Lecture 2B:

Linear Motion & Stability

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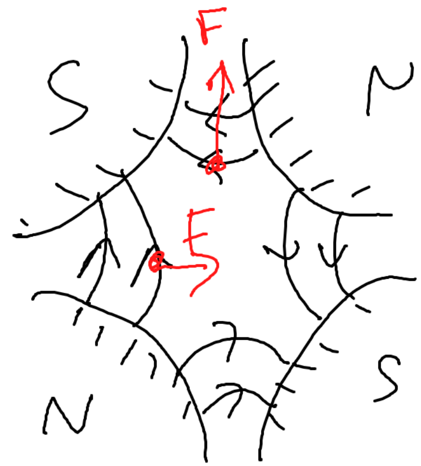
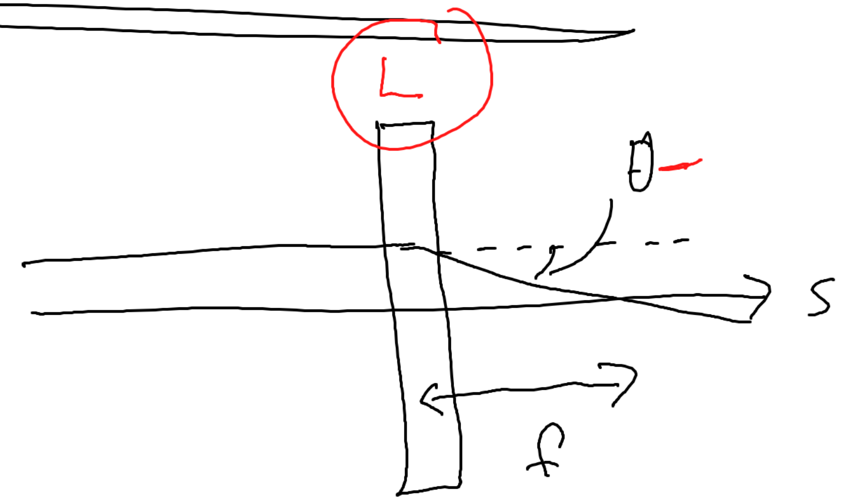
LECT. 2B : LINEAR MOTION & STABILITY

QUADRUPOLE

$$B_x = B' y$$
$$B_y = B' x$$

constant



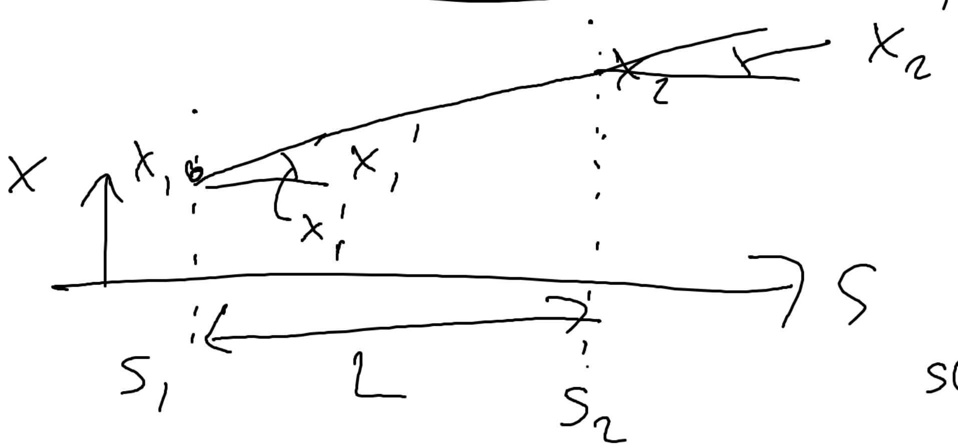


$$\boxed{\theta = \frac{q B_y}{p_0} L = \frac{B'}{(B\rho)} \cdot L \cdot x}$$

Focal STRENGTH $\boxed{\frac{1}{f} = KL}$ where $\boxed{K = \mathrm{sgn}(q) \frac{B'}{(B\rho)}}$

