sin\left(\mu^{(0)}\right) &= \frac{1}{2}\mathrm{Tr}\left[\int_0^C M^{(0)}(s_1;C)A^{(1)}(s_1)M^{(0)}(0;s_1)\mathrm{d}s_1\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\int_0^C M^{(0)}(0;s_1)M^{(0)}(s_1;C)A^{(1)}\mathrm{d}s_1\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\int_0^C M^{(0)}(0;C)A^{(1)}\mathrm{d}s_1\right] \\
&= -\frac{1}{2}\int_0^C \beta^{(0)}(s)\sin(\mu^{(0)})K^{(1)}(s)\mathrm{d}s
\end{aligned}$$

$$Tr[XYZ] = Tr[ZXY]$$

$$M(0;C) = \begin{pmatrix} \cos(\mu) + \alpha\sin(\mu) & \beta\sin(\mu) \\ -\gamma\sin(\mu) & \cos(\mu) - \alpha\sin(\mu) \end{pmatrix}$$

$$A^{(1)} \equiv \begin{pmatrix} 0 & 0 \\ -K^{(1)} & 0 \end{pmatrix}$$

So:

$$\boxed{\mu^{(1)} = \frac{1}{2}\int_0^C \beta^{(0)}(s)K^{(1)}(s)\mathrm{d}s}$$

1st order chromaticity

$$\cos(\mu) = \frac{1}{2}\mathrm{Tr}[M(0;C)]$$

$$\cos(\mu) = \cos(\mu^{(0)}) - \sin(\mu^{(0)})\Delta\mu - \cos(\mu^{(0)})\frac{(\Delta\mu)^2}{2} + \dots$$

$$\mu = \mu^{(0)} + \Delta\mu = \mu^{(0)} + \mu^{(1)}\delta + \mu^{(2)}\delta^2 + \dots$$

Select 2nd order:

$$-\mu^{(2)}\sin\left(\mu^{(0)}\right) - \frac{\left(\mu^{(1)}\right)^2}{2}\cos\left(\mu^{(0)}\right)$$

$$= \frac{1}{2}\mathrm{Tr}\left[\int_0^C \mathrm{d}s_1 M^{(0)}(s_1;C)A^{(2)}(s_1)M^{(0)}(0;s_1) + \int_0^C \mathrm{d}s_2 M^{(0)}(s_2;C)A^{(1)}(s_2)M^{(1)}(0;s_2)\right]$$

$$= \frac{1}{2}\mathrm{Tr}\left[\int_0^C \mathrm{d}s_1 M^{(0)}(s_1;C)A^{(2)}(s_1)M^{(0)}(0;s_1) + \int_0^C ds_2 \int_0^{s_2} \mathrm{d}s_1 M^{(0)}(s_2;C)A^{(1)}(s_2)M^{(0)}(s_1;s_2)A^{(1)}(s_1)M^{(0)}(0;s_1)\right]$$

$$= -\frac{1}{2}\int_0^C \beta^{(0)}(s)\sin\left(\mu^{(0)}\right)K^{(2)}(s)\mathrm{d}s - \frac{1}{2}\int_0^C ds_2 \int_0^{s_2} \mathrm{d}s_1 K^{(1)}(s_1)\beta^{(0)}(s_1)K^{(1)}(s_2)\beta^{(0)}(s_2)\sin\left(\psi(s_2)-\psi(s_1)\right)\sin\left(\psi(s_2)-\psi(s_1)-\mu^{(0)}\right)$$

$$= -\frac{1}{2}\int_0^C \beta^{(0)}(s)\sin\left(\mu^{(0)}\right)K^{(2)}(s)\mathrm{d}s - \frac{1}{4}\int_0^C ds_2 \int_0^C \mathrm{d}s_1 K^{(1)}(s_1)\beta^{(0)}(s_1)K^{(1)}(s_2)\beta^{(0)}(s_2)\sin\left(|\psi(s_2)-\psi(s_1)|\right)\sin\left(|\psi(s_2)-\psi(s_1)|-\mu^{(0)}\right)$$

$$= -\frac{1}{2}\int_0^C \beta^{(0)}(s)\sin\left(\mu^{(0)}\right)K^{(2)}(s)\mathrm{d}s - \frac{1}{8}\int_0^C ds_2 \int_0^C \mathrm{d}s_1 K^{(1)}(s_1)\beta^{(0)}(s_1)K^{(1)}(s_2)\beta^{(0)}(s_2)\left[\cos\left(\mu^{(0)}\right) - \cos\left(2|\psi(s_2)-\psi(s_1)|-\mu^{(0)}\right)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$-\mu^{(2)}\sin\left(\mu^{(0)}\right) = -\frac{1}{2}\int_0^C \beta^{(0)}(s)\sin\left(\mu^{(0)}\right)K^{(2)}(s)\mathrm{d}s + \frac{1}{8}\int_0^C ds_2 \int_0^C \mathrm{d}s_1 K^{(1)}(s_1)\beta^{(0)}(s_1)K^{(1)}(s_2)\beta^{(0)}(s_2)\cos\left(2|\psi(s_2)-\psi(s_1)|-\mu^{(0)}\right)$$

$$\boxed{\mu^{(2)} = \frac{1}{2}\int_0^C \beta^{(0)}(s)K^{(2)}(s)\mathrm{d}s - \frac{1}{8\sin\left(\mu^{(0)}\right)}\int_0^C ds_2 \int_0^C \mathrm{d}s_1 K^{(1)}(s_1)\beta^{(0)}(s_1)K^{(1)}(s_2)\beta^{(0)}(s_2)\cos\left(2|\psi(s_2)-\psi(s_1)|-\mu^{(0)}\right)}$$

$$\boxed{\mu^{(2)} = \frac{1}{2}\int_0^C \beta^{(0)}(s)K^{(2)}(s)\mathrm{d}s - \frac{1}{8\sin\left(\mu^{(0)}\right)}\int_0^C ds_2 \int_0^C \mathrm{d}s_1 K^{(1)}(s_1)\beta^{(0)}(s_1)K^{(1)}(s_2)\beta^{(0)}(s_2)\cos\left(2|\psi(s_2)-\psi(s_1)|-\mu^{(0)}\right)}$$

Substitute in: $$\frac{\beta^{(1)}(s)}{\beta^{(0)}(s)} = -\frac{1}{2\sin\left(\mu^{(0)}\right)}\int_0^C K^{(1)}(s')\beta^{(0)}(s')\cos\left(2|\psi(s')-\psi(s)|-\mu^{(0)}\right)\mathrm{d}s'$$

Homework exercise:
Derive this

$$\boxed{\mu^{(2)} = \frac{1}{2}\int_0^C \beta^{(0)}(s)K^{(2)}(s)\mathrm{d}s + \frac{1}{4}\int_0^C \beta^{(1)}(s)K^{(1)}(s)\mathrm{d}s}$$

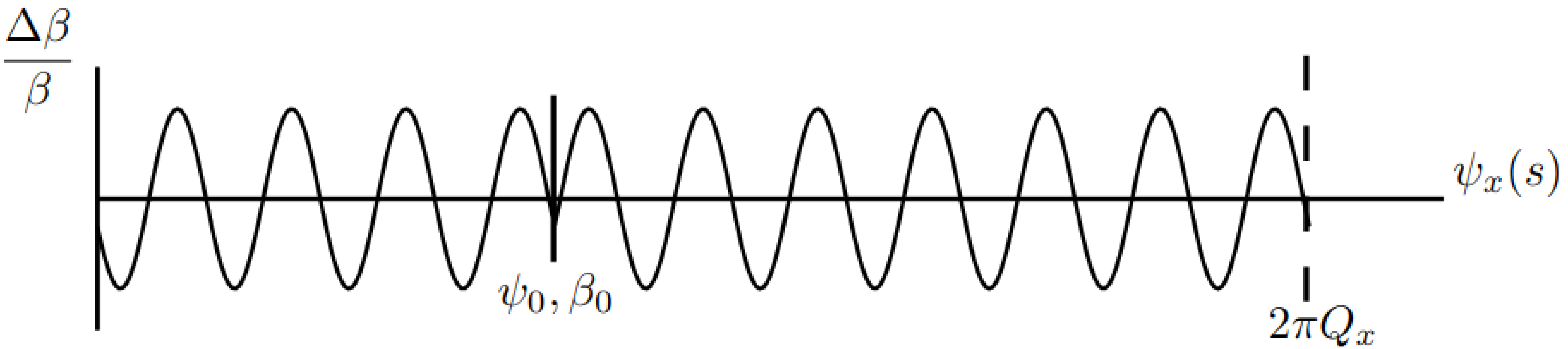

Fig. 8.3 (Peggs & Satogata)

Expand K in vertical plane:

$$(1+\delta)^{-1} = 1 - \delta + \delta^2 - \ldots$$

$$\Delta K_y = -K_1\left(\frac{1}{1+\delta} - 1\right) - \frac{K_2}{1+\delta}\left(\eta\delta + \eta^{(2)}\delta^2 + \ldots\right)$$

$$= \underbrace{(K_1 - K_2\eta)}_{K_y^{(1)}}\,\delta + \underbrace{\left(-K_1 + K_2\eta - K_2\eta^{(2)}\right)}_{K_y^{(2)}}\,\delta^2 + \mathcal{O}(\delta^3)$$

K_1: Quadrupole strength

K_2: Sextupole strength

η: Linear dispersion

$\eta^{(2)}$: 2nd-order dispersion

$$\boxed{\mu_y^{(1)} = \left.\frac{\mathrm{d}\mu}{\mathrm{d}\delta}\right|_{\delta=0} = 2\pi\frac{\mathrm{d}Q}{\mathrm{d}\delta} = \frac{1}{2}\int_0^C \beta_y^{(0)} K_y^{(1)}\mathrm{d}s = \frac{1}{2}\int_0^C \beta_y^{(0)}\,(K_1 - K_2\eta)\,\mathrm{d}s}$$

$$\mu_y^{(2)} = \frac{1}{2}\int_0^C \beta_y^{(0)} K_y^{(2)}\mathrm{d}s + \frac{1}{4}\int_0^C \beta_y^{(1)} K_y^{(1)}\mathrm{d}s$$

$$= \frac{1}{2}\int_0^C \beta_y^{(0)}\left(-K_1 + K_2\eta - K_2\eta^{(2)}\right)\mathrm{d}s + \frac{1}{4}\int_0^C \beta_y^{(1)}\,(K_1 - K_2\eta)\,\mathrm{d}s$$

$$\boxed{\mu_y^{(2)} = \frac{1}{2}\left.\frac{\mathrm{d}^2\mu}{\mathrm{d}\delta^2}\right|_{\delta=0} = \pi\frac{\mathrm{d}^2 Q}{\mathrm{d}\delta^2} = -\mu_y^{(1)} - \frac{1}{2}\int_0^C \beta_y^{(0)} K_2\eta^{(2)}\mathrm{d}s + \frac{1}{4}\int_0^C \beta_y^{(1)}\,(K_1 - K_2\eta)\,\mathrm{d}s}$$

