Lecture 3:
Strong Focusing Optics

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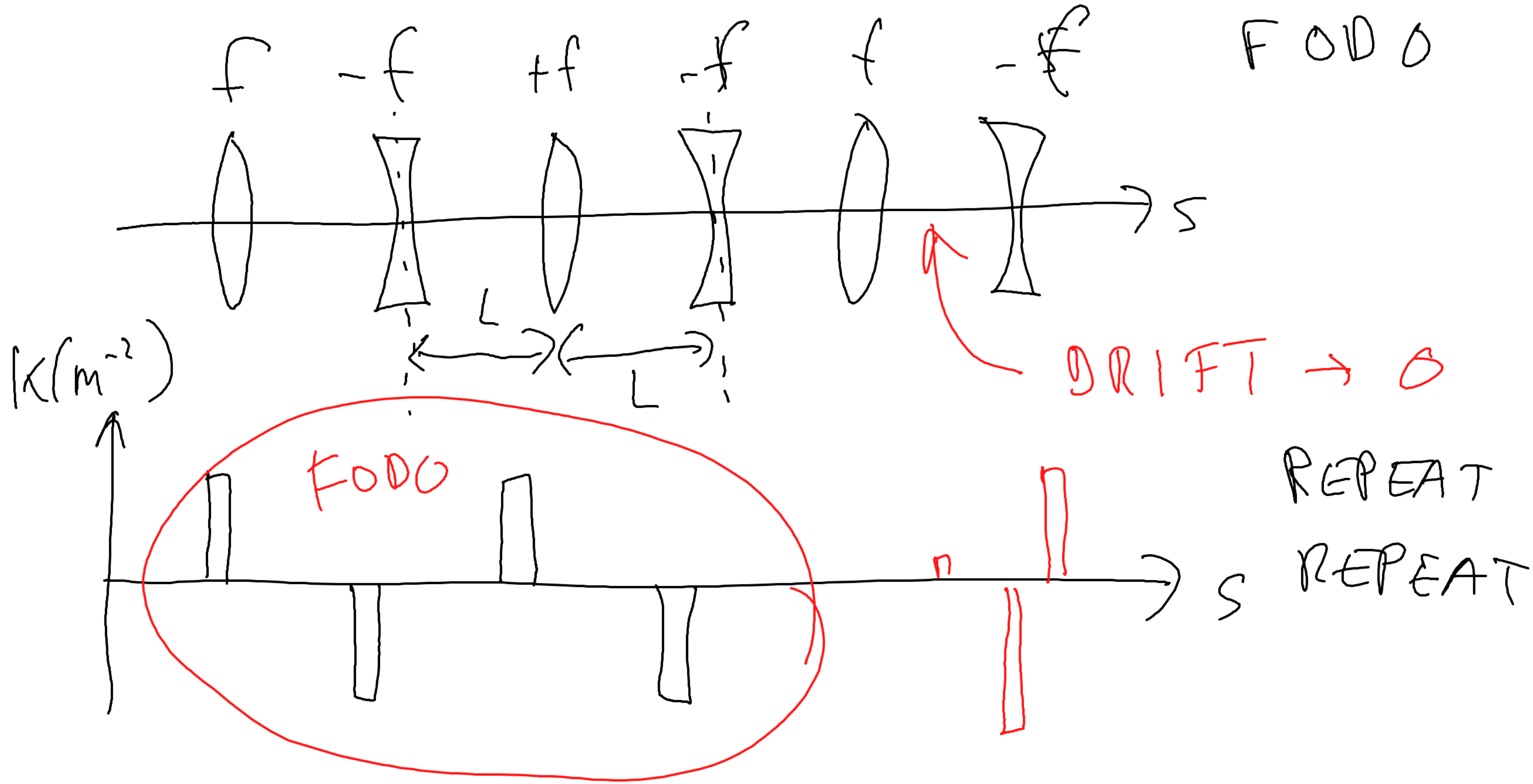
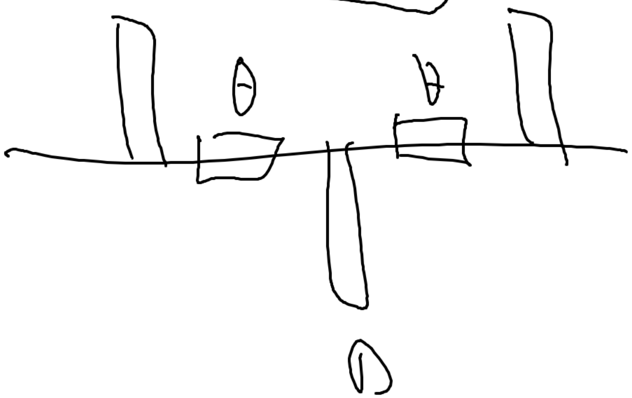
3: STRONG FOCUSING OPTICS