MATRICES: DRIFT

$$\boxed{x' \equiv dx/ds}$$



$$x_2 = x_1 + L\,x_1'$$

$$x_2' = x_1'$$

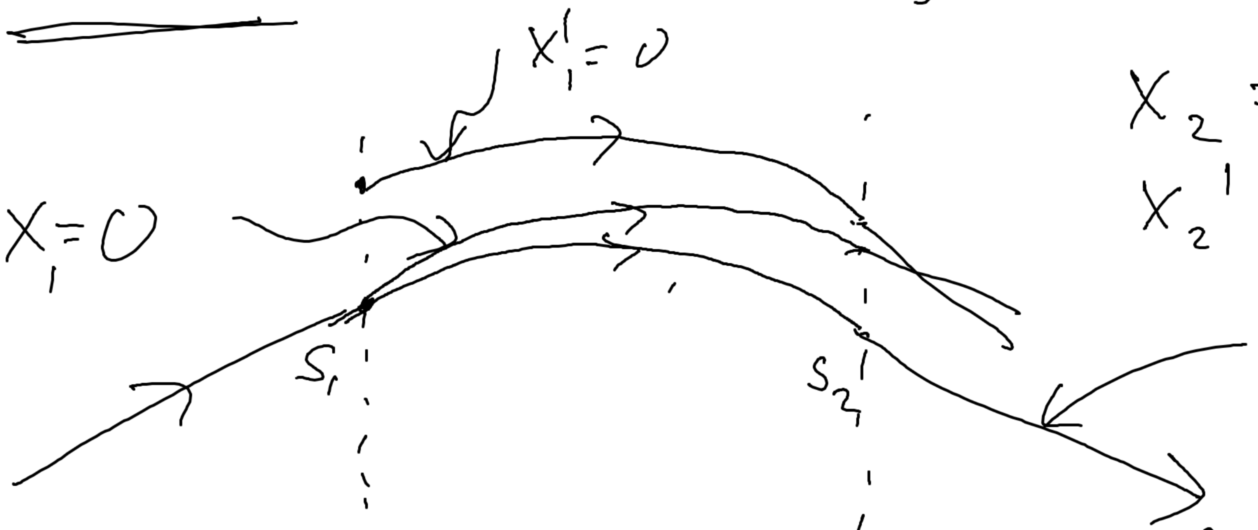
so

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = M_{DRIFT} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1$$

where

$$\boxed{M_{DRIFT} = \begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

DIPOLE (RECTANGULAR)



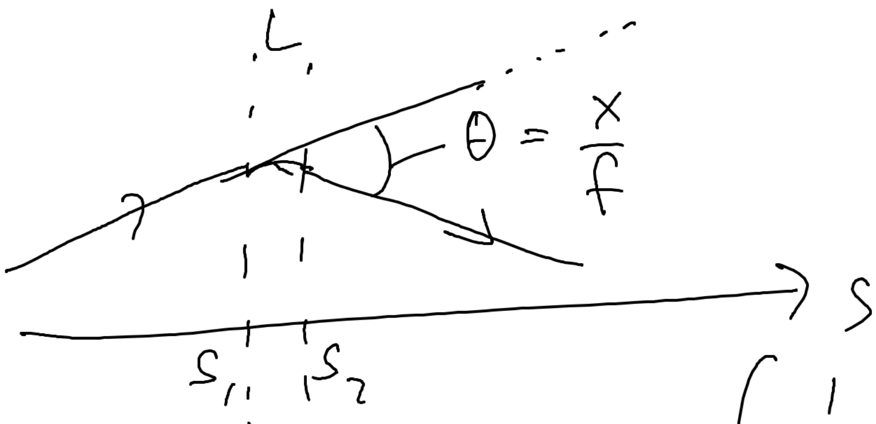
$$x_2 = x_1 + L x_1'$$
$$x_2' = x_1'$$

Bend angle is independent of x_1 (RBEND)

Co-ordinate frame rotates

$$\Rightarrow \quad \boxed{M_{RBEND} = M_{DRIFT}} \quad !!$$

QUADRUPOLE



THIN QUAD $(L \ll f)$

$$x_2 \simeq x_1$$

$$x_2' = x_1' - \frac{x}{f}$$

$$M_{\text{THIN QUAD}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & +\frac{1}{f} & 1 \end{pmatrix}$$

focus/defocus!

$$M_{\text{QUAD}} \begin{pmatrix} \cos(kL) & \frac{1}{k}\sin(kL) & 0 & 0 \\ -k\sin(kL) & \cos(kL) & & \\ 0 & 0 & \cosh(kL) & \frac{1}{k}\sinh() \\ 0 & 0 & k\sinh() & \cosh() \end{pmatrix}$$

where $k = \sqrt{K}$

LEGO Lattice of drift, dipole & quads

bend by 2π

Q1 : How to test if motion is stable?

Q2 : How to deal with the F/D problem?

Q3 : How to make beam small in some places, large in others?