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W-vector (Montague) formulation

We know:

$$\frac{\mathrm{d}\mu}{\mathrm{d}s} = \frac{1}{\beta} \qquad \frac{\mathrm{d}\beta}{\mathrm{d}s} = -2\alpha$$

$$\frac{\mathrm{d}^2\sqrt{\beta}}{\mathrm{d}s^2} + K\sqrt{\beta} - \beta^{-3/2} = 0 \quad (8.35)$$

Differentiation:

$$\frac{\mathrm{d}^2\sqrt{\beta}}{\mathrm{d}s^2} = -\frac{\mathrm{d}\alpha}{\mathrm{d}s}\beta^{-1/2} - \alpha^2\beta^{-3/2}$$

Substitute into (8.35) & simplify:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = K\beta - \frac{1+\alpha^2}{\beta}$$

Define:

$$B = \frac{(\beta_1 - \beta_0)}{(\beta_0\beta_1)^{1/2}} \qquad A = \frac{(\alpha_1\beta_0 - \alpha_0\beta_1)}{(\beta_0\beta_1)^{1/2}}$$

$$\psi = \frac{1}{2}(\mu_0 + \mu_1) \qquad \Delta K = K_1 - K_0$$

(not the same K_1 as slide 17!)

0: central orbit
1: off-momentum orbit

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \frac{1}{2}\frac{(\beta_0 + \beta_1)}{\beta_0\beta_1}$$

After differentiation and some algebra:

$$\frac{\mathrm{d}B}{\mathrm{d}s} = -2A\frac{\mathrm{d}\psi}{\mathrm{d}s}$$

$$\frac{\mathrm{d}A}{\mathrm{d}s} = 2B\frac{\mathrm{d}\psi}{\mathrm{d}s} + (\beta_0\beta_1)^{1/2}\Delta K$$

In an achromatic region, $\Delta K = 0$

From previous slide:

$$\frac{\mathrm{d}B}{\mathrm{d}s} = -2A\frac{\mathrm{d}\psi}{\mathrm{d}s}$$

$$\frac{\mathrm{d}A}{\mathrm{d}s} = 2B\frac{\mathrm{d}\psi}{\mathrm{d}s} + \cancel{(\beta_0\beta_1)^{1/2}\Delta K}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\psi} = -2A \qquad \frac{\mathrm{d}A}{\mathrm{d}\psi} = 2B$$

$$\frac{\mathrm{d}^2B}{\mathrm{d}\psi^2} + 4B = 0 \qquad \frac{\mathrm{d}^2A}{\mathrm{d}\psi^2} + 4A = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(A^2 + B^2) = 0$$

$$\therefore (A^2 + B^2) = \mathrm{Constant}$$

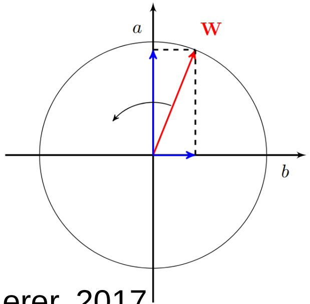
A and *B* oscillate at twice ψ

Now redefine variables in limit $\delta \to 0$

$$b = \lim_{\delta\to 0}\frac{B}{\delta} = \lim_{\delta\to 0}\frac{1}{\delta}\frac{(\beta_1 - \beta_0)}{(\beta_0\beta_1)^{1/2}}$$

$$a = \lim_{\delta\to 0}\frac{A}{\delta} = \lim_{\delta\to 0}\frac{1}{\delta}\frac{(\alpha_1\beta_0 - \alpha_0\beta_1)}{(\beta_0\beta_1)^{1/2}}$$

$$\psi \to \mu_0 \qquad \Delta k = \lim_{\delta\to 0}\frac{\Delta K}{\delta}$$



Haerer, 2017

Define:

$$\vec{W} = b + ia$$

In achromatic regions *W* has a constant amplitude and rotates at twice the average betatron phase advance.
When passing through a quad or sext its amplitude is modified.

Now consider what happens when we pass through a thin quad or sext

$$\frac{\mathrm{d}b}{\mathrm{d}s} = -2a\frac{\mathrm{d}\mu_0}{\mathrm{d}s}$$

$$\frac{\mathrm{d}a}{\mathrm{d}s} = 2b\frac{\mathrm{d}\mu_0}{\mathrm{d}s}0 + (\beta_0\beta_1)^{1/2}\Delta k$$

$\Delta\mu_0 = 0$ β-functions same before and after thin lens

So

$$\Delta b = 0$$

$$\Delta a = (\beta_0\beta_1)^{1/2}\Delta k_n \Delta s \approx \beta_0 K_1 L_q \text{ for quad}$$

$$\Delta a = (\beta_0\beta_1)^{1/2}\Delta k_n \Delta s \approx -\beta_0 \eta K_2 L_s \text{ for sext}$$

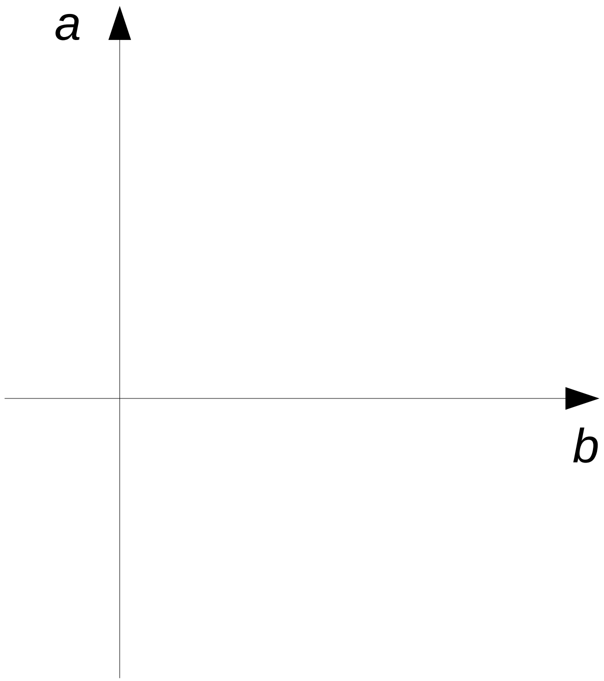
An observer downstream of quad would see:

$$\begin{cases} \Delta a(\mu) & \approx \beta_0 K_1 L_q \cos(2\mu) \\ \Delta b(\mu) & \approx \beta_0 K_1 L_q \sin(2\mu) \end{cases}$$

An observer downstream of sext would see:

$$\begin{cases} \Delta a(\mu) & \approx -\beta_0 \eta K_2 L_s \cos(2\mu) \\ \Delta b(\mu) & \approx -\beta_0 \eta K_2 L_s \sin(2\mu) \end{cases}$$

What happens to vertical W-vector in a 60° FODO cell?



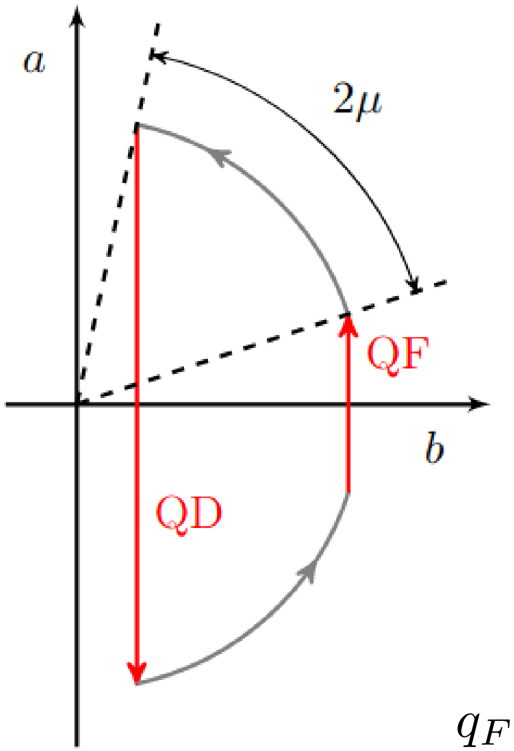
$$\Delta b = 0$$

$$\Delta a \approx \beta_0 K_1 L_q \text{ for quad}$$

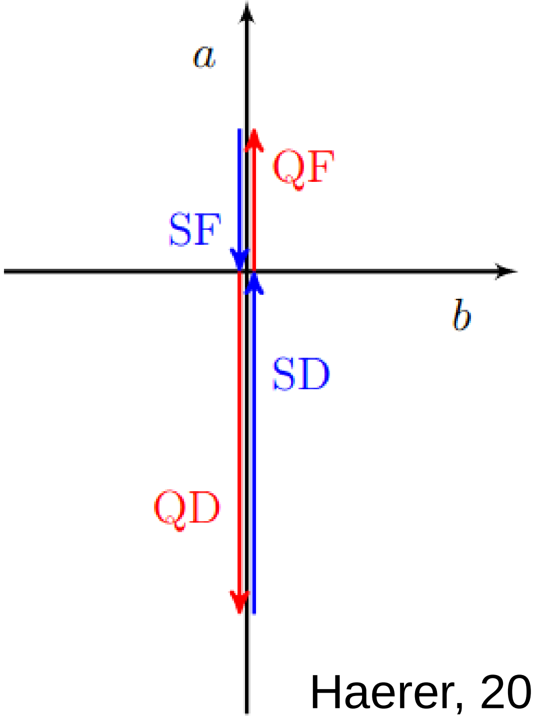
$$\Delta a \approx -\beta_0 \eta K_2 L_s \text{ for sext}$$

What happens to vertical W-vector in a 60° FODO cell?