Q: When is a FODO STABLE?

A: WHEN $\boxed{L < 2f}$ (see below)

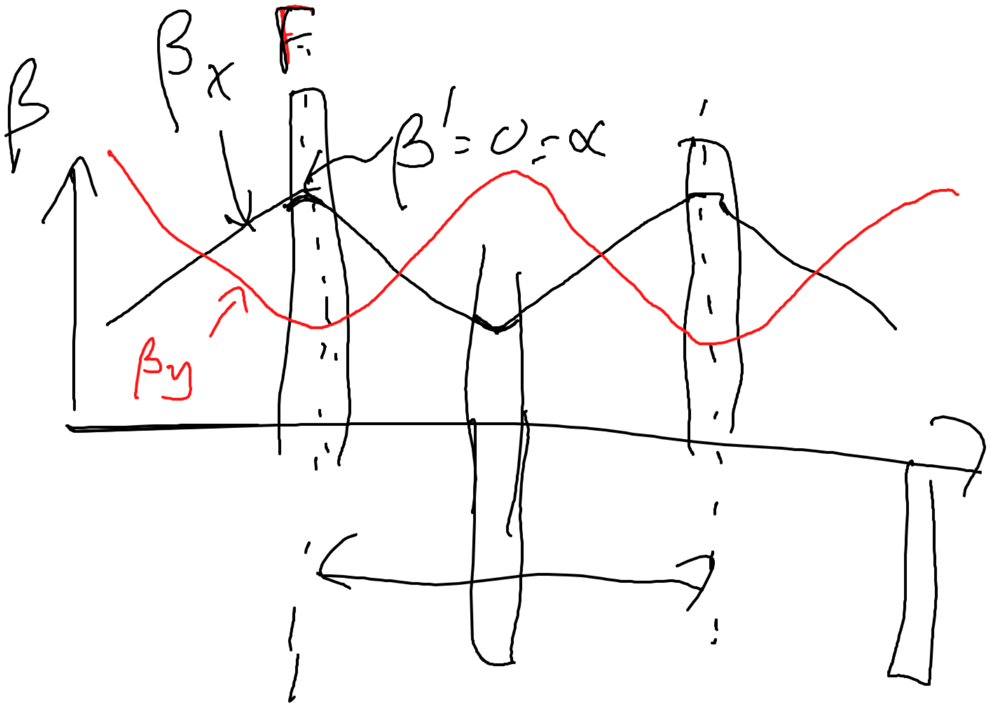
F O D O F



NOTE:

① $\beta(s)$ is PERIODIC w period $2L$ (ASSUME)

② $\beta' = \frac{d\beta}{ds} \sim \alpha = 0$ in centre of quad



$$M_{FODO} = \begin{pmatrix} \cos(\Delta\phi) & \beta_F \sin(\Delta\phi) \\ -\frac{\sin(\Delta\phi)}{\beta_F} & \cos(\Delta\phi) \end{pmatrix}$$

$$\left[\alpha_F = 0\right]$$

EXPLICITLY (thin quads ① $L_Q \ll f$ ② $L_Q \ll L$)

$$M_{FODO,x,y} = \begin{pmatrix} 1 & 0 \\ \mp q & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 2q & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mp q & 1 \end{pmatrix}$$