Q4 : Why (& when) must nonlinear magnets be included (esp. sextupoles)?

LINEAR STABILITY

Reference point

ONE-TURN MATRIX

$$M = M_{n,n-1} \cdots\cdots M_{21} M_{10}$$

The 4x4 matrices M_{ij} are block diagonal! So horizontal stability becomes a 2x2 problem.

$$\boxed{\bar{X}_n = M^n \cdot \bar{X}_0}$$ where $\bar{X}_0 = \begin{pmatrix} x \\ x' \end{pmatrix}$

M has 2 complex eigenvectors $\bar{V}_1$, $\bar{V}_2$ such that $\boxed{M\bar{v} = \lambda \bar{v}}$ complex scalar

Write the initial vector as

$$\bar{x}_0 = A\bar{v}_1 + B\bar{v}_2$$

Both sides are real, but $A, B, \bar{v}_1, \bar{v}_2$ are complex.

On turn # n

$$\bar{x}_n = M^n \bar{x}_0 = A\lambda_1^n \bar{v}_1 + B\lambda_2^n \bar{v}_2$$

<u>IF</u> $\bar{x}_n$ is to be bounded for all n, <u>THEN</u> so also must λ_1^n & λ_2^n be bounded

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SOLVE the CHARACTERISTIC EQN. $\boxed{\det(M - \lambda \bar{I}) = 0}$

2x2 unit matrix

Start by writing $\boxed{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

then get quadratic eq^n

$$\underbrace{(ad-bc)}_{\det(M)\,=\,1}-\underbrace{(a+d)}_{\mathrm{Trace}(M)}\lambda+\lambda^2=0$$

so $\boxed{\lambda^{-1}+\lambda=\mathrm{Tr}(M)}=a+d$

By inspection the eigenvalues are reciprocals

$$\lambda_1 e^{i\mu},\ \lambda_2=e^{-i\mu}=\cos(\mu)-i\sin(\mu)$$

μ may be complex! Find it by solving

$$\boxed{2\cos(\mu) = \mathrm{Tr}(M)}$$

If μ IS complex, then λ_1^n or λ_2^n goes to ∞

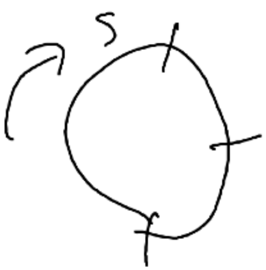
THEREFORE CONDITION FOR STABILITY IS

$$\boxed{-1 \leq \frac{1}{2}\mathrm{Tr}(M) \leq 1}$$

If the motion <u>IS</u> stable, then write the one turn matrix as

$$\boxed{M(s) \equiv \begin{pmatrix} \cos(\mu) + \alpha(s)\sin(\mu) & \beta(s)\sin(\mu) \\ -\gamma(s)\sin(\mu) & \cos(\mu) - \alpha(s)\sin\mu \end{pmatrix}}$$

NO s!!



Where "Twiss" functions or "Courant-Snyder" functions here $\boxed{\gamma \equiv \frac{1+\alpha^2}{\beta}}$

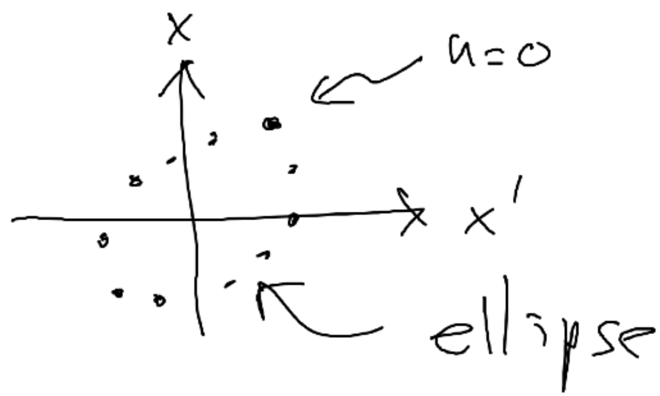
to guarantee $\boxed{\det(M) = 1}$

NOTE

① $M(s)$, $\beta(s)$, $\alpha(s)$ + $\gamma(s)$ are all functions of s

② BUT μ is NOT a function of s !!

Q: WHAT DOES MOTION LOOK like in phase space?