Uncompensated



Compensated



$$\Delta b = 0$$
$$\Delta a \approx \beta_0 K_1 L_q \text{ for quad}$$
$$\Delta a \approx -\beta_0 \eta K_2 L_s \text{ for sext}$$

$$q_F = -q_D$$

Haerer, 2017

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Chromaticity correction in a collider

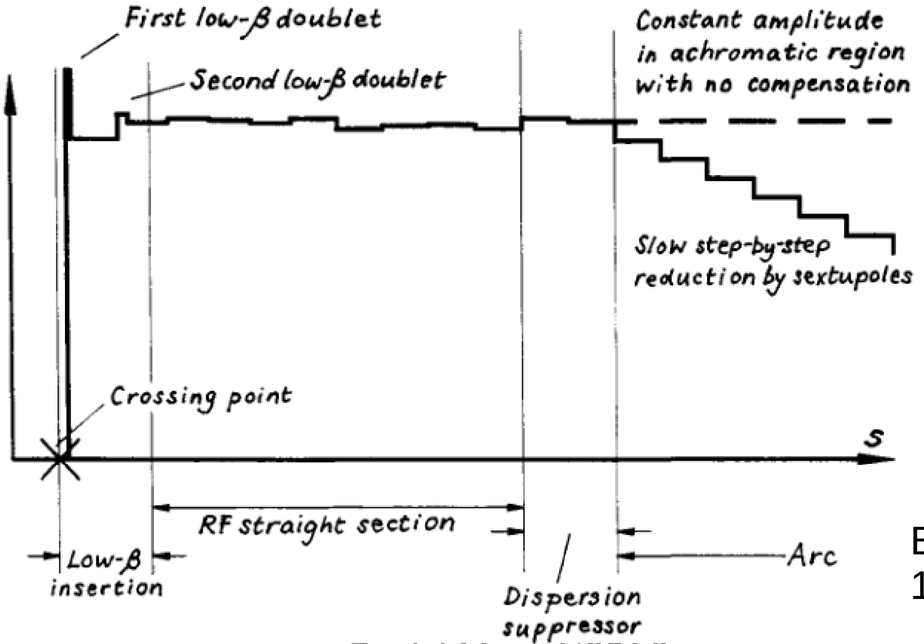
- Colliders have low-β insertions that contribute greatly to chromaticity
- Low-β quads chromaticity cannot be compensated locally, as they are in dispersion-free region, so need to use arcs
- By dividing sexts into families can progressively reduce *W* to zero over an arc

$$\frac{\mathrm{d}\mu}{\mathrm{d}\delta} = \frac{1}{2}\int_0^C \beta_y^{(0)}\left(K_1 - K_2\eta\right)\mathrm{d}s$$

$$|\vec{W}| = \sqrt{a^2 + b^2}$$

$$\Delta a \approx \beta_0 K_1 L_q \text{ for quad}$$

$$\Delta a \approx -\beta_0 \eta K_2 L_s \text{ for sext}$$



Bryant & Johnsen, 1993

- We want a sextupole scheme that builds up the amplitude of *W* to compensate the IP chromaticity

- Building up the amplitude of *W* **slowly** keeps individual sextupole strengths to a minimum, reducing nonlinearities and resonance excitation

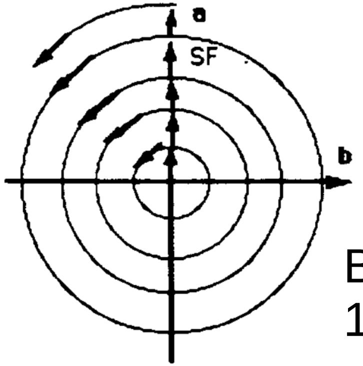
- In order for a series of sextupoles to add constructively, *W* needs to rotate by $2n\pi$ between sextupoles

- Interleaved scheme:

$$
\begin{array}{ll}
\{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\} & \{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\} \\
\leftarrow \quad \text{1st group} \quad \rightarrow & \leftarrow \quad \text{2nd group} \quad \rightarrow \\
\{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\}\ldots. & \{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\} \\
\leftarrow \quad \text{3rd group} \quad \rightarrow & \leftarrow \quad \text{Last group} \quad \rightarrow
\end{array}
$$

- Start with all F-sexts equal and all D-sexts equal, set to compensate natural chrom

- Increment SF family by $\Delta k'_{\mathrm{SF}}$



Bryant, 1995

Cell betatron phase advance = μ_0

$(N+1)$ cells in each group

$\{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\}\{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\}$

$\leftarrow$ 1st group $\rightarrow \leftarrow$ 2nd group $\rightarrow$

$\{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\}\ldots\{S_F,S_D;S_{F1},S_{D1};S_{F2},S_{D2};\ldots;S_{FN},S_{DN}\}$

$\leftarrow$ 3rd group $\rightarrow \leftarrow$ Last group $\rightarrow$

So phase condition is:

$$(N+1)\mu_0 = n\pi, \text{ where } n = \text{integer}$$

We do not want to excite $3Q = p$ integer resonance. So

$$3(N+1)\mu_0 = (2m+1)\pi, \text{ where } m = \text{integer}$$

The 2 conditions mean: $3n = 2m+1$

n has to be an odd integer

$\mu_0 = \pi/2$ for $n=1$, $N=1$ and $m=1$ i.e. 4 families

$\mu_0 = \pi/3$ for $n=1$, $N=2$ and $m=1$ i.e. 6 families

$\mu_0 = \pi/4$ for $n=1$, $N=3$ and $m=1$ i.e. 8 families

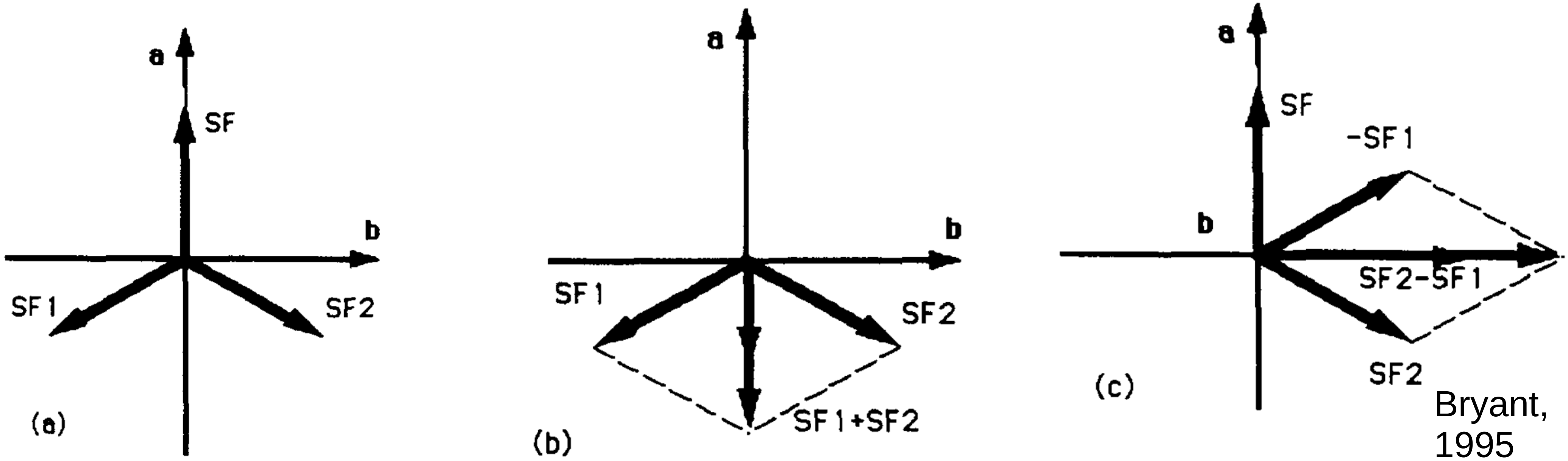
$\downarrow$ etc.

$\mu_0 = 3\pi/4$ for $n=3$, $N=3$ and $m=4$ i.e. 8 families

$\mu_0 = 3\pi/5$ for $n=3$, $N=4$ and $m=4$ i.e. 10 families

$\downarrow$ etc.

60° lattice



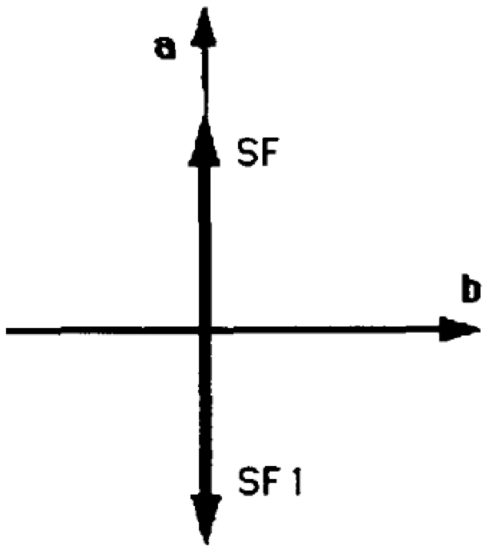
Bryant, 1995

All 3 SF families incremented equally. No net *W* vector excited, but chromaticity changes.

By setting SF1 & SF2, we can choose our net *W* vector. Here we use (SF1+SF2).

By setting SF1 & SF2, we can choose our net *W* vector. Here we use (SF2-SF1).