$\frac{1}{2}F$ O D O $\frac{1}{2}F$

$$\boxed{q = \frac{1}{2}\frac{1}{f}}$$

$$M_{FODO} = \begin{pmatrix} 1-2(qL)^2 & 2L\left(1+(qL)\right) \\ -2q(qL)(1-qL) & 1-2(qL)^2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Define $s = \sin\left(\frac{\Delta\phi}{2}\right)$ $c = \cos\left(\frac{\Delta\phi}{2}\right)$

$$M_{11} = \cos(\Delta\phi) = c^2 - s^2 = 1 - 2s^2 = 1 - 2(qL)^2$$

$$\boxed{\sin\left(\frac{\Delta\phi}{2}\right) = qL}$$

So NEED $\boxed{L < \frac{1}{q} = 2f}$ for stability

$$M_{12} = \beta_F \sin(2\phi) = \beta_F \cdot s \cdot c$$

$$= 2L(1+s) \checkmark$$

so

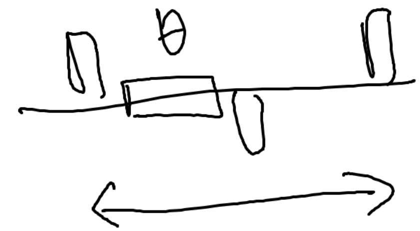
$$\boxed{\beta_F = L \cdot \frac{(1+s)}{sc} = L\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}}}$$

$$\boxed{\beta_D = L \cdot \frac{(1-s)}{sc} = L\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}}}$$

EG $q \to 0$: $\beta_F \to L$, $\beta_D \to L$

EXAMPLE : RHIC

$$q = \frac{1}{2}\frac{1}{f} = 0.05\ \mathrm{m}^{-1}$$



$$L_{RHIC} = 14.5\ \mathrm{m}$$

$$L_{LHC} \approx 6 \times 15 \approx 90\ \mathrm{m}$$

$$\Delta\phi \simeq 90^\circ \quad (83^\circ, 84^\circ)$$

$$s = \frac{1}{\sqrt{2}}$$

$$\beta_F = \sqrt{2}L\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)^{\frac{1}{2}}$$

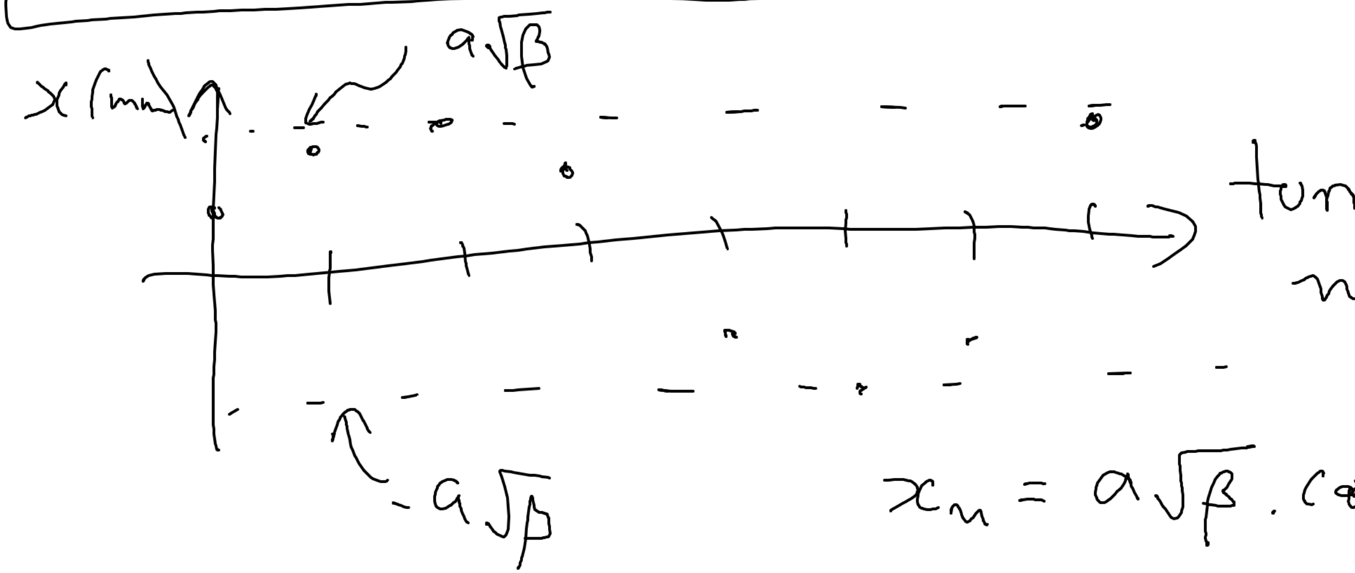
$$\beta_F = (2+\sqrt{2})L = 3.41L \approx 50\ \mathrm{m}$$

