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