90° lattice



Bryant, 1995

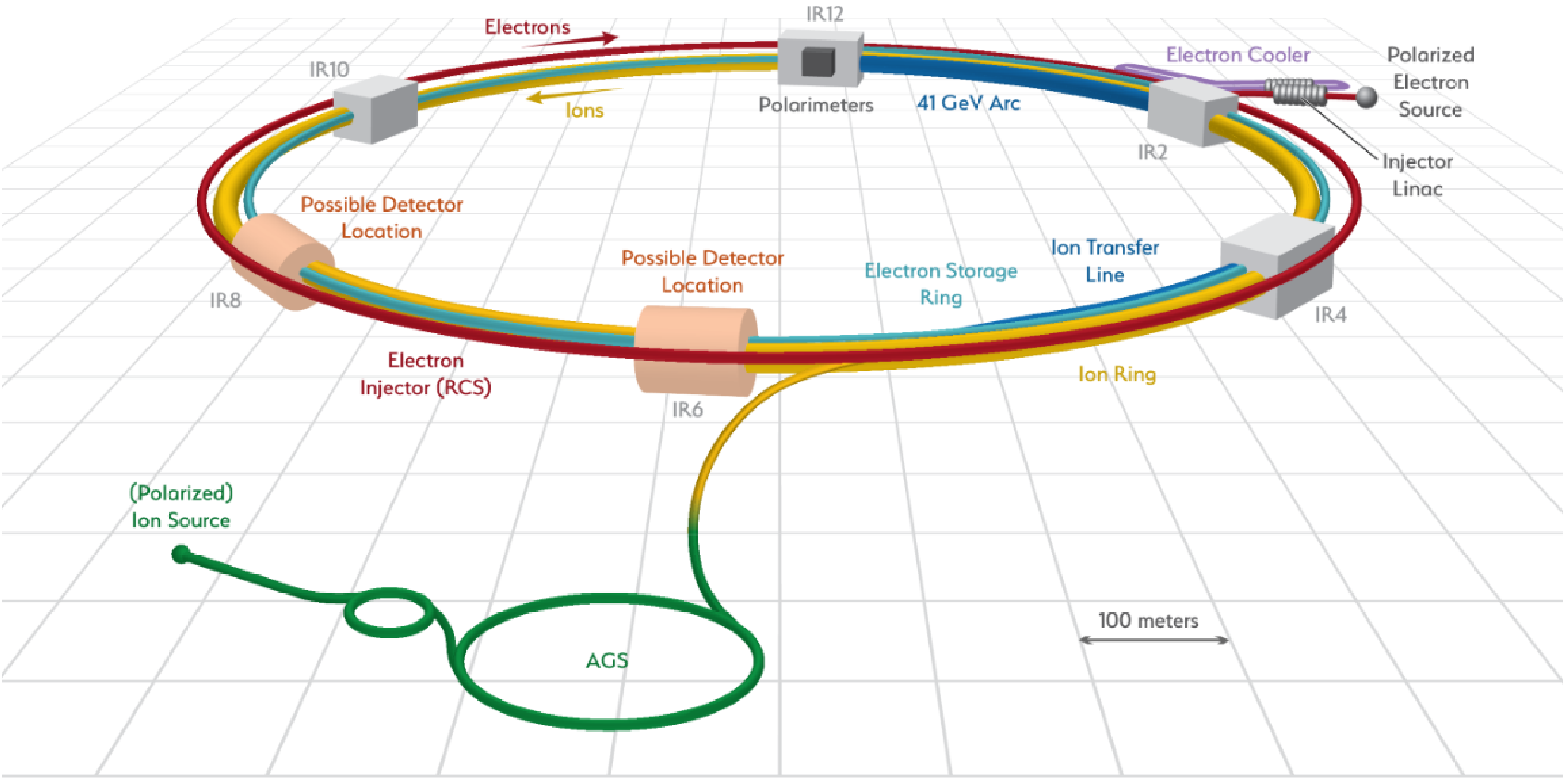
With a 90° lattice it is not possible to choose the direction of the *W* vector.

The phase advance must be set such that the *W* vector arrives parallel to the *a* axis.

Outline

- Recap of yesterday's lecture
- Derivation of 1st & 2nd order chromaticity
- W-vector (Montague) formalism
- Chromaticity correction in a collider
- **Chromaticity correction and dynamic aperture in the EIC ESR**

Chromaticity correction in the EIC ESR

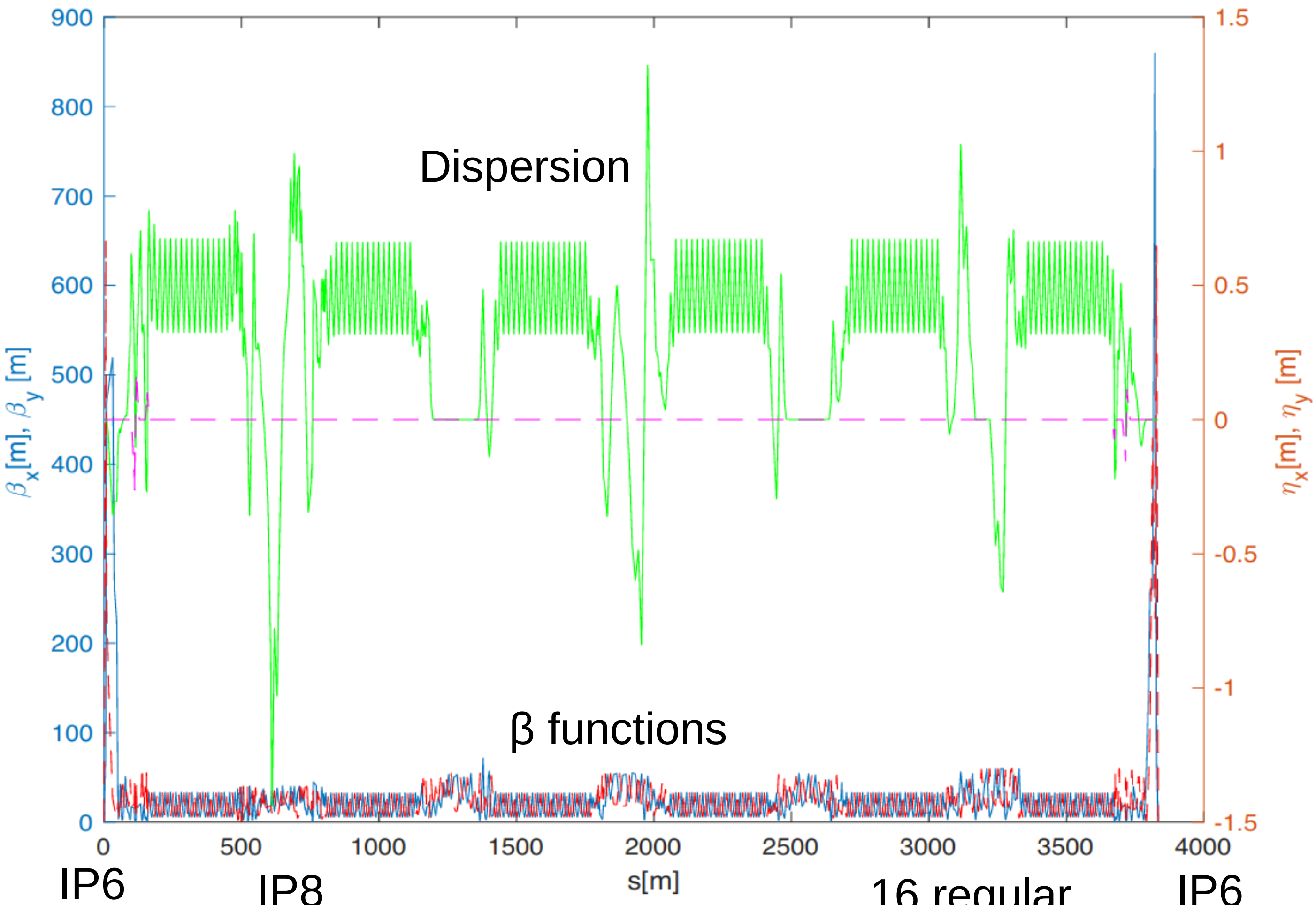


$$\epsilon_x\,[\mathrm{m}] \;=\; F(\Delta\phi)\,\frac{E^2\,[\mathrm{GeV}^2]}{J_x N_d^3} \qquad (12.31)$$

EIC: Electron Storage Ring (ESR)

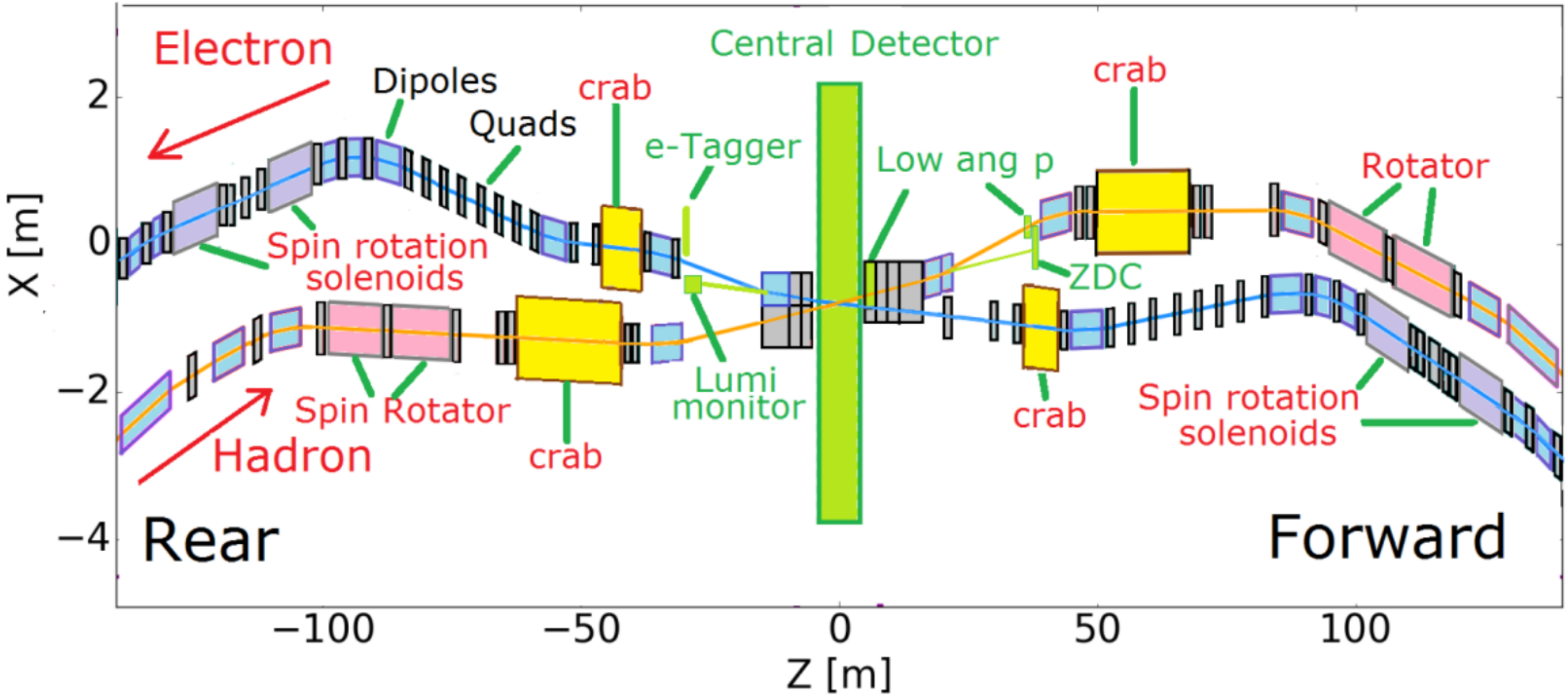
Parameter	60°	90°
Beam energy, E_0 [GeV]	10	18
Circumference, C [m]	3834	3834
Emittance, ϵ_x [nm]	24.0	28.3
Energy spread, σ_δ [10^{-4}]	5.54	9.83
Betatron tunes, ν_x/ν_y	45.12/36.1	48.12/43.1
Chromaticity, ξ_{0x}/ξ_{0y}	$-83/-91$	$-85/-94$
IP betas, β_x^*/β_y^* [m]	0.42/0.05	0.42/0.05
Distance from IP to quad, L^* [m]	5.3	5.3