$$\beta_D \quad (2-\sqrt{2}) \rightarrow 10\ \mathrm{m}$$

WHAT DOES $\beta(s)$ MEAN?

4, (s) answers
perspectives

① TURN-BY-TURN



$$x_n = a\sqrt{\beta}\cdot\cos(2\pi Q_x n + \phi_0)$$

NOTE: (1) Max displacement $\sim \sqrt{\beta}$

(2) Integer part of Q is irrelevant

Q = integer + q

(RHIC: $Q_x = 28.195$) → 0.805

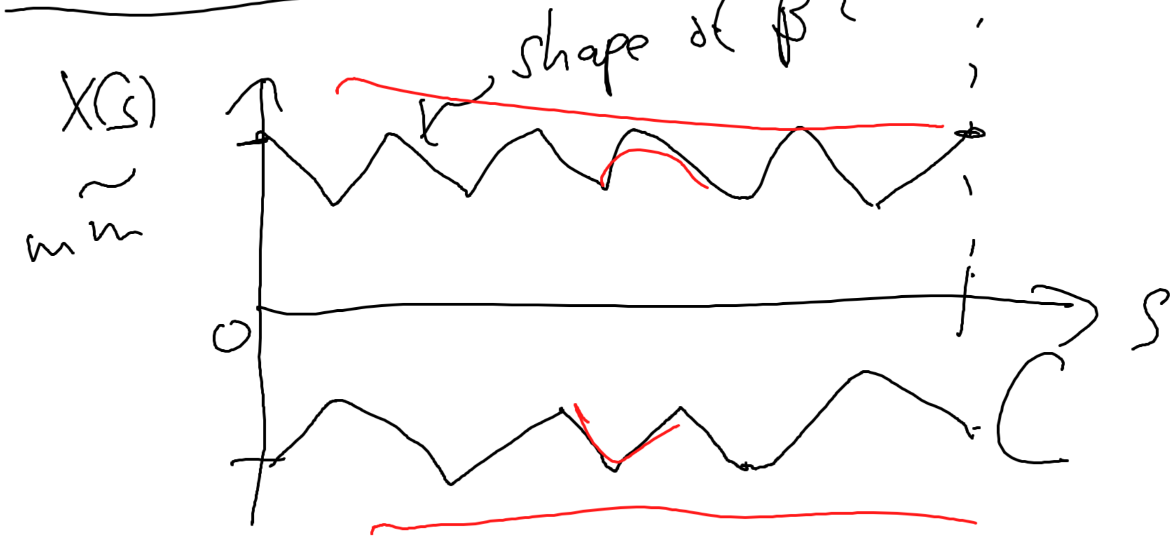
$Q_y = 29.205$

(3) A diagnostic measuring x_n can't distinguish q from $1-q$

Nyquist theorem?

