Lecture 12

Sextupoles & Chromaticity 1

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Lecture 12: Sextupoles & chromaticity - I

Chromaticity

1. **FODO lattice**
2. **RHIC uncorrected**
3. **Sextupole next to every quadrupole**
4. **Chromaticity correction**
5. **RHIC tune footprint**

Henon map

1. **Todd's on-line simulation ...**
2. **Triangles near Q = 1/3**
3. **Taxonomy of dynamical behavior**

Chaos - Lyapunov exponent

Dynamic Aperture

Catch-22 .. specified that a concern for one's own safety in the face of dangers that were real and immediate was the process of a rational mind. Orr was crazy and could be grounded. All he had to do was ask; and as soon as he did, he would no longer be crazy and would have to fly more missions.”

J. Heller, “Catch-22”.

- SEXTUPOLES are (usually) necessary to avoid NON-linear resonances that are often driven by sextupoles themselves!!
- Betatron motion of a particle with $\delta = \frac{\Delta p}{p}$ (constant)

$$x'' + \frac{k}{(1+\delta)} x = 0$$

$$y'' - \frac{k}{(1+\delta)} y = 0$$

?

$K(\delta)$ "weakens" with increasing δ

- CHROMATICITY measures the rate of change of tune wrt momentum:

$$\boxed{\chi = \frac{dQ}{d\delta}}$$

Both χ_x & χ_y are NATURALLY negative

① FODO LATTICE with N identical cells, each with ϕ phase advance

$$Q = \frac{N}{2\pi}\,\phi$$

so natural (uncorrected) chromaticity is

$$\chi_{NAT} = \frac{N}{2\pi}\,\frac{d\phi}{d\delta}$$

and use EQN. 3.48 to get

$$\frac{d\phi}{d\delta} = -2\tan\left(\frac{\phi}{2}\right)$$

so

$$\boxed{\chi_{x,NAT} = \chi_{y,NAT} = \frac{-Q\tan(\phi/2)}{\phi/2} \simeq -Q}$$

Rule of thumb OFTEN holds: Q: When does it fail?

(2) RHIC NATURAL $\chi_{NAT} \simeq -50$

RMS momentum spread $\frac{\sigma_p}{p} = \langle \delta^2 \rangle^{\frac{1}{2}} \simeq 2 \times 10^{-3}$

UNCORRECTED tune spread: $\boxed{\sigma_Q = |\chi_{NAT}| \frac{\sigma_p}{p} \simeq 0.1}$

UNACCEPTABLY shrinks available space in tune plane.

$\boxed{\text{MUST DECREASE } \chi}$ $|\chi| \to \sim 1$ or 2

3 SEXTUPOLE next to every QUADRUPOLE

FIRST separate closed orbit + betatron oscillations:

$$x_{TOT} = \underbrace{\eta \cdot \delta}_{\text{constant}} + x$$

oscillates

THEN expand quad-sext kick

$$\Delta x'_{TOT} = -q(1-\delta)(\eta\delta + x) - S(1-\delta)(\eta\delta + x)^2 + \dots$$

AND collect terms in $x^n \delta^m$:

$$(\Delta \eta')\delta + \Delta x' = -q\cdot\eta\delta \qquad x^0\delta^1$$

$$-qx - Sx^2 \qquad x^m\delta^0 \ \text{GEOMETRIC}$$

$$+q\cdot x\delta - S\,2\eta\cdot x\delta + Sx^2\delta$$

$$+O(\delta^2)$$

UP TO ORDER $x^1 \delta^1$ $\boxed{\Delta x' = -q\,x + (q - 2\eta S)\,x\delta}$

IF $\boxed{S = \frac{q}{2\eta}}$ then 0 and effective quad strength is independent of δ $\Rightarrow$ $\boxed{\chi = 0}$!!

In a FODO lattice $\eta_F \simeq 2\eta_D$

so $S_F \simeq -\frac{1}{2} S_D$

Q: What happens if $\eta = 0$?

[Q: What do geometric terms do?]