Lattice functions in the ESR for 1 IP



IP6 IP8 16 regular FODO cells per arc IP6

Interaction region



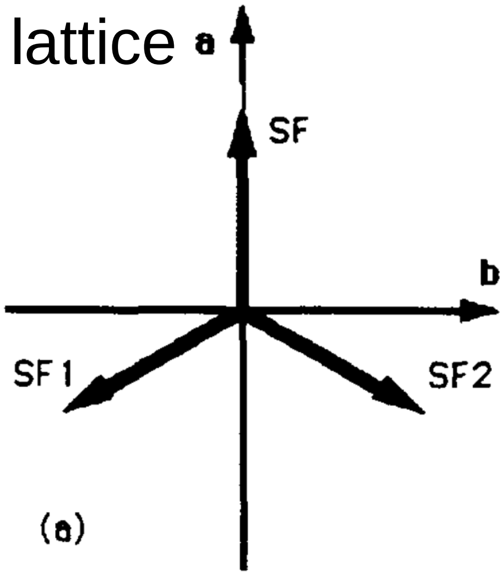
Different solutions are required for 60° and 90° lattices

- For 60° lattice can use 6 families (3 families per plane)
- For 90° lattice can use 4 families (2 families per plane)

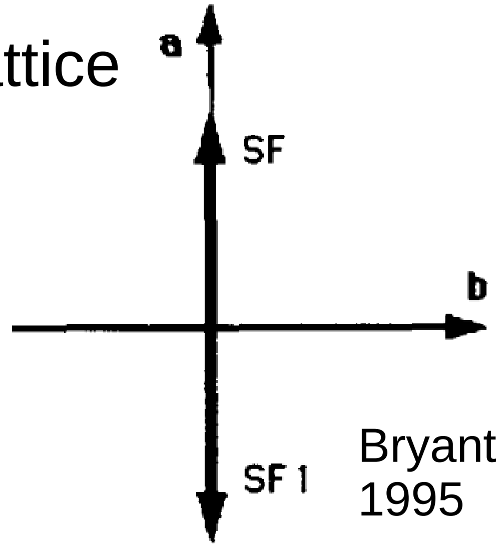
90° lattice is most challenging:

- Additional constraints on phase advance to first sextupoles in arcs
- Larger rms momentum spread

60° lattice



90° lattice



Bryant, 1995

Ultimately it's all about maximizing lifetime!

- We want there to be a large volume in phase space in which a particle survives
- Dynamic aperture describes the region in *x*-*y* space in which a particle will survive for many turns
- Momentum acceptance describes the momentum range in which a particle will survive for many turns
- Touschek scattering describes the interaction of electrons inside a bunch with a transfer of transverse momentum to longitudinal momentum
- Particles will survive for a longer time if the momentum acceptance is larger
- In the ESR, each electron bunch will be frequently replaced (every 6 mins), so long beam lifetime is only required to keep the charge variation small

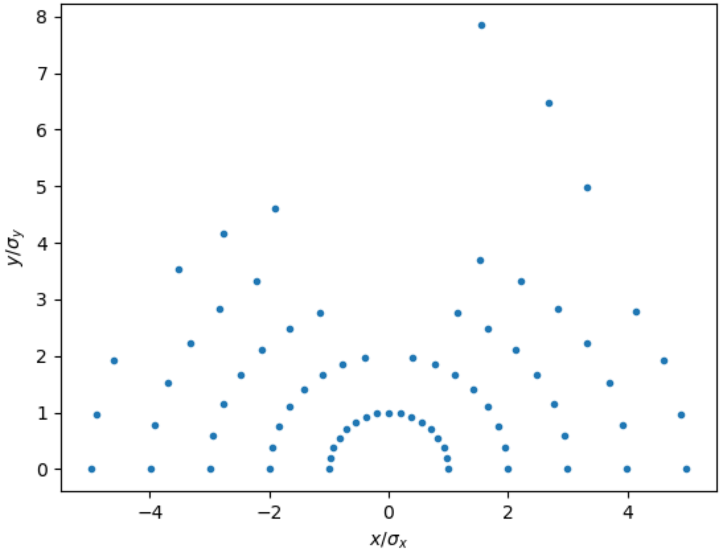
$$\frac{1}{\tau} \propto \frac{N}{\gamma^2 \sigma_x \sigma_y \sigma_z \delta_{\mathrm{acc}}^3}$$

The best way to compute dynamic aperture is to track over many turns

- For hadrons need to track over millions or billions of turns
- For electrons only need to track over thousands of turns due to synchrotron damping

A survival plot shows the particles that survive after many turns.

Need to do this for various momentum offsets.

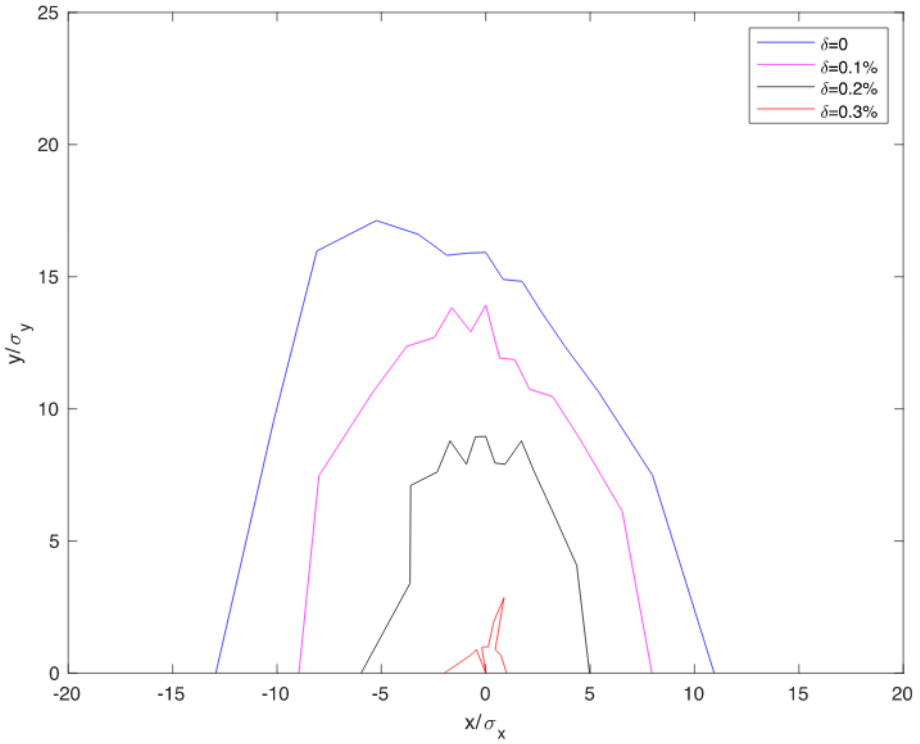


Goals:

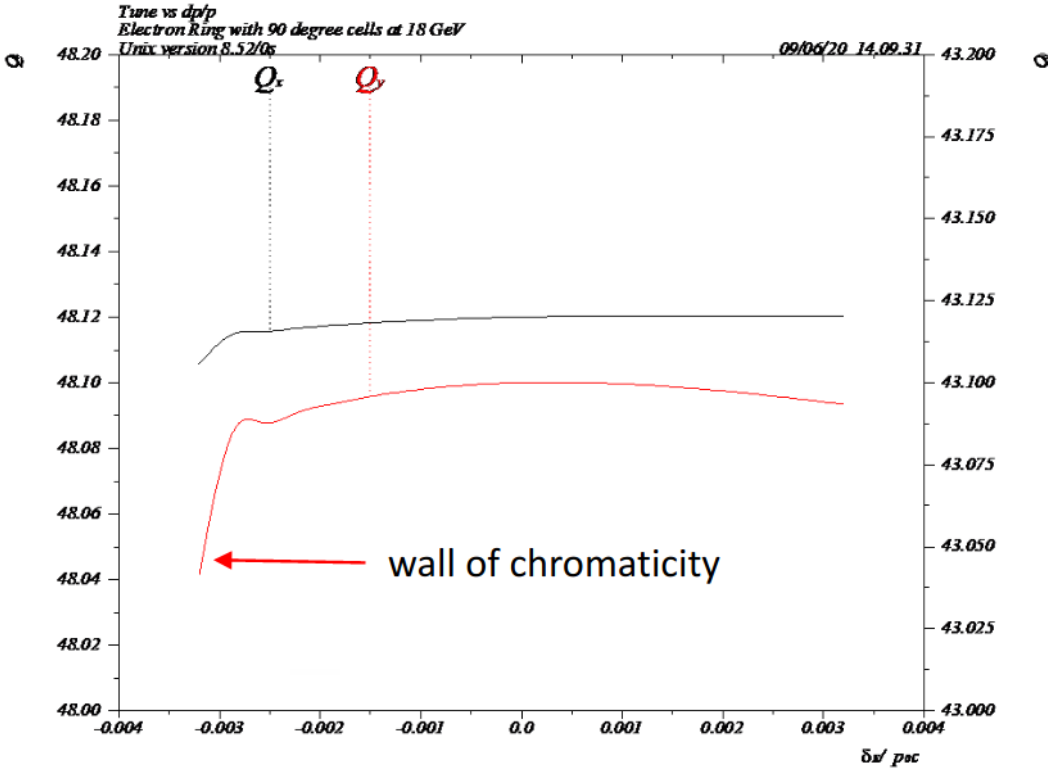
- Correct the linear chromaticity to +1 in both planes
- Achieve 10σ dynamic aperture on-momentum
- Achieve 10σ momentum acceptance (0.6% for 60°; 1% for 90°)

In order to achieve sufficient on-momentum dynamic aperture, need to keep sextupole strengths down and ensure there is no build-up of nonlinear resonances

Dynamic aperture at 18 GeV, 90° lattice, 1IP
Starting point: 2 families of sextupoles