sampling theory

R : CONTINUOUS



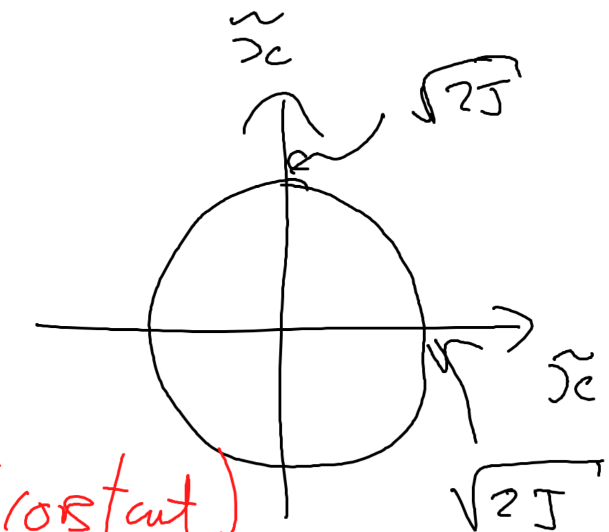
$$\boxed{x(s) = \sqrt{2J\beta}\cos(\phi(s))}$$

$\beta^{\frac{1}{2}}(s_1)$ + $\beta^{\frac{1}{2}}(s_2)$ represent relative amplitude requirements

③ SINGLE PARTICLE (ACTION)

Normalized co-ordinates

$$\begin{pmatrix}\tilde{x}\\ \tilde{x}'\end{pmatrix}_t = \sqrt{2J_t}\begin{pmatrix}\sin(\phi_t)\\ \cos(\phi_t)\end{pmatrix}$$



Action (constant)

Where $J_{\cancel{t}} = J =$ constant

$$\phi_t = 2\pi Q\cdot t + \phi_0$$

Angle

"Action-angle" co-ordinates (J_t, ϕ_t) v. convenient when we go nonlinear

SINCE $\begin{pmatrix} x \\ x' \end{pmatrix} \equiv T^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix}$

PHYSICAL CO-ORD

$$\boxed{x_t = \sqrt{\beta}\, \tilde{x}_t = \sqrt{2J\beta}\, \sin(2\pi Q\, t + \phi_0)}$$

RMS displacement of this 1 particle over many turns

$$\sigma_1 \equiv \langle x_t^2 \rangle^{\frac{1}{2}}$$

$$\boxed{\sigma_1 = \sqrt{\beta J}}$$

L L L

e.g. J (μm)

4 A BUNCH OF PARTICLES

Suppose a bunch has an action distribution $\rho(J)$

$$N_{bunch} = \int_0^\infty \rho(J)\,.\,dJ \simeq \begin{array}{l} 10^{11} \text{ protons (RHIC)} \\ 10^{9} \text{ Au ions (RHIC)} \end{array}$$

Bunch size is σ where

$$\sigma^2 = \frac{1}{N}\int_0^\infty \sigma_i^2 \cdot \rho(J)\, dJ = \beta \cdot \frac{1}{N}\int_0^\infty J \cdot \rho(J)\, dJ$$