4 CHROMATICITY CORRECTION (general case)

PREVIOUSLY (EQN 8.29): If a quad increases strength by Δq

$$\Delta Q = \frac{\beta_0}{4\pi} \cdot \Delta q$$

QUAD weakens with δ $\therefore$ $\frac{dq}{d\delta} = -q$

SO NATURAL chromaticities are

$$\boxed{\begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix}_{NAT} = -\frac{1}{4\pi} \begin{pmatrix} \sum_{quads} q\beta_x \\ -\sum q\beta_y \end{pmatrix}}$$

Strength q is usually +ve (-ve) when β_x (β_y) is large

$\Rightarrow$ $\boxed{\text{NATURAL CHROMS ARE NEGATIVE}}$

SEXTUPOLES contribute a "QUADRUPOLE" component

$$\frac{dq}{dL} = \eta S$$

So, ONE FAMILY of sextupole contribute

$$\Delta \begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix} = \frac{S_1}{2\pi} \begin{pmatrix} \Sigma \eta \beta_x \\ - \Sigma \eta \beta_y \end{pmatrix}$$

different directions

TWO FAMILIES placed near F & D quads:

EQN 9.17

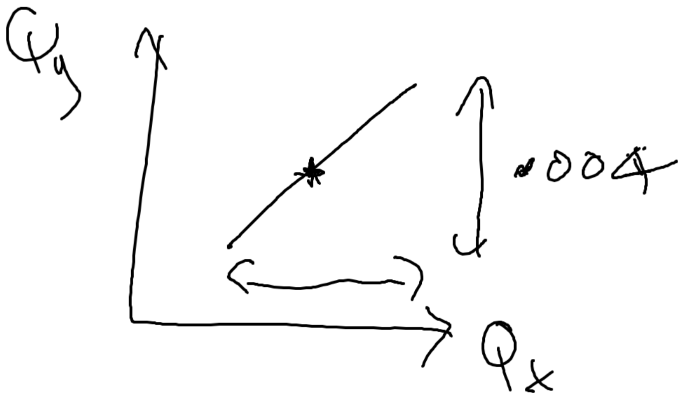
$$\boxed{\begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} \Sigma_F \eta \beta_x, & \Sigma_D \eta \beta_x \\ -\Sigma_F \eta \beta_y, & -\Sigma_D \eta \beta_y \end{pmatrix} \begin{pmatrix} S_F \\ S_D \end{pmatrix} + \begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix}_{NAT}}$$

SOLVE for S_F + S_D by inverting the matrix!

⑤ CHROMATICITY CORRECTION

$\chi_{GOAL} = +2$ THV

$\frac{\sigma_p}{p} = 2 \times 10^{-3}$



Q: Why keep χ slightly +ve? INSTABILITIES e.g. Head-Tail

Q: Why is TUNE FOOTPRINT a blob, not a line?

A: Also have tune shifts $\Delta Q_x(a_x, a_y)$ ...

Q: What drives the resonance lines in tune plane?

HÉNON MAP: GEOMETRIC TERMS ($\delta = 0$)

"exhibits $\boxed{\text{ALL TYPICAL PROPERTIES}}$ of ... dynamical systems"

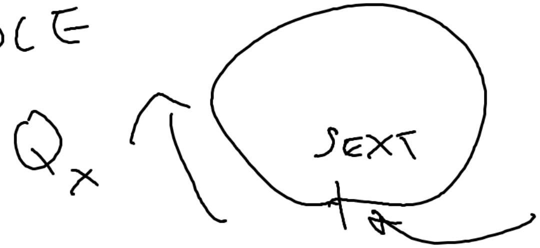
until finished {

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q), & \sin(2\pi Q) \\ -\sin(2\pi Q), & \cos(2\pi Q) \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix} + \begin{pmatrix} 0 \\ z^2 \end{pmatrix}$$

}

- The ONLY control parameter Q is like horizontal tune
- The kick z^2 is like a unit strength sextupole (horizontal)

ONE SEXTUPOLE



reference point just before sextupole

KICK: $\begin{pmatrix} x \\ x' \end{pmatrix}_{0+\epsilon} = \begin{pmatrix} x \\ x' - g x^2 \end{pmatrix}_0$

ROTATE: $\begin{pmatrix} x \\ x' \end{pmatrix}_1 = R(2\pi Q_x) \begin{pmatrix} x \\ x' \end{pmatrix}_{0+\epsilon}$

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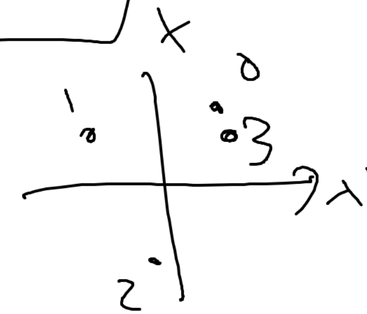
① TUNE IS CLOSE TO 1/3 $\quad \boxed{Q = \frac{1}{3} + \delta Q}$