Simulations in LEGO with RF on;
no synchrotron radiation; no errors



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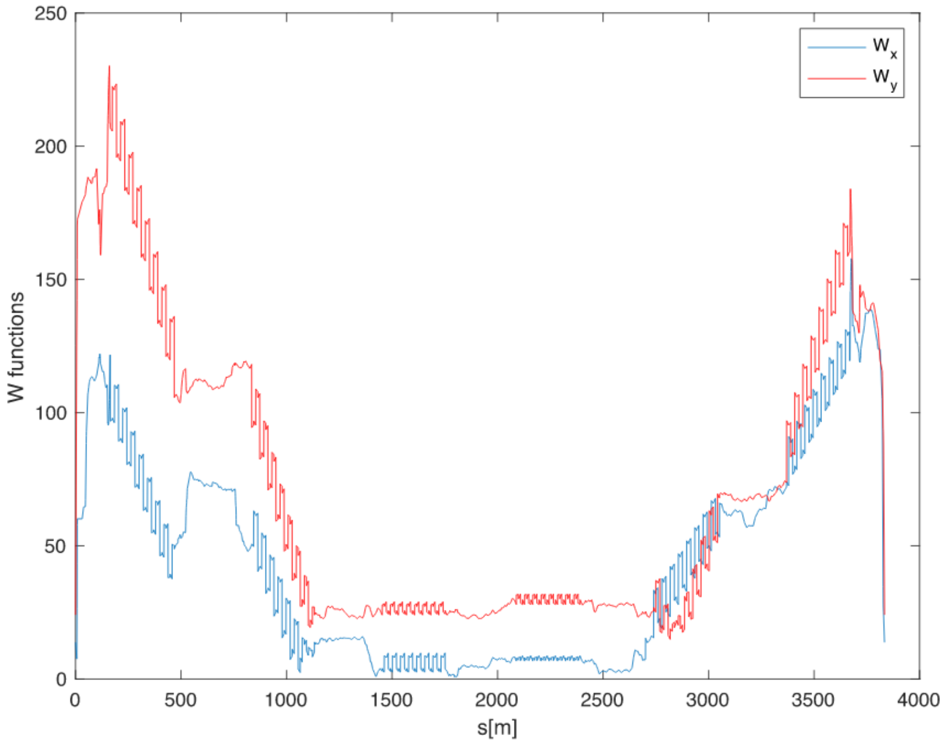
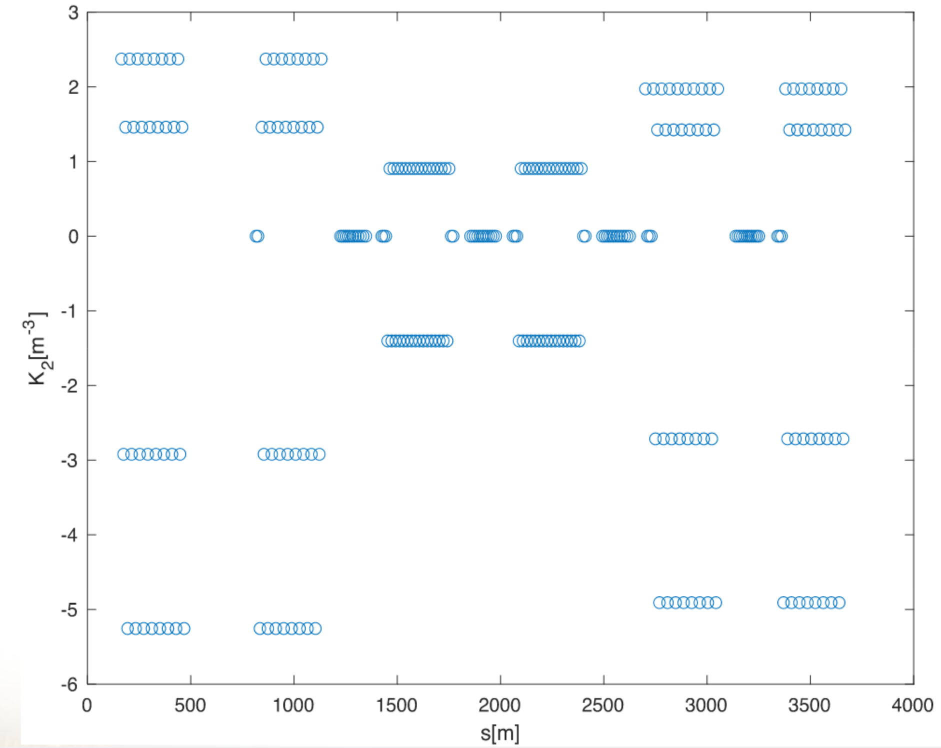
Dynamic aperture at 18 GeV, 90° lattice, 1IP
After a lot of work: 10 families of sextupoles





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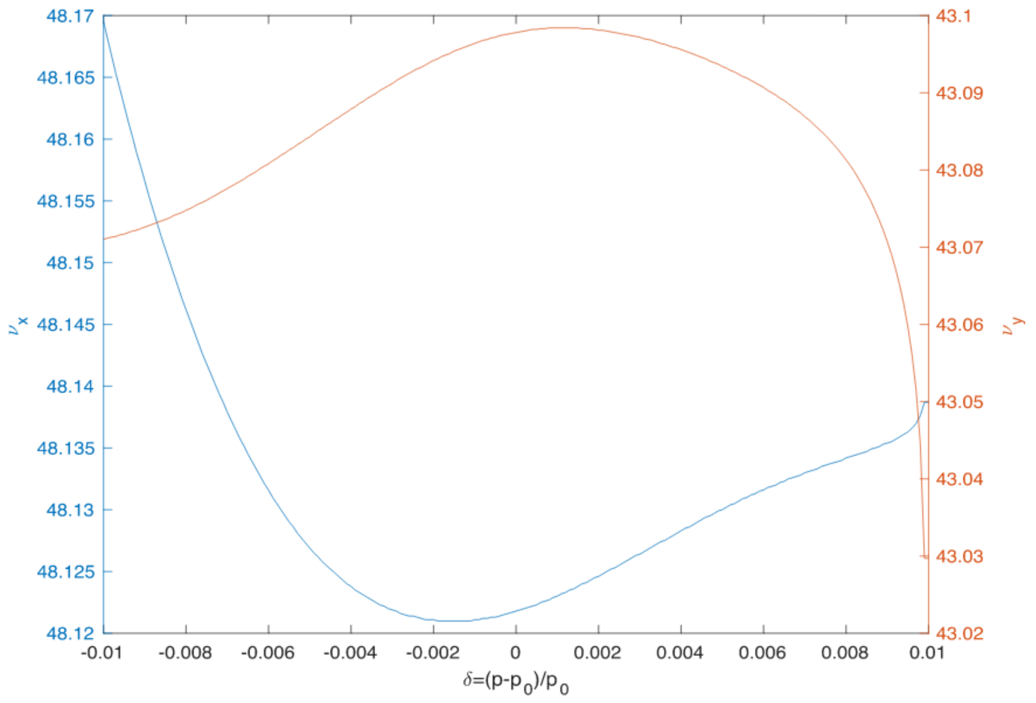
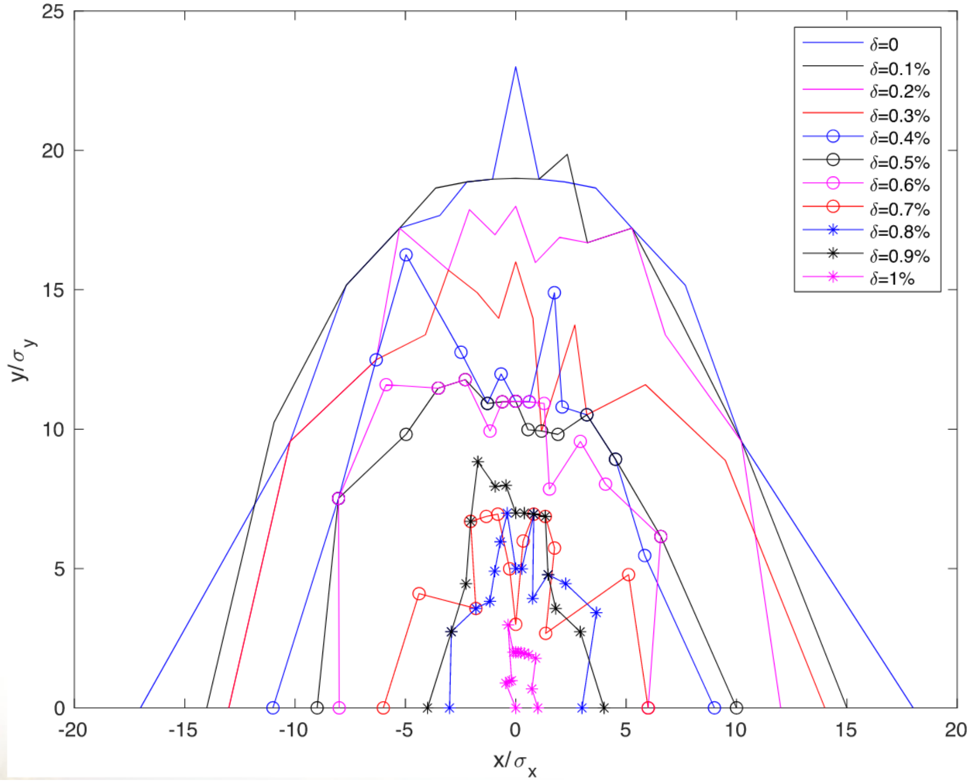
Dynamic aperture at 18 GeV, 90° lattice, 1IP
10 families of sextupoles