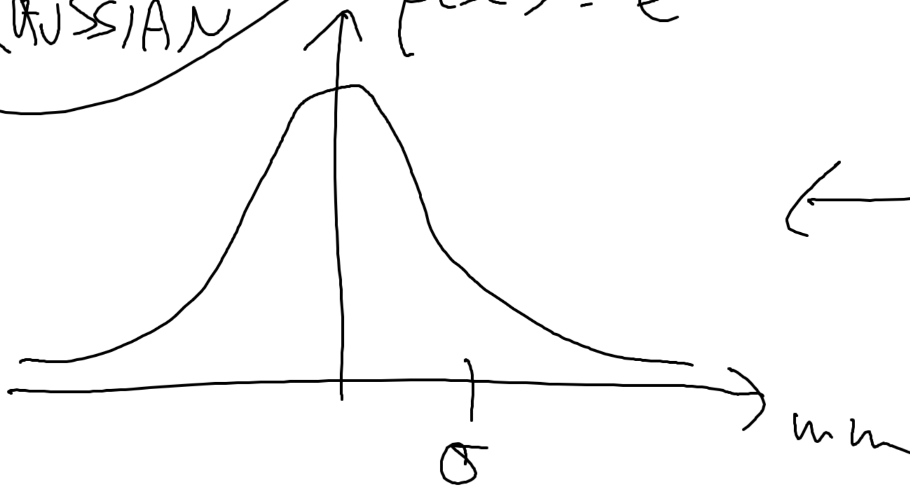
PROPERTY OF ACCELERATOR

$$\boxed{\sigma^2 \equiv \beta \langle J \rangle} \sim \frac{1}{\gamma}$$

PROP. OF BEAM

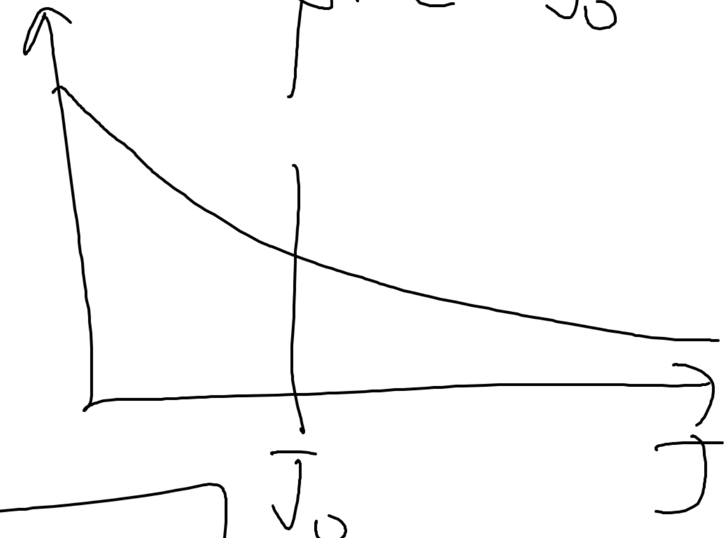
EXAMPLE
GAUSSIAN

$\rho(x) = e^{-\frac{x^2}{2\sigma^2}}$



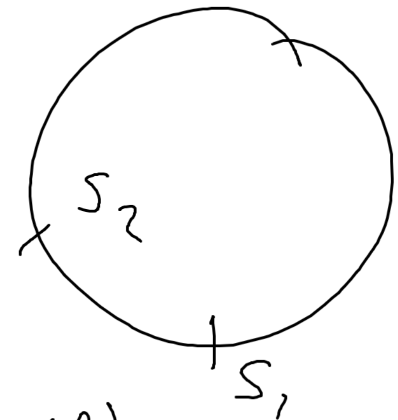
$\rho(J)$

$\rho = e^{-\frac{J}{J_0}}$



$\langle J \rangle = J_0$, so $\boxed{\sigma = \sqrt{\beta J_0}}$

5 PROPOGATE TWISS FUNCTIONS



IF $\quad M_{21} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

"IT CAN BE SHOWN"

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} m_{11}^2, & -2m_{11}m_{12}, & m_{12}^2 \\ -m_{21}m_{11}, & 1+m_{12}m_{21}, & -m_{12}m_{22} \\ m_{21}^2, & -2m_{22}m_{21}, & m_{12}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

EXAMPLE $\quad M_{21}$ is a drift $= \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

$$\beta_2 = \beta_1 - 2L\alpha_1 + L^2\gamma_1 \qquad \boxed{\gamma_1 = \frac{1+\alpha_1}{\beta_1}}$$

IF $\alpha = 0$ $\quad \boxed{\beta_2 = \beta_1 + L^2/\beta}$

$$\tan(\Delta\phi) = \frac{m_{12}}{m_{11}\beta_1 - m_{12}\cancel{\alpha_1}}$$

$\to 0$

$$\boxed{\tan(\Delta\phi) = \frac{L}{\beta_1}} \to \infty$$

$$\Delta\phi \to \frac{\pi}{2}$$