BUT CAN CONVERT ELLIPSES TO CIRCLES

$$M = \begin{pmatrix} c+\alpha s & \beta s \\ -\gamma s & c-\alpha s \end{pmatrix} \qquad \begin{aligned} c &= \cos(\mu) \\ s &= \sin(\mu) \end{aligned}$$

$$= \underbrace{\begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}}_{T^{-1}} \underbrace{\begin{pmatrix} c & s \\ -s & c \end{pmatrix}}_{R(\mu)} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}}_{T}$$

T — Floquet matrix

$$\boxed{M = T^{-1} R T}$$

Rotation matrix has the property

$$R^n(\mu) = R(n\mu)$$

after n turns

PHYSICAL PHASE SPACE

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$= (T^{-1}RT)(T^{-1}RT) \ldots\ldots (T^{-1}RT) \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$= T^{-1}(s)\, R^n(\mu)\, T(s) = T^{-1}(s)\, R(n\mu)\, T(s)$$

Better, use NORMALISED PHASE SPACE

$$\boxed{\begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix} \equiv T \begin{pmatrix} x \\ x' \end{pmatrix}}$$

If $\begin{pmatrix}\tilde{x}\\ \tilde{x}'\end{pmatrix} = a\begin{pmatrix}\sin(\phi_0)\\ \cos(\phi_0)\end{pmatrix}$

on turn n

$$\begin{pmatrix}\tilde{x}\\ \tilde{x}'\end{pmatrix}_n = R(n\mu)\begin{pmatrix}\tilde{x}\\ \tilde{x}'\end{pmatrix}_0$$

$\mu = 2\pi Q$

$\cancel{Q = 2\pi\mu}$

Betatron TUNE.

SOLN. TO EQN. OF MOTION IN NORMALISED SPACE

$$\boxed{\begin{aligned}\tilde{x}_n &= a\sin(2\pi Q\cdot n + \phi_0)\\ x'_n &= a\cos(2\pi Q\cdot n + \phi_0)\end{aligned}}$$

NORMALISED MOTION

PHYSICAL SPACE, Solution becomes

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = T^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix}$$

$$\boxed{\begin{aligned} x_n &= a\sqrt{\beta}\,\sin(2\pi Q.n + \phi_0) \\ x_n' &= \frac{a}{\sqrt{\beta}} \cos(2\pi Q.n + \phi_0) - \alpha \sin(2\pi Q.n + \phi_0) \end{aligned}}$$

① If $\alpha = 0$, ~~matrix~~ ellipse is erect

② β is ratio of ellipse extreme : ~~physical~~ displacement size to angular size

m

MOTION FROM s_1 TO s_2

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{21} \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

$$\boxed{M_{21} = T_2^{-1} \, R(\phi_2 - \phi_1) \, T_1}$$

$$\boxed{M_{21} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(C_{21} + \alpha_1 S_{21}\right) & \sqrt{\beta_2\beta_1}\; S_{21} \\ \frac{-\left((1+\alpha_1\alpha_2)S_{21} + (\alpha_2-\alpha_1)C_{21}\right)}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}\left(C_{21} - \alpha_2 S_{21}\right) \end{pmatrix}}$$

$$S_{21} = \sin(\phi_2 - \phi_1)$$

$$C_{21} = \cos(\qquad)$$

EXAMPLE

$$s_2 = s_1 + \Delta s$$

$$\phi_2 = \phi_1 + \Delta\phi$$

$$C_{21} \simeq 1$$

$$S_{21} \simeq \Delta\phi$$

$$\beta_2 = \beta_1 + \Delta\beta \quad \therefore \quad \sqrt{\frac{\beta_2}{\beta_1}} \simeq 1 + \frac{1}{2}\frac{\Delta\beta}{\beta_1}$$

$$M_{21} = \begin{pmatrix} \left(1+\frac{1}{2}\frac{\Delta\beta}{\beta}\right)\left(1+\alpha\Delta\phi\right) & \beta\cdot\Delta\phi \\ \sim & \left(1-\frac{1}{2}\frac{\Delta\beta}{\beta}\right)\left(1-\alpha\Delta\phi\right) \end{pmatrix}$$

As $\Delta s \to 0$, this becomes a DRIFT $M_{21} \begin{pmatrix} 1 & \Delta s \\ \sim & 1 \end{pmatrix}$

Showing that

$$\boxed{\frac{d\phi}{ds} = \frac{1}{\beta(s)}}$$

$$\boxed{\alpha = -\frac{1}{2}\frac{d\beta}{ds}}$$

PHASE ADVANCE

In general

$$\boxed{\phi(s_2) - \phi(s_1) = \int_1^2 \frac{ds}{\beta(s)}}$$

In particular, tune.

$$\boxed{Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}}$$

Integer does matter!!