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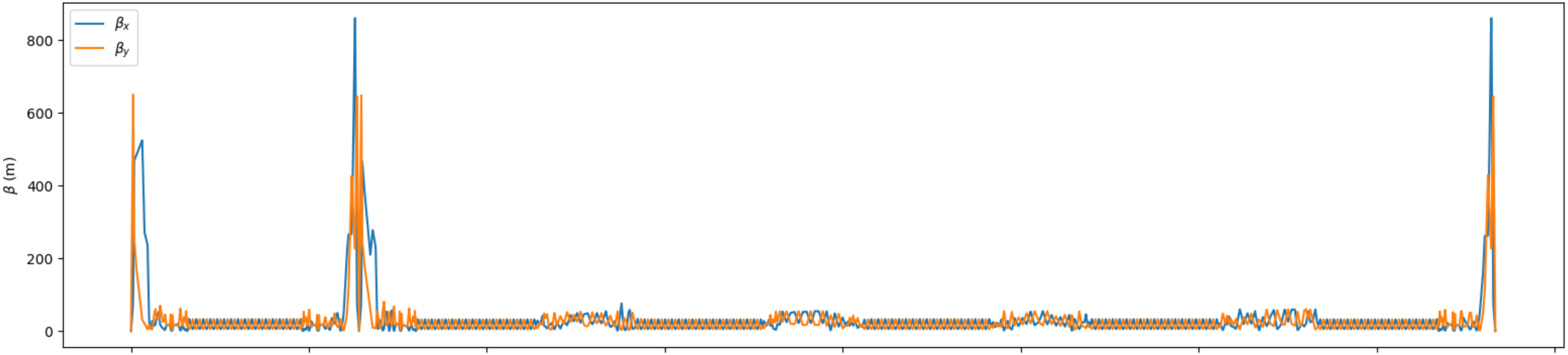
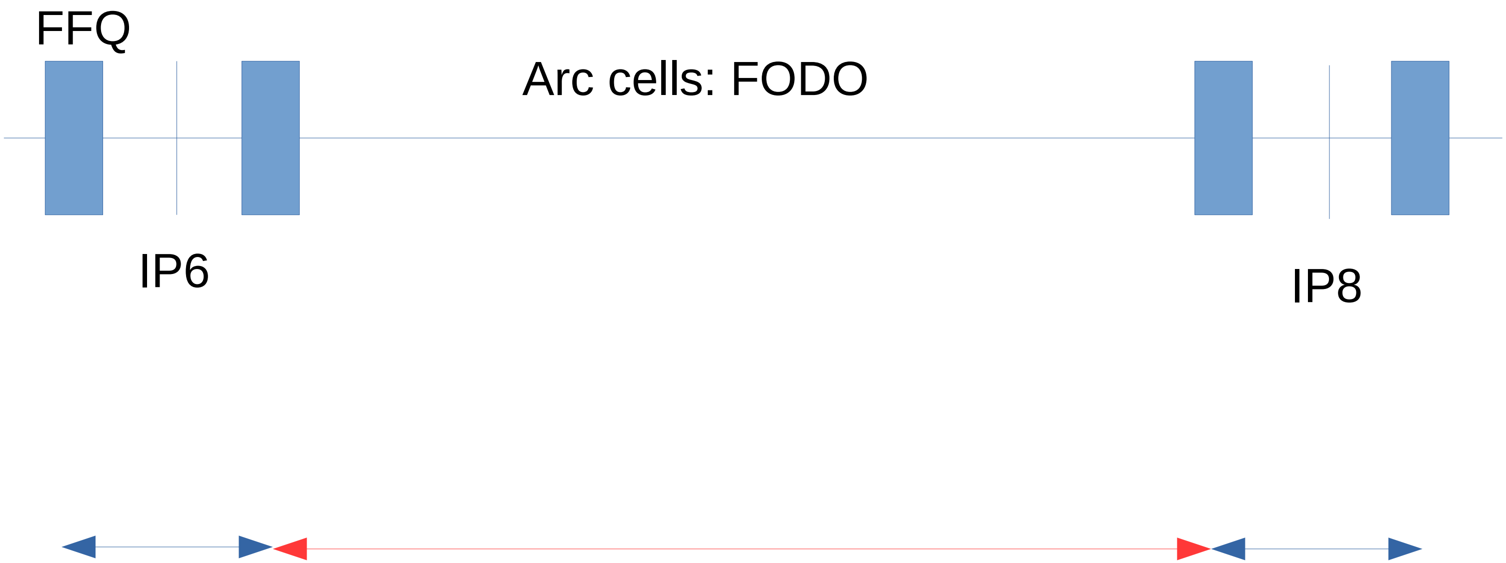
Dynamic aperture at 18 GeV, 90° lattice, 2IP

- There is the option of including a second IP (and detector)
- This is even more challenging, as chromaticity is even higher (-125 in vertical plane)
- Work is in progress to achieve sufficient dynamic aperture



It is possible to use the 2 IPs to cancel out the β beat by setting the betatron phase advance between the IPs?
What value is required?



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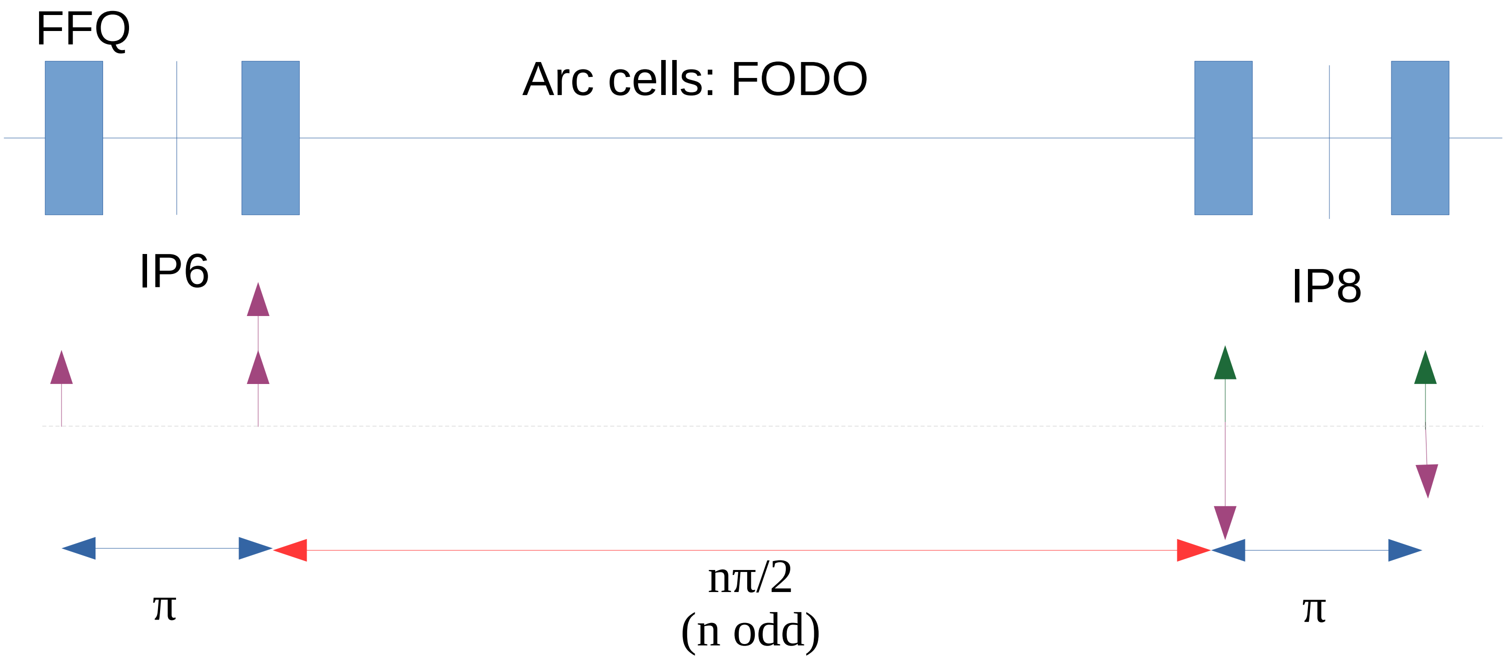
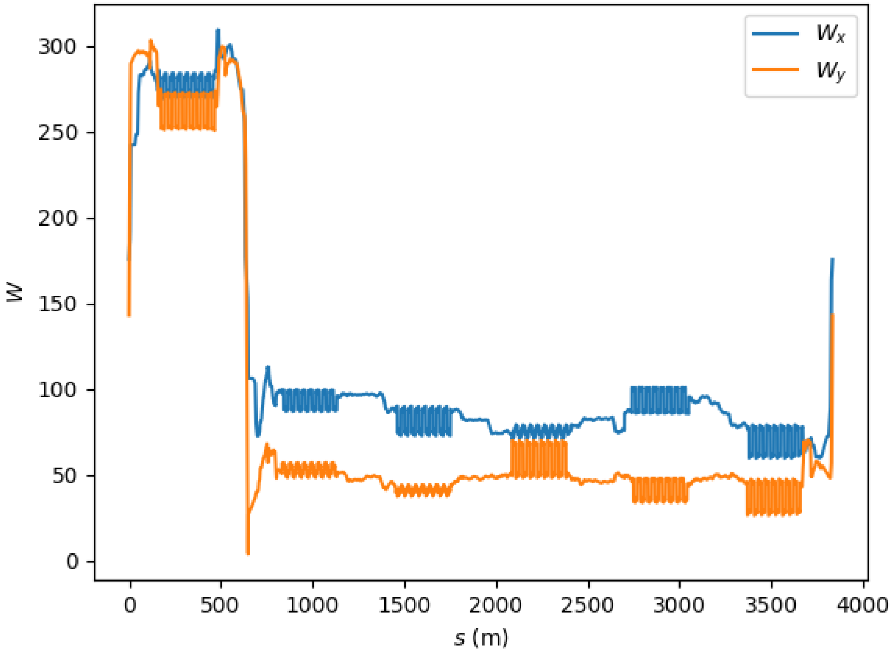
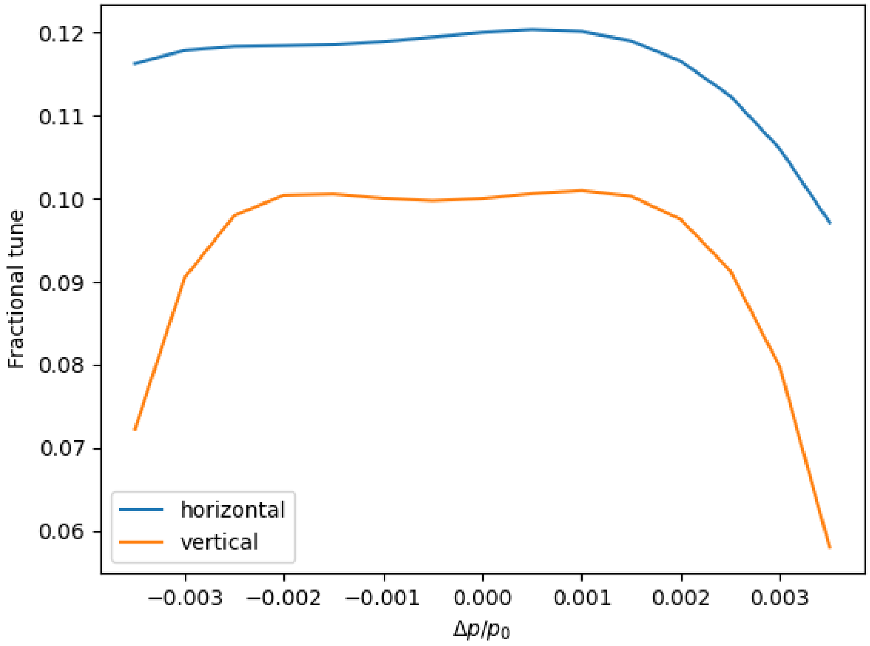


