USPAS Accelerator Physics 2021
(Virtually) Texas A&M University

+

Ch 10: Octupoles, Detuning, Slow Extraction

More Equations **with an occasional pretty picture**

Todd Satogata (Jefferson Lab and ODU) / satogata@jlab.org
Steve Peggs (BNL) / peggs@bnl.gov
Daniel Marx (BNL) / dmarx@bnl.gov and Nilanjan Banerjee / nb522@cornell.edu

http://www.toddsatogata.net/2021-USPAS

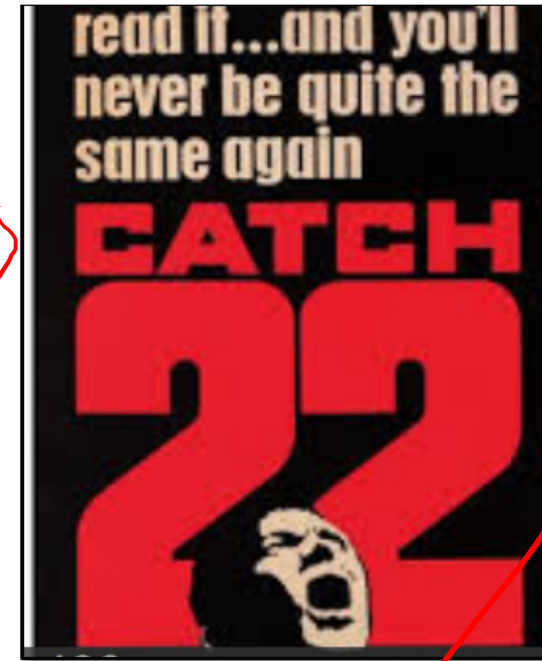
Username test / Password test

Overview

- Useful nonlinearities
- 10.1: Octupoles and detuning
- 10.2: Discrete motion in action-angle (J,ϕ) space
  - Difference (Kobayashi) Hamiltonian
  - More lecturer self-indulgence
- 10.3: Motion near half-integer tunes
  - Contours of constant Hamiltonian (energy)
- 10.4: Half-integer slow extraction
  - A useful application of first-order octupole perturbation theory
- 10+: Extending to third-integer extraction
- Modern use: resonance island extraction at CERN

Useful Nonlinearities

(!)

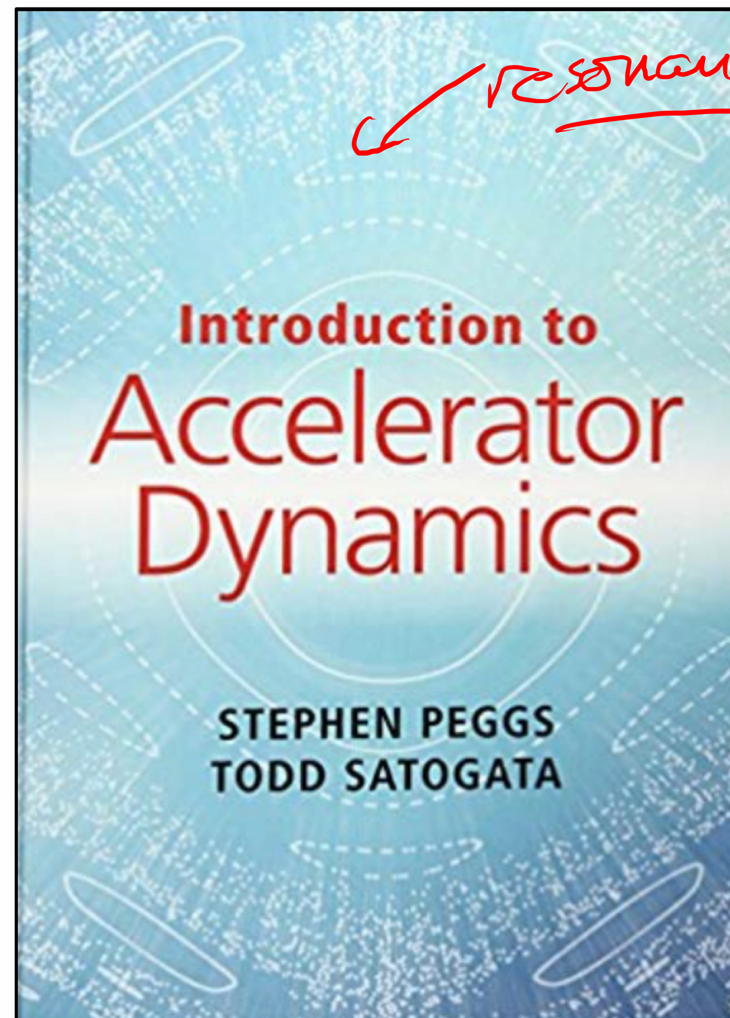


- Catch-22 revisited → even linear $\propto \frac{1}{1+\delta}$ → NL!
  - Nonlinearities are unavoidable in accelerators
  - Nonlinearities can correct motion – to a degree
  - Nonlinearities add higher "order" nonlinear behavior
  - But nonlinearities can be used for good!
  - Octupoles introduce new first-order behavior

"ORDER": ORDER IN TAYLOR/PS EXPANSION

→ Perturbation theory

resonances



Review: 1D Normalized/Action-Angle Coordinates

(x_p, x_p') : physical coordinates

(x, x') : normalized coordinates

$$\begin{pmatrix} x_p \\ x_p' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

inverse Floquet transformation, book Equation 3.22

Units
Phase space
Connection to SHO

Physical: $[x_p] = m$ $[x_p'] = (\text{rad})$

Normalized: $[x] = [x'] = m^{1/2}$

LINEAR

used to $\beta(s)$

$\alpha(s) \neq 0$

$(x x')$

$2\pi Q$ (1 turn)

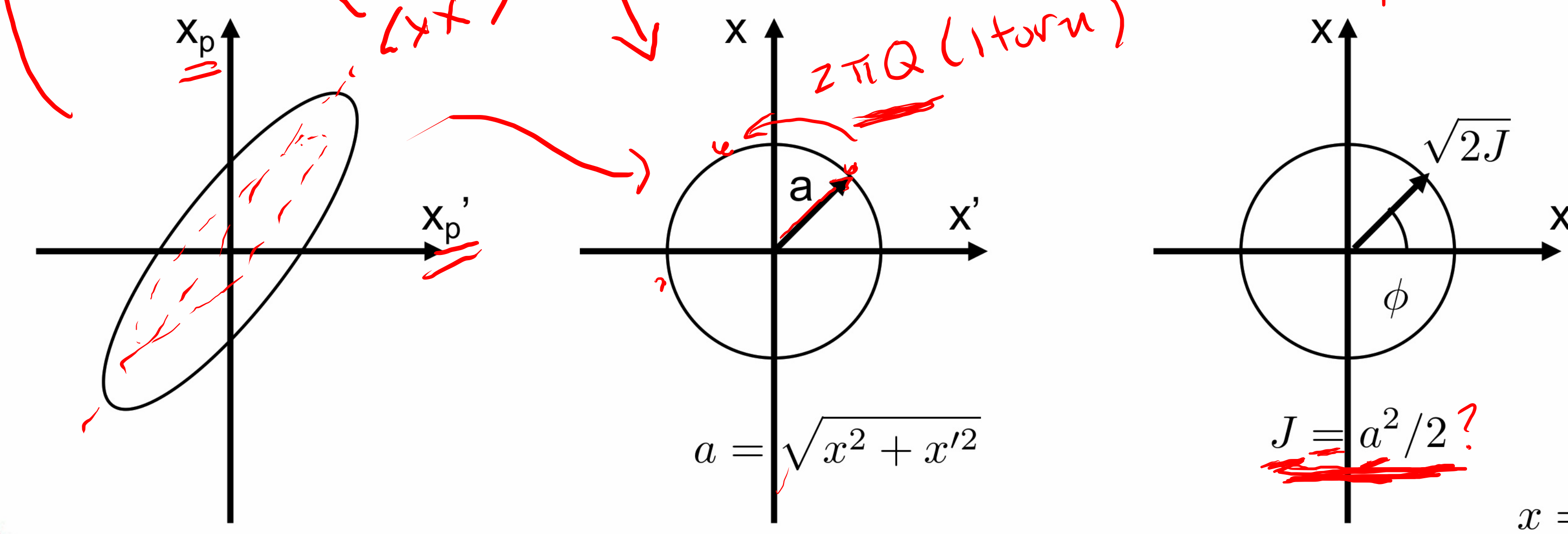
SHO Simple harmonic oscillator

$\ddot{x} + kx = 0$

sin ( )

cos ( )

$p_x \propto \dot{x}$



Physical (x_p, x'_p) Normalized (x,x') Action-Angle

$x = \sqrt{2J} \sin\phi$

$x' = \sqrt{2J} \cos\phi$

Linear Phase advance/turn

$$\Delta\phi = 2\pi Q_0$$

$H = \frac{1}{2} k x^2 + \frac{p_x^2}{2m}$

$H = \ldots 2\pi Q_0 J$

$\frac{1}{2} a^2$

10.1: 1D Single Octupole Kick

(x_p, x'_p) : physical coordinates

(x, x') : normalized coordinates

$$\begin{pmatrix} x_p \\ x'_p \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

inverse Floquet transformation, book Equation 3.22

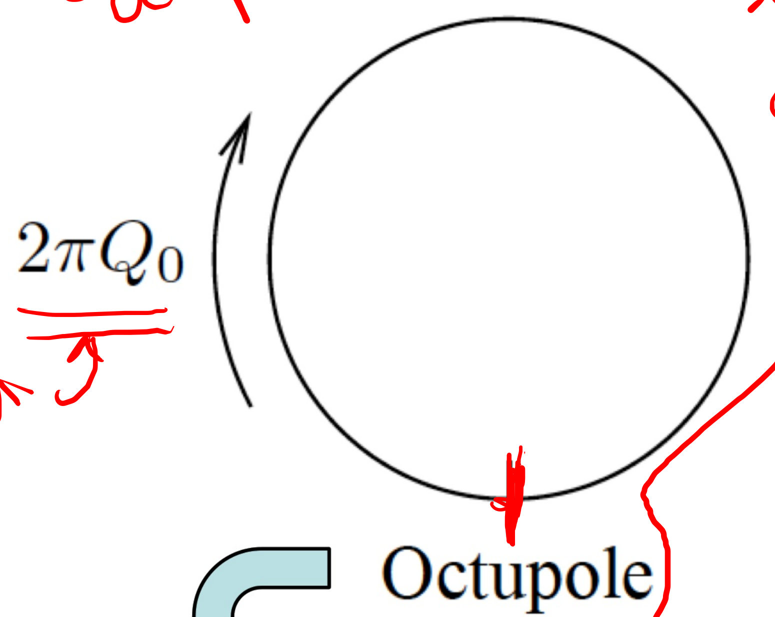
single octupole in linear ring

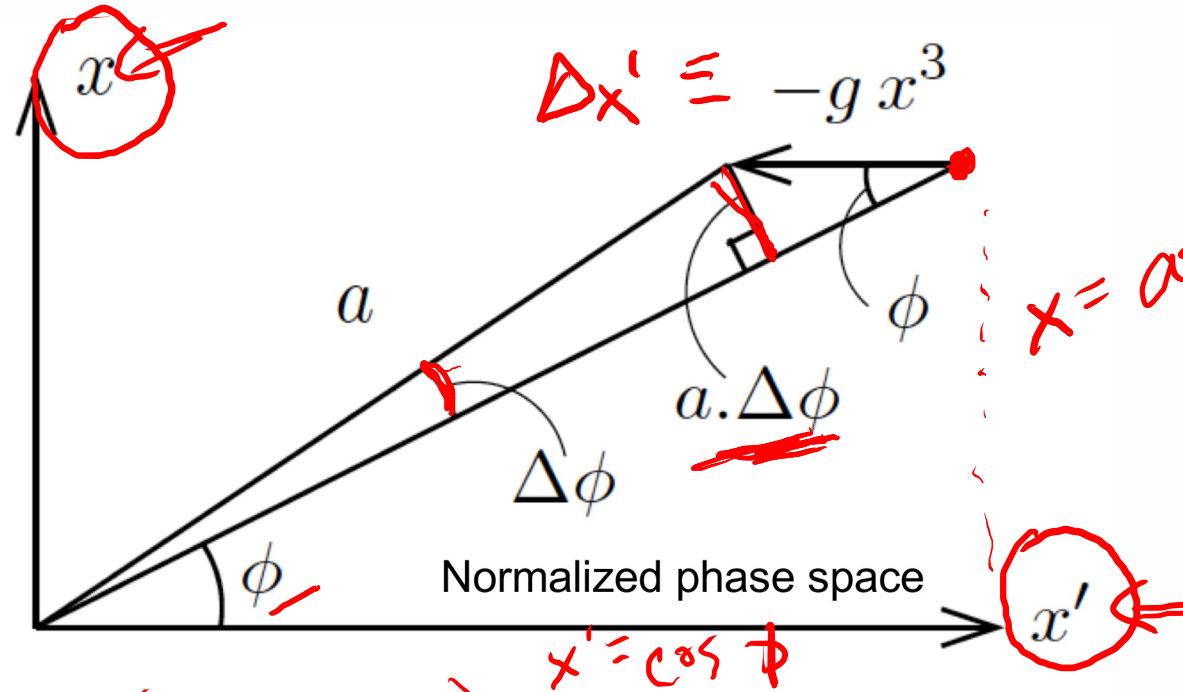
quad $\Delta x'_p \propto x_p$

sext $\Delta x'_p \propto x_p^2$

oct

linear phase advance per turn





$$\Delta x'_p = -g_p x_p^3 \qquad g_p \equiv \frac{B'''L}{(B\rho)} \text{ (be careful)}$$

L_{sext} (Mag lecture)

~ rigidity

madx definitions (manual eqn 1.8)

$$B_y(x,0) = \sum_{n=0}^{\infty} \frac{B_n\, x^n}{n!}$$

Taylor series

Octupole coefficient $B_3 = (\partial^3 B_y/\partial x^3)$.

$$\Rightarrow g_p = \frac{B_3}{3!}$$

- Linear 1D lattice with single octupole kick

$$\Delta x' = -g x^3 \qquad g \equiv g_p \beta^2$$

$\beta = 1\,m$

1D Single Octupole Detuning and Resonances

$$\Delta x' = -gx^3 \qquad g \equiv g_p \beta^2$$

5

- Use the normalized phase space figure (using triangles) to show that

$\sin^4 > 0$

$$\boxed{\sin\phi = \frac{a\,\Delta\phi}{gx^3}} \Rightarrow \quad \Delta\phi = gx^3 \sin\phi / a = ga^2 \sin^4\phi$$

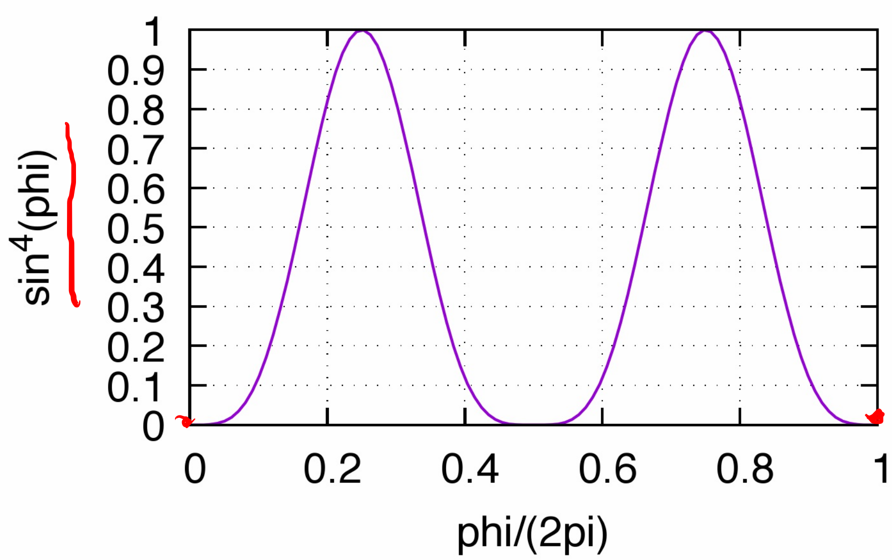
Use $x = a\sin\phi$

$$\Delta\phi = ga^2 \sin^4(\phi)$$
$$= ga^2 \left(\underbrace{\frac{3}{8}} - \underbrace{\frac{1}{2}\cos(2\phi) + \frac{1}{8}\cos(4\phi)} \right)$$

Amplitude-dependent detuning: doesn't depend on phase!

Resonant driving: periodic in betatron phase ϕ

$$\boxed{\sin^4 x = \frac{1}{8}[3 - 4\cos(2x) + \cos(4x)]}$$



(Useful Euler Trick)

$$\sin^n(\phi) = \underbrace{\left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)^n}_{\text{binomial expansion}} = \frac{1}{(2i)^n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{(m+1)} (e^{i\phi})^{n-m} (e^{-i\phi})^m$$

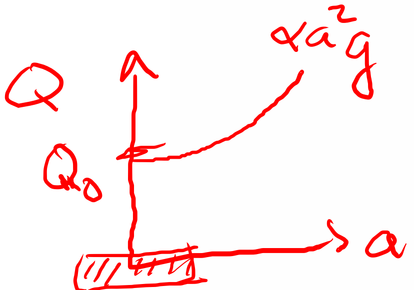
$$\sin^2 x = \frac{1}{2}[1 - \cos(2x)] \qquad \text{quad} \qquad \cos^2 x = \frac{1}{2}[1 + \cos(2x)]$$

$$\sin^3 x = \frac{1}{4}[3 \sin x - \sin(3x)] \qquad \text{sext} \qquad \cos^3 x = \frac{1}{4}[3 \cos x + \cos(3x)]$$

$$\sin^4 x = \frac{1}{8}[3 - 4\cos(2x) + \cos(4x)] \qquad \text{octopole} \qquad \cos^4 x = \frac{1}{8}[3 + 4\cos(2x) + \cos(4x)]$$

$$\sin^2\phi = \left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)^2 = \left[\frac{e^{2i\phi} + e^{-2i\phi} - 2}{-4}\right] = \frac{1}{2} - \cos(2\phi)$$

Octupole Detuning Amplitude Dependence



$\Delta\phi = \frac{3}{8} g a^2$ + stuff that depends on ϕ

⇒ tune spread

- $\Delta\phi$ is an additional phase advance every turn
  - Dependent on amplitude a but not dependent on phase ϕ

- This is fundamentally a shift in the tune
  - Base (small-amplitude) tune is defined to be Q_0
  - Tune of particles at amplitude a from octupoles is

$$Q = Q_0 + \frac{3}{16\pi} g a^2$$

Why do $\Delta\phi$ and Q differ by 2π factor?

Q : tune, normalized phase advance over 1 turn

  - Nicely first order in octupole strength g
  - Turns out to be first order for quadrupoles, octupoles, dodecapoles, ... (you can see the pattern; dodecapole homework)
    - (Second order in nonlinearity strength for sextupoles, decapoles, ...)

10.2: Discrete Motion in (J,ϕ) Space

- Using action-angle space where $J \equiv a^2/2$

$$Q = Q_0 + \frac{3}{8\pi} g J$$

- We can work out the general behavior in action along with phase to find general time evolution for **well-behaved particles**:

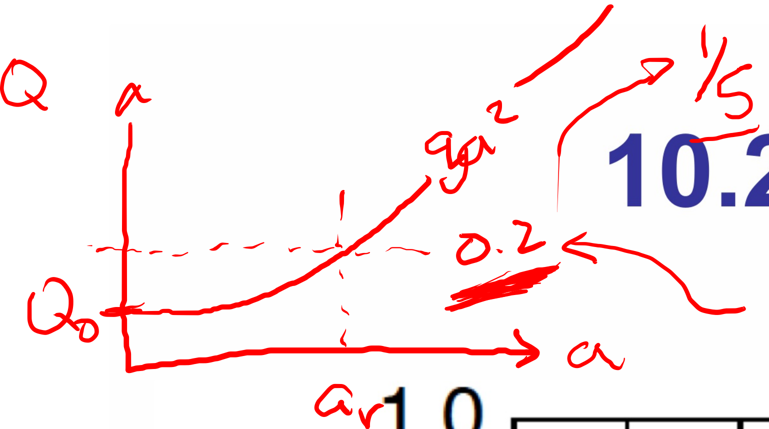
$$J_t = J_0 + \sum_{k=1}^{\infty} u_k \cos(k\phi_t + \phi_k) \qquad (10.10)$$

continue with octopoles

$$\boxed{\phi_t = \phi_0 + 2\pi Q_0 t} + \sum_{k=1}^{\infty} v_k \cos(k\, 2\pi Q\, t + \theta_k)$$

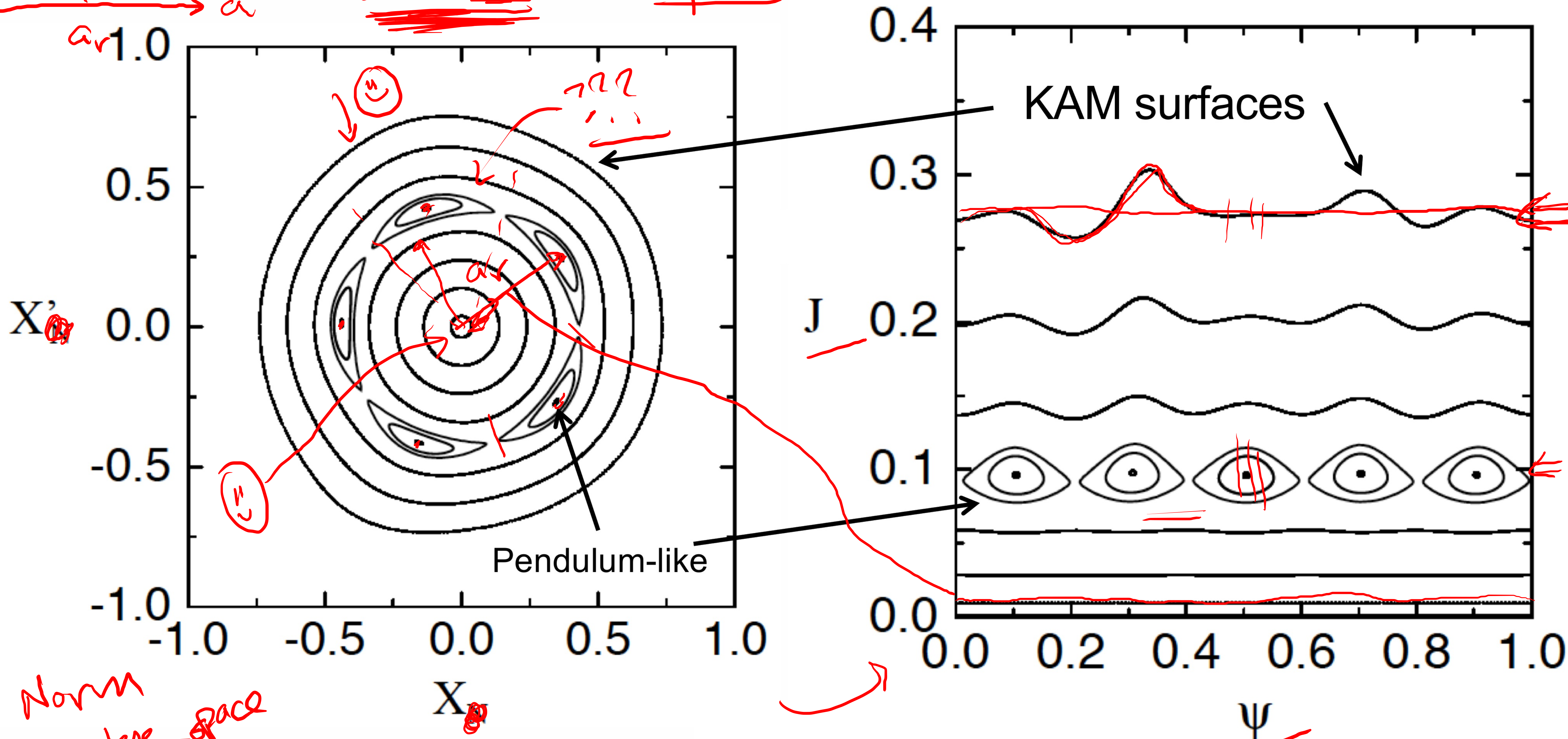
$u_k, v_k, \phi_k, \theta_k$ depend on nonlinearities

J no longer constant



10.2: Discrete Motion in (J,ϕ) Space

$\sin(5\phi)$ => in phase => nonlinear resonances



Norm phase space

- "Smear", and Collins distortion functions (T.L. Collins, FNAL report 84/114)

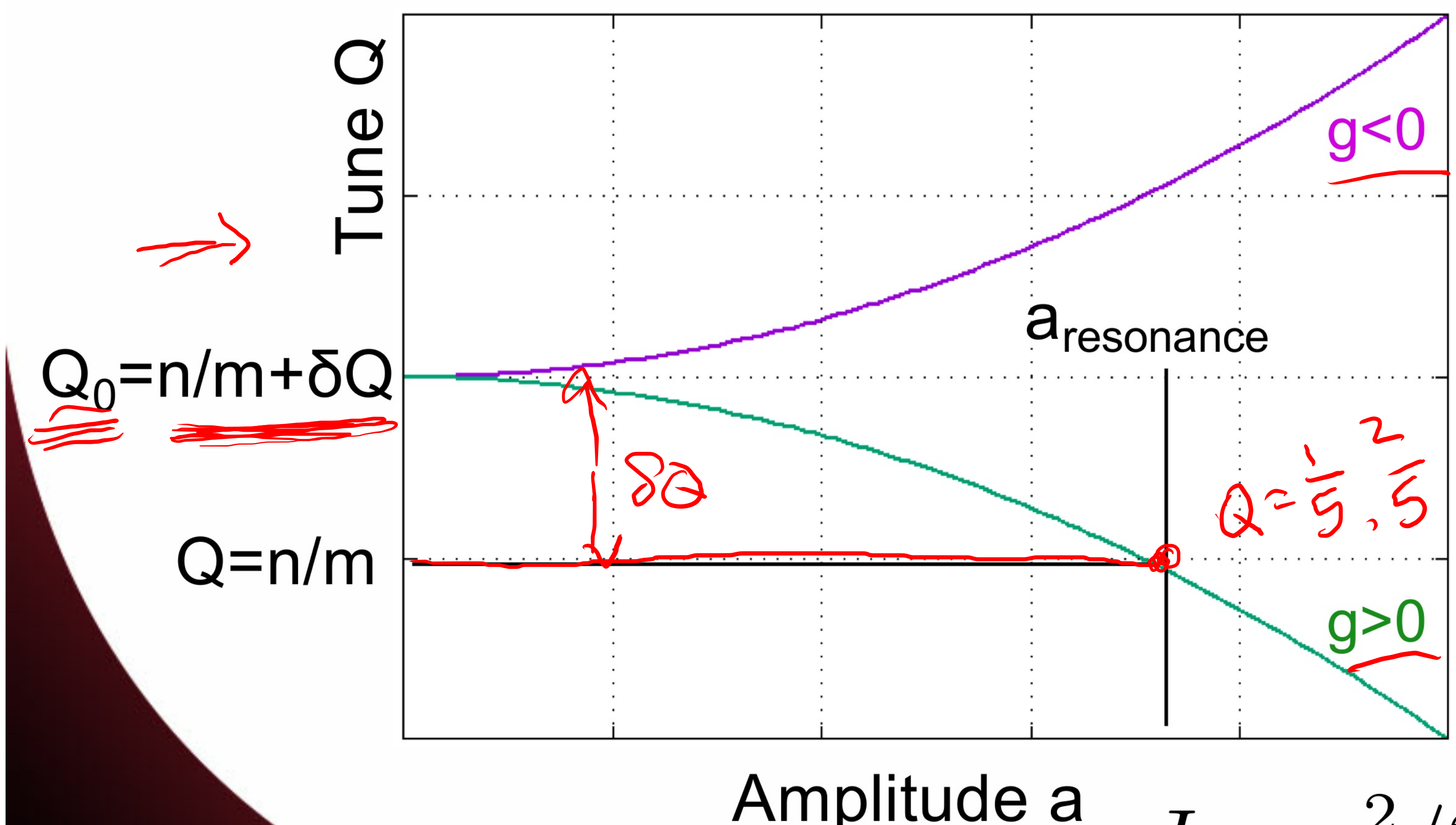
What is Really Happening Here?

sin(5φ) washes out
"phase averages to 0"

- Tune varies with amplitude depending on nonlinearity

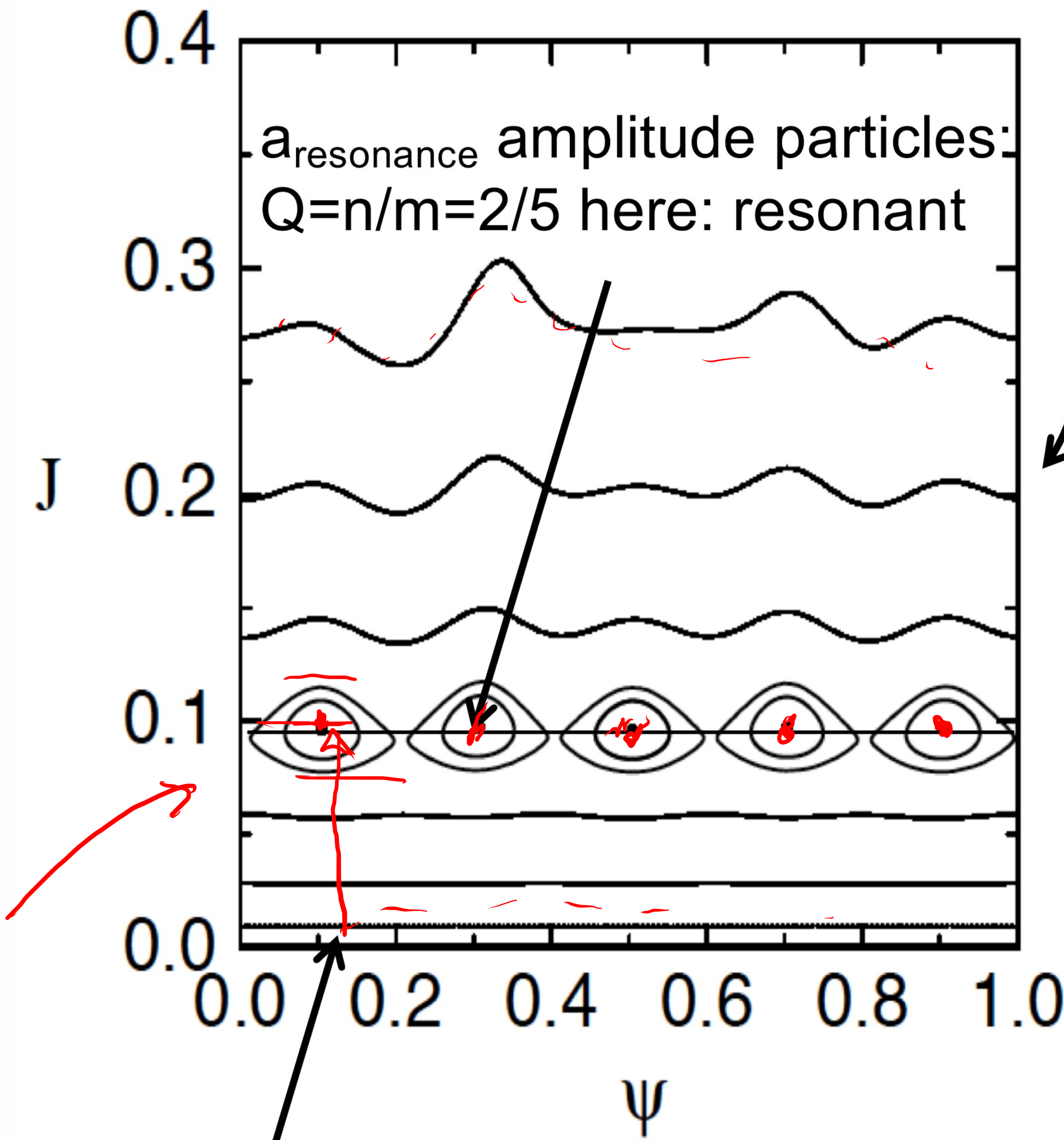
$$Q = Q_0 + \frac{3}{8\pi} gJ \quad \text{for octupoles}$$

- When Q_0 is near resonance, particles with amplitude $a_{resonance}$ have resonant tunes



$$J \equiv a^2/2$$

Large amplitude particles: tune curved below n/m, not resonant



Small amplitude particles: Q_0=n/m+δQ – not resonant

One-Turn Discrete Kobayashi "Hamiltonian"

1D

- Conservation suggest that we can write a "conserved" quantity and apply Hamiltonian dynamics to our discrete dynamical system $H_1(\phi, J)$ Linear: $\boxed{H = 2\pi Q_0 J}$

$$\Delta\phi = \frac{\partial H_1}{\partial J} \qquad \Delta J = -\frac{\partial H_1}{\partial \phi}$$

$\Delta J = 0$ (conserved)

$\Delta\phi = 2\pi Q_0$

- Here H_1 is a "one-turn" discrete Kobayashi Hamiltonian. More generally we can include all 2D nonlinearities:

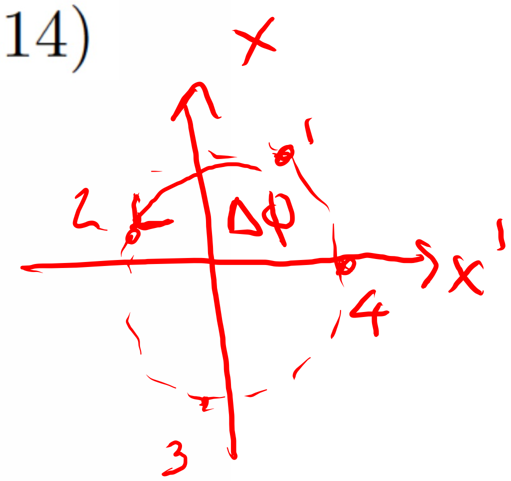
$$H_1 = 2\pi(Q_{x0} J_x + Q_{y0} J_y) + \sum_{ijkl} V_{ijkl} J_x^{i/2} J_y^{j/2} \sin(k\phi_x + l\phi_y + \phi_{ijkl}) \tag{10.14}$$

2D

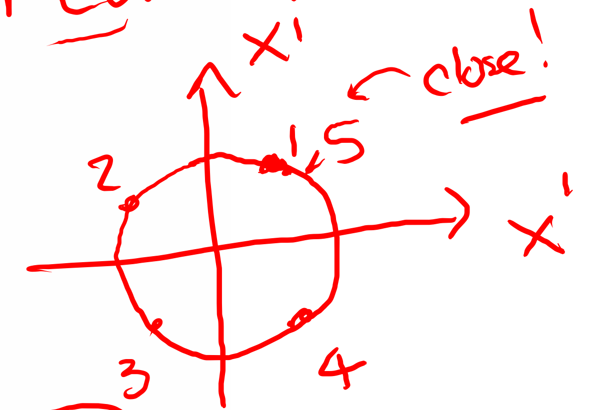
$$\text{Amplitude} - \text{dependent detuning when } k = l = 0, i \text{ and/or } j \neq 0$$

10.3: Motion Near Half-Integer Tunes

$$H_1 = 2\pi(Q_{x0}\,J_x + Q_{y0}\,J_y) + \sum_{ijkl} V_{ijkl}\,J_x^{i/2} J_y^{j/2} \sin(k\phi_x + l\phi_y + \phi_{ijkl}) \tag{10.14}$$



- One-turn maps from the one-turn "Hamiltonian" are still pretty jumpy → $\boxed{\Delta\phi = 2\pi Q_0}$ (discrete jumps)
  - The fractional part of the tunes can be big even if everything else is perturbatively small
- But we can integrate the above equation and handwave an "N-turn" map → Near resonance
  - Near Q=k/N values, the phase advance is nearly 2π
  - All motion in N turns becomes perturbatively small



4-turn "map"

small Δφ over 4 turns

⇒ differentials ☺

One-Turn Single Octupole Kobayashi Hamiltonian

Only from octupole!

octupole

octupole kick ($\Delta\phi$)

$$\Delta\phi = ga^2 \sin^4(\phi)$$
$$= ga^2 \left(\frac{3}{8} - \frac{1}{2}\cos(2\phi) + \frac{1}{8}\cos(4\phi) \right)$$

$$J \equiv a^2/2$$

$$\Delta\phi = gJ \left(\frac{3}{4} - \cos(2\phi) + \frac{1}{4}\cos(4\phi) \right)$$

Difference Hamiltonian

$$\boxed{\Delta\phi = \frac{\partial H_1}{\partial J}} \qquad \Delta J = -\frac{\partial H_1}{\partial \phi}$$

linear ring

1 octupole kick

$$H_1 = 2\pi Q_0 J + \frac{gJ^2}{2}\left(\frac{3}{4} - \cos(2\phi) + \frac{1}{4}\cos(4\phi) \right)$$

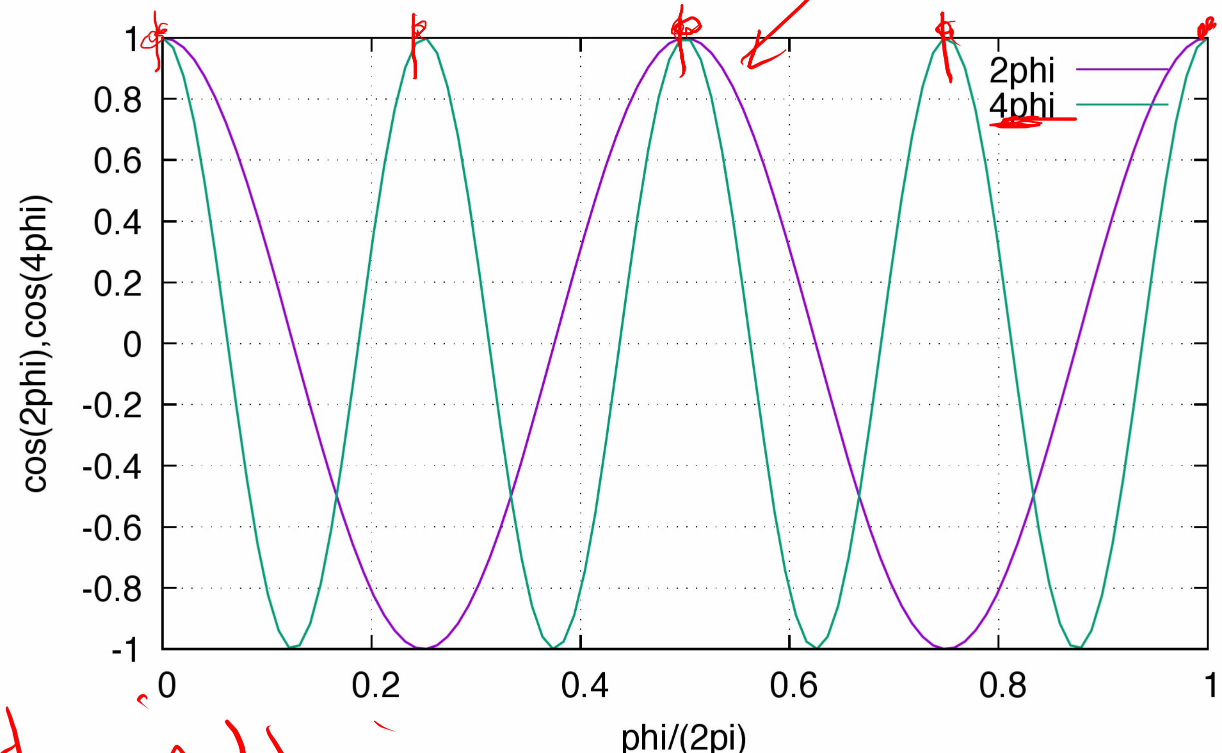
$$\Delta\phi = 2\pi Q_0 + gJ[\ldots] \qquad \Delta J = \frac{gJ^2}{2}[2\sin(2\phi) - \sin(4\phi)]$$

https://www.toddsatogata.net/2021-USPAS/SingleOctupoleMap.html

(Phase Averaging: Throwing Terms Away)

- Krylov-Bogoliubov(-Mitropolsky) Averaging
  - Far from resonance, driving terms average to zero

half-integer 2ϕ



octopole first-order Hamiltonian

$$H_1 = 2\pi Q_0 J + \frac{gJ^2}{2}\left(\frac{3}{4} - \cos(2\phi) + \frac{1}{4}\cos(4\phi)\right)$$

1D Motion Near Half-Integer Tunes

First ordering

one turn $\rightarrow$ $$H_1 = 2\pi Q_0 J + \frac{gJ^2}{2}\left(\frac{3}{4} - \cos(2\phi) + \frac{1}{4}\cos(4\phi)\right)$$

two turn average $\rightarrow$ $$\boxed{Q_0 = \frac{1}{2} + \delta Q} \qquad \delta Q \ll 1$$ $\Leftarrow$ ignore other resonances

$$H_2 \;=\; 2\pi\,\delta Q\,J \;+\; \left[\frac{3}{8} - \frac{1}{2}\cos(2\phi) + \frac{1}{8}\cos(4\phi)\right] gJ^2$$ $\Leftarrow$ look at $\Delta\phi$, ΔJ

perturbative

or more generally, in the presence of many octupoles

$$H_2 \;=\; \underbrace{2\pi\,\delta Q\,J}_{\text{Tune difference from 1/2}} \;+\; [\underbrace{V_0}_{\text{octupole amplitude dependent detuning}} + \underbrace{V_2\cos(2\phi+\phi_2)}_{\text{Half-integer “resonance driving”}} + \underbrace{V_4\cos(4\phi+\phi_4)}_{\text{Quarter-integer “resonance driving”}}]\; J^2$$

Fixed Points, Resonance Driving, Pendula

2-turn average

$$H_2 \approx 2\pi\,\delta Q\;J + \left[\frac{3}{8} - \frac{1}{2}\cos(2\phi)\right] gJ^2$$

(averaged out or ignored 4ϕ)

$$\Delta\phi = \frac{\partial H_2}{\partial J} = 2\pi\,\delta Q + \frac{3}{4} gJ - gJ^2\cos(2\phi)$$

$$\Delta J = -\frac{\partial H_2}{\partial \phi} = gJ^2\sin(2\phi)$$

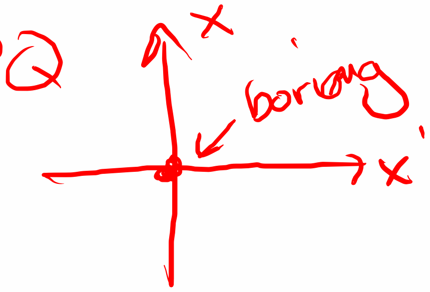
$\sin 2\phi = 0$

Where is $\Delta J = 0$? $J = 0$ is trivial; $\phi = 0, \pi$ is not

expand around FP ...

$\Delta\phi = 2\pi\,\delta Q$

$\Delta J = 0$



$\sin 2\phi = 0$

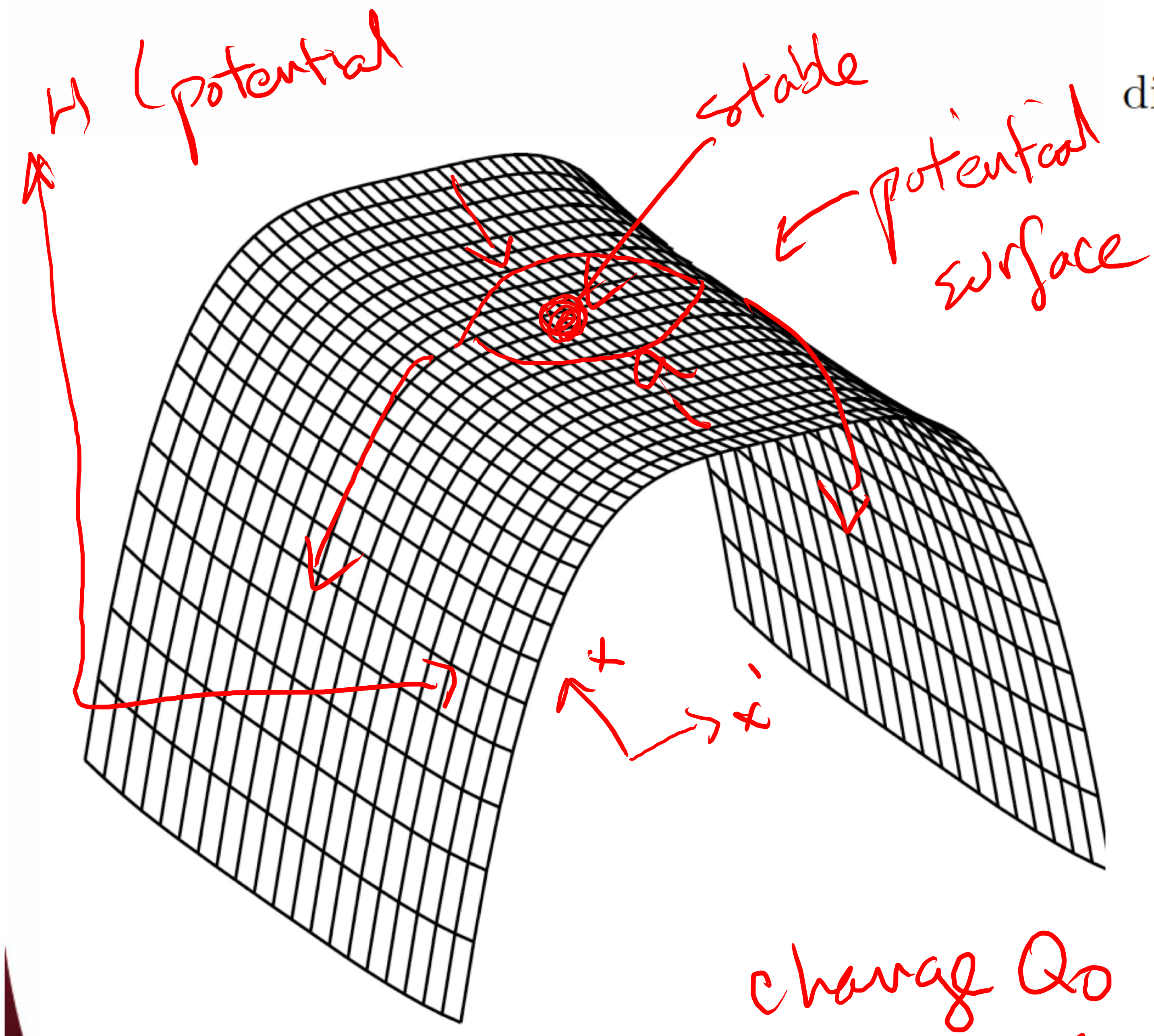
$\phi = 0, \pi$

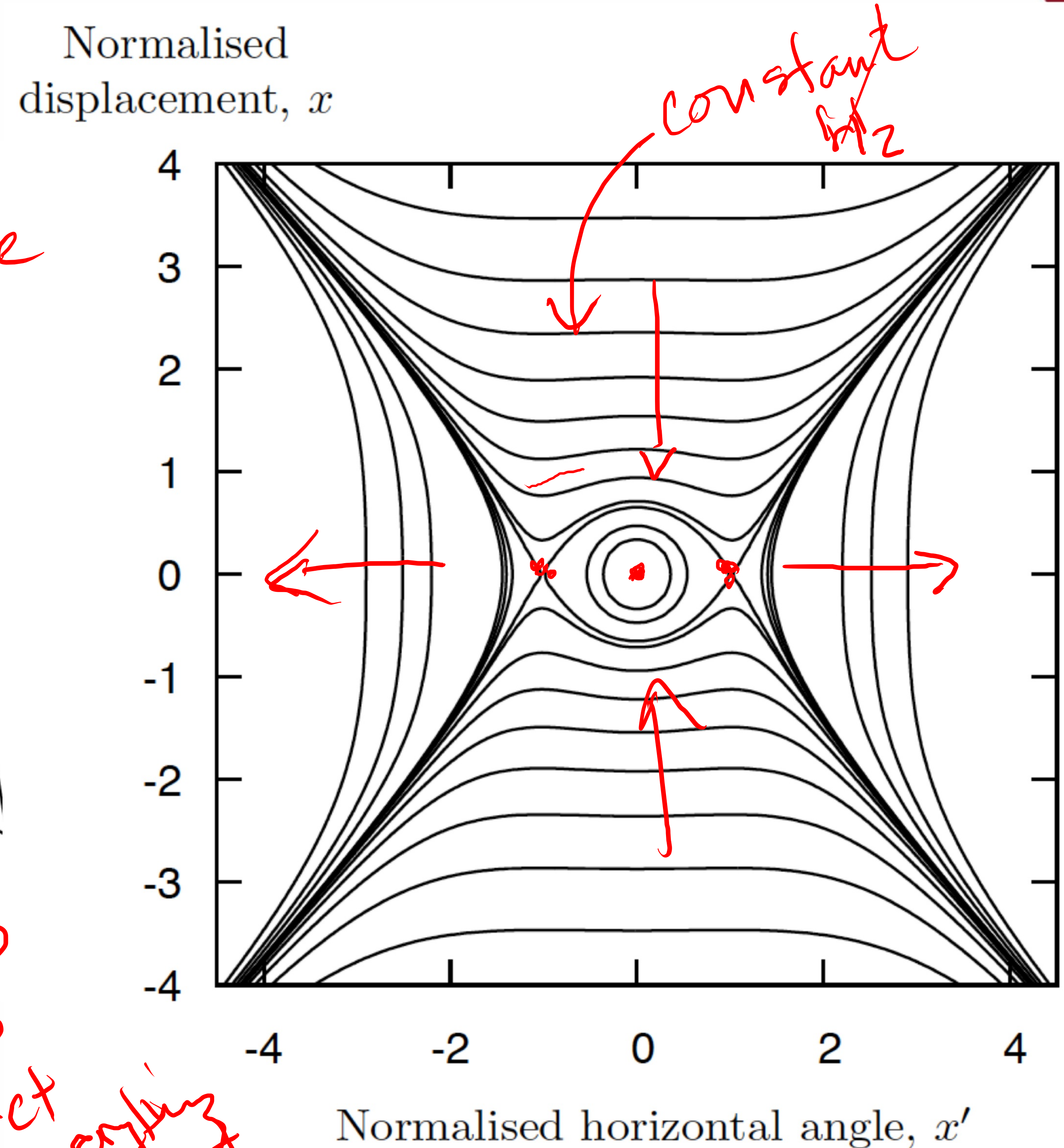
$\Delta J = 0$

$\Delta\phi = 2\pi\,\delta Q + \frac{3}{4} gJ - gJ^2 = 0?$

$J > 0$ maybe

Motion Near Half-Integer Tunes: Figs 10.3-4

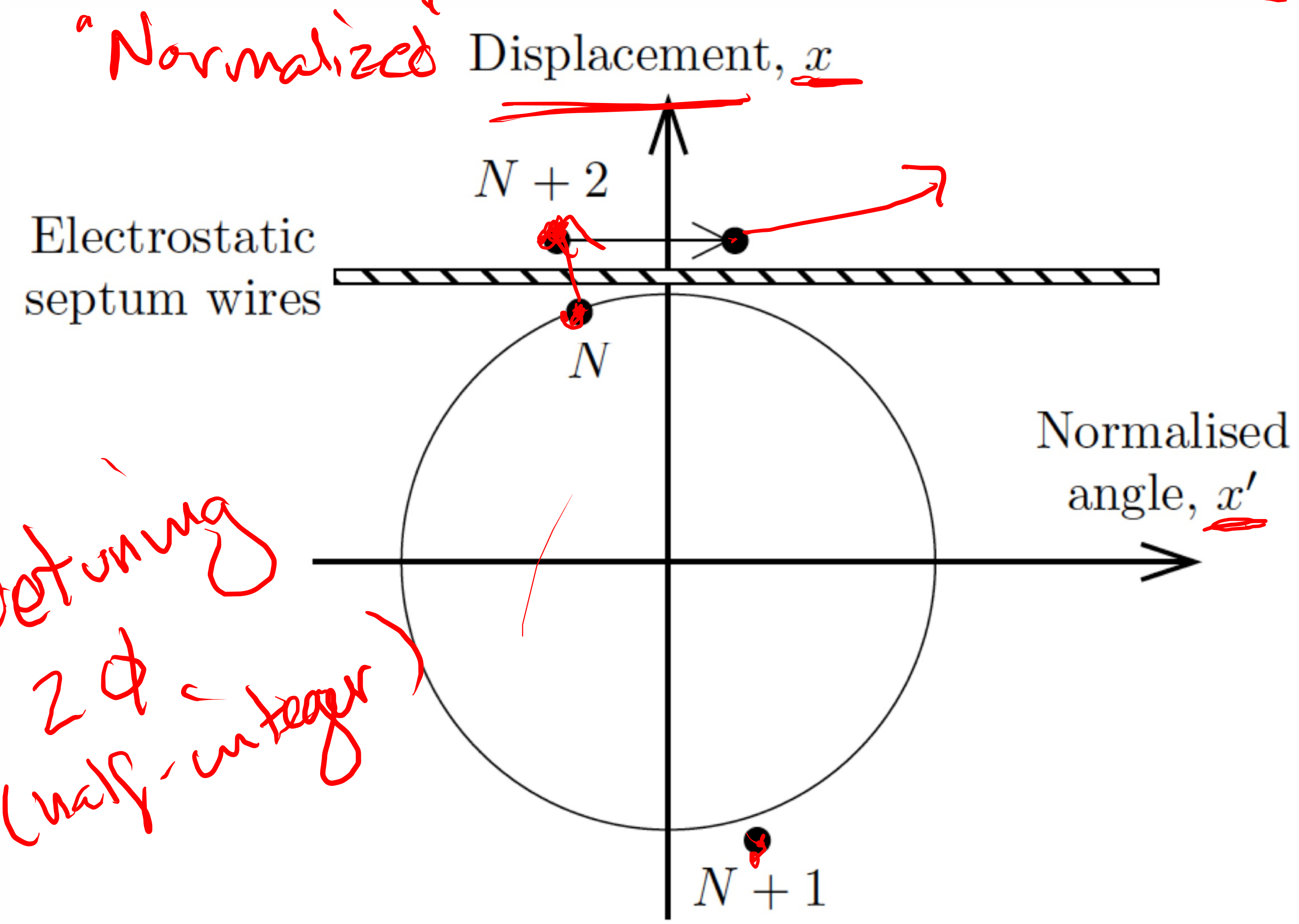




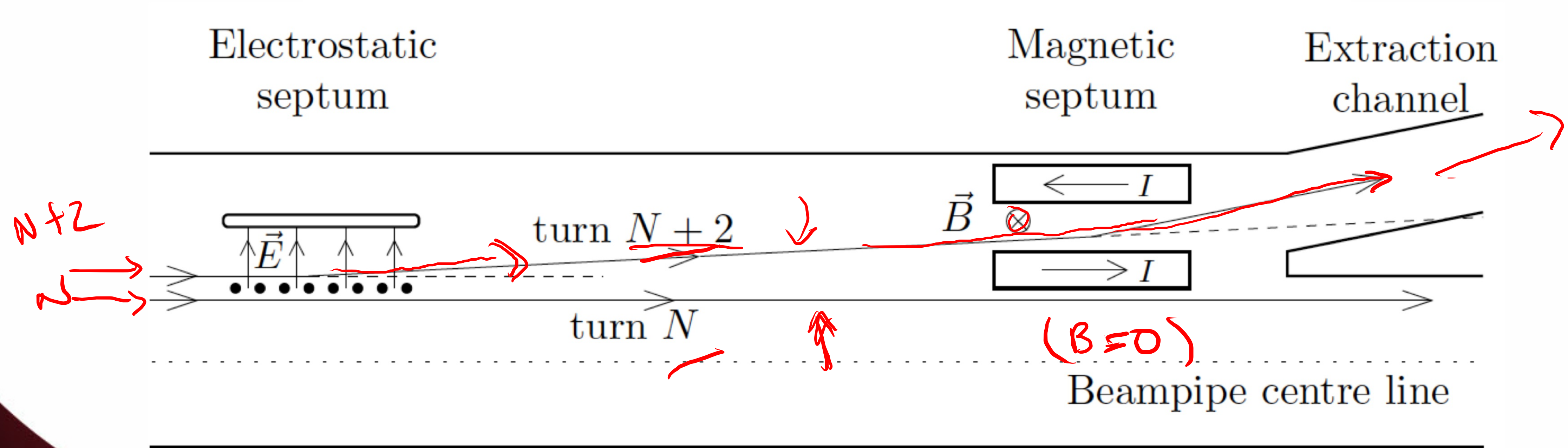
change Q_0
→ 0.5
→ extract everything

Can be used for slow extraction

Half-Integer Slow Extraction



Octupoles
first order detuning
$2\phi \approx$ (half-integer)



Entire USPAS courses on injection/extraction

- http://uspas.fnal.gov/programs/2017/niu/courses/injection-extraction.shtml

Injection and Extraction of Beams

Instructor:
Uli Wienands, Argonne National Lab and Edu Marin-Lacoma, CERN

Purpose and Audience
This course provides an introduction to the physics and design of ring injection and extraction systems. Proton, ion, and electron systems will be covered. The course is appropriate for anyone with some background in accelerator physics and technology and with an interest in injection and extraction of beams, including operational staff.

- http://uspas.fnal.gov/materials/17NIU/niu-injection-and-extraction.shtml

Course Materials - NIU - June 2017

Injection and Extraction of Beams

course given by Uli Wienands, Argonne National Lab and Edu Marin-Lacoma, CERN

Updated pdf of the lecture hand-outs: Accelerator Injection and Extraction

Zipped archive of the Mad-X and Python scripts for the Mad-X injection-design exercise: MADX Exercise files.zip (Windows and Linux users can ignore the _MACOSX folder that will be there after unzipping the file.)

Resonance Islands Revisited

- Todd's dissertation: E778 in the Fermilab Tevatron

- 5^{th} order resonance islands driven to "second order" in sextupole strength

- Modern usage: resonance island extraction at CERN (Gioavanozzi slides)