Net motion after 3 turns is SMALL ...



$$\begin{pmatrix} x \\ x' \end{pmatrix}_3 - \begin{pmatrix} x \\ x' \end{pmatrix}_0 \approx 3\mu \begin{pmatrix} x' \\ -x \end{pmatrix}_0 - g\left[\begin{pmatrix} S_3 \\ C_3 \end{pmatrix} x_0^2 + \begin{pmatrix} S_2 \\ C_2 \end{pmatrix} x_1^2 + \begin{pmatrix} S_1 \\ C_1 \end{pmatrix} x_2^2\right]$$

WHERE $\mu = 2\pi\delta Q$, $C_k = \cos\left(k\frac{2\pi}{3}\right)$, $S_k = \sin\left(k\frac{2\pi}{3}\right)$

MORE SUCCINCTLY, 3-turn discrete KOBAYASHI HAMILTONIAN

$$\boxed{H_3 = \frac{\mu}{2}\left(x^2 + x'^2\right) + \frac{g}{9}\sum_{k=1}^{3}\left(C_k x + S_k x'\right)^3}$$

where $\Delta x \approx \frac{\partial H_3}{\partial x'} \cdot \Delta t$ with $\Delta t = 3$

$\Delta x' \approx -\frac{\partial H_3}{\partial x} \cdot \Delta t$

- H_3 (value of) is approximately constant along a trajectory

- Factorising H_3 gives

$$\boxed{H_3 = \frac{g}{3}\left(\frac{2\mu}{g}\right)^3 + \frac{g}{12}\underbrace{\left(x+\sqrt{3}x'+\frac{4\mu}{g}\right)}_{\text{straight line}}\underbrace{\left(x-\sqrt{3}x'+\frac{4\mu}{g}\right)}_{0\,?}\underbrace{\left(x-\frac{2\mu}{g}\right)}_{0\,?}}$$

So, if $H_3 = \frac{g}{3}\left(\frac{2\mu}{g}\right)^3$

$$\boxed{x+\sqrt{3}x'+\frac{4\mu}{g} = 0}$$

TRIANGLE !!

HÉNON'S TAXONOMY of 1-D nonlinear behavior

1. REGULAR NON-RESONANT
   - Roughly circular motion (near center, e.g.)
   - Enough turns (dots) make "continuous" lines
2. REGULAR RESONANT
   - Island hopping: only some phases accessible
3. RAPIDLY DIVERGENT (e.g. Q = 0.324)
   - Amplitude increases rapidly & regularly
4. CHAOS: Fly specks: visually distinctive, but

   Q: WHAT IS IT??

LYAPUNOV EXPONENT

- Start 2 test trajectories VERY close together (e.g. one digit in double precision) 13 digits $\to 10^{-13}$

Q: How do they BEGIN to diverge in time?

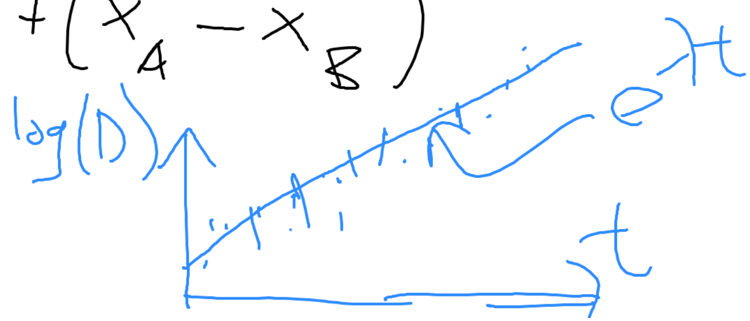
A1: LINEARLY $\Rightarrow$ Regular

A2: EXPONENTIALLY $\Rightarrow$ Chaotic fly specks

e.g. $D \equiv (x_A - x_B)^2 + (x'_A - x'_B)^2$

$$\boxed{D(t) \sim D_0 e^{\lambda t}}$$

Lyapunov exponent



DYNAMIC APERTURE (DA)

- IN SIMULATION there is (usually) a clear cut maximum amplitude beyond which particles rapidly get lost (hit the beam pipe)
- IN REALITY there is noise/excitation/diffusion/damping that drive particles across the DA
  - beam current decreases with time.
- EVEN IN 2-D (not 3-D) WITH 1 SEXTUPOLE this is a complex story!

Q: Which tunes give larger DAs?

Q: Why is this naive? What is missing?