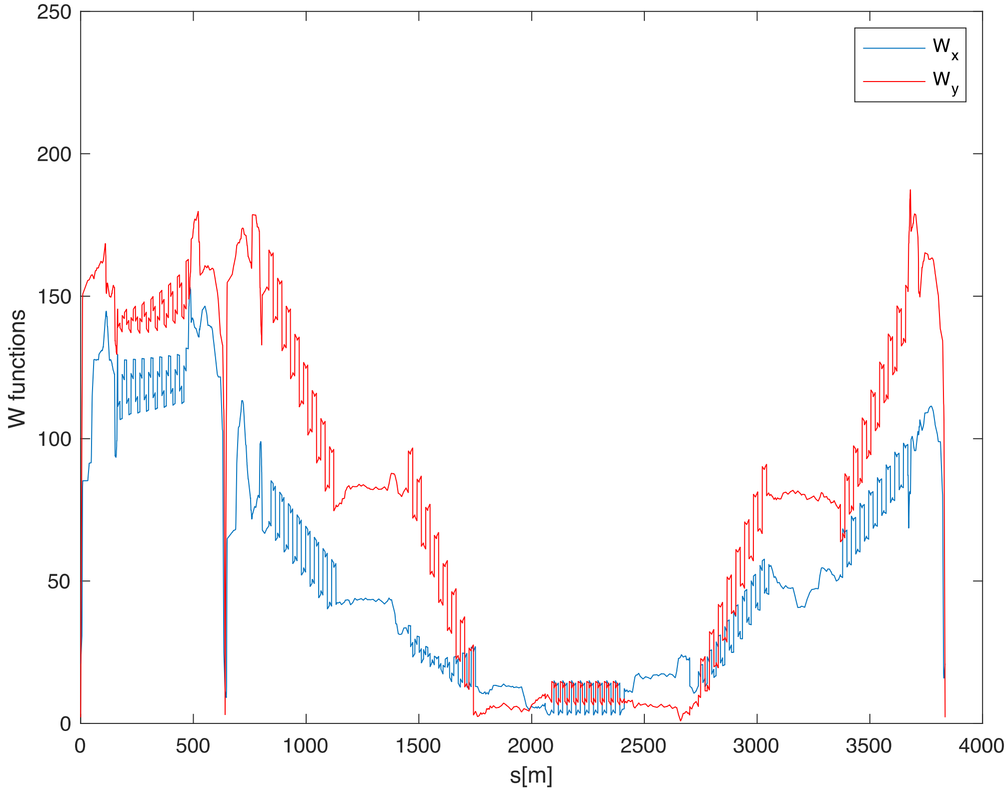
Dynamic aperture at 18 GeV, 90° lattice, 2IP

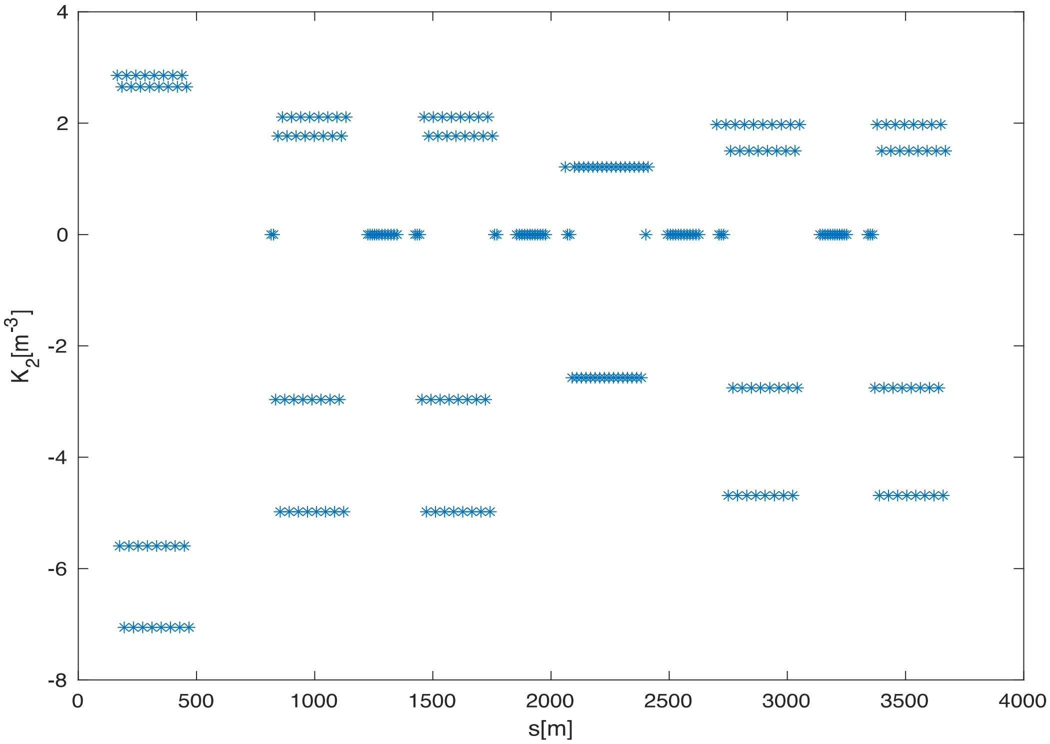
Setting phase between IPs
2 families of sextupoles in rest of ring





After a lot of work:
14 independent families of sextupoles
Phase trombones to adjust phase between arcs





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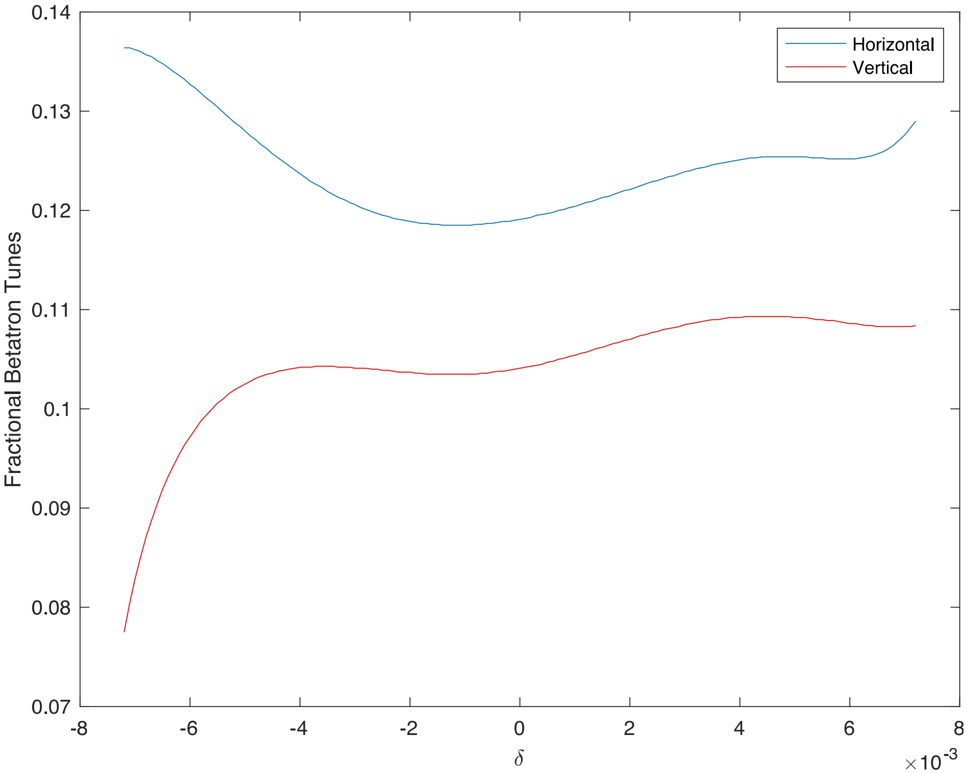

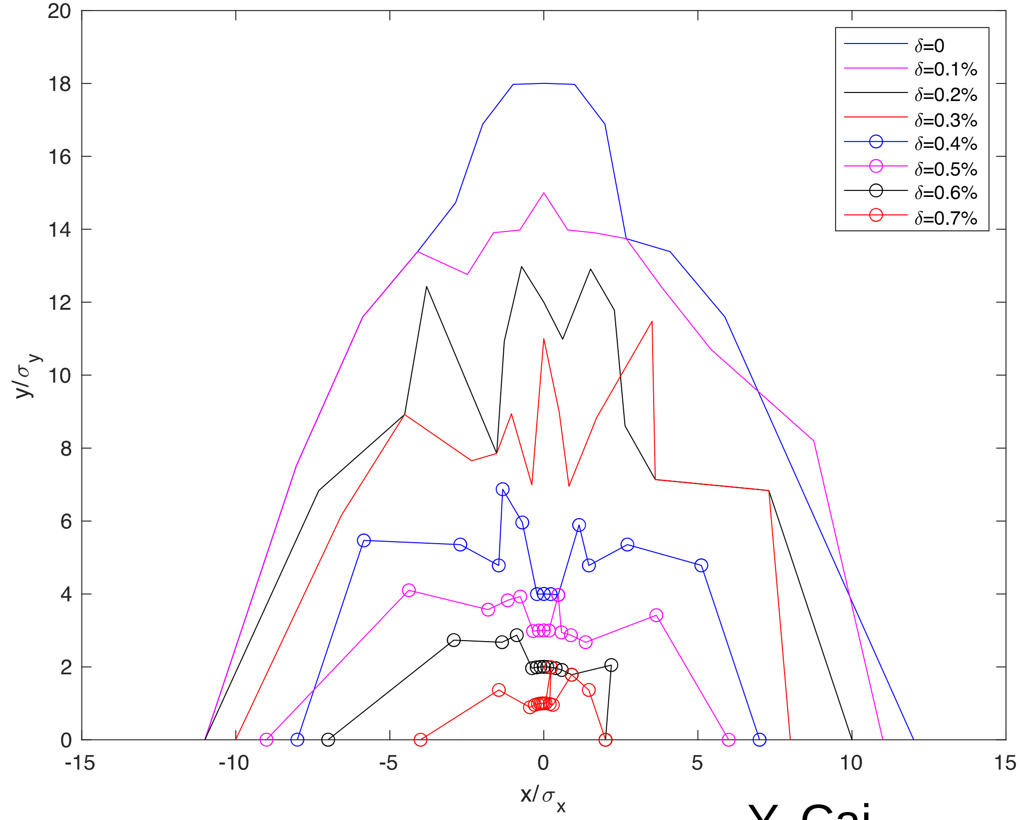

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Summary

- Chromaticity correction is essential for all circular, strong-focusing synchrotrons
- Chromaticity is not just the linear term – higher order terms may be important as well
- Derived formulae for 1st and 2nd order chromaticity – 2nd order term depends on β beat
- Introduced W functions – a useful tool for correcting 2nd order chromaticity
- In colliders the interaction region contributes greatly to chromaticity and introduces a β beat, which we can compensate with sextupole families in the arcs
- Chromaticity compensation in the EIC ESR is challenging due to the large natural chromaticity and many constraints in the lattice
- Dynamic aperture optimization is important to maximize beam lifetime
- There are many ways to approach dynamic aperture optimization – often you just have to try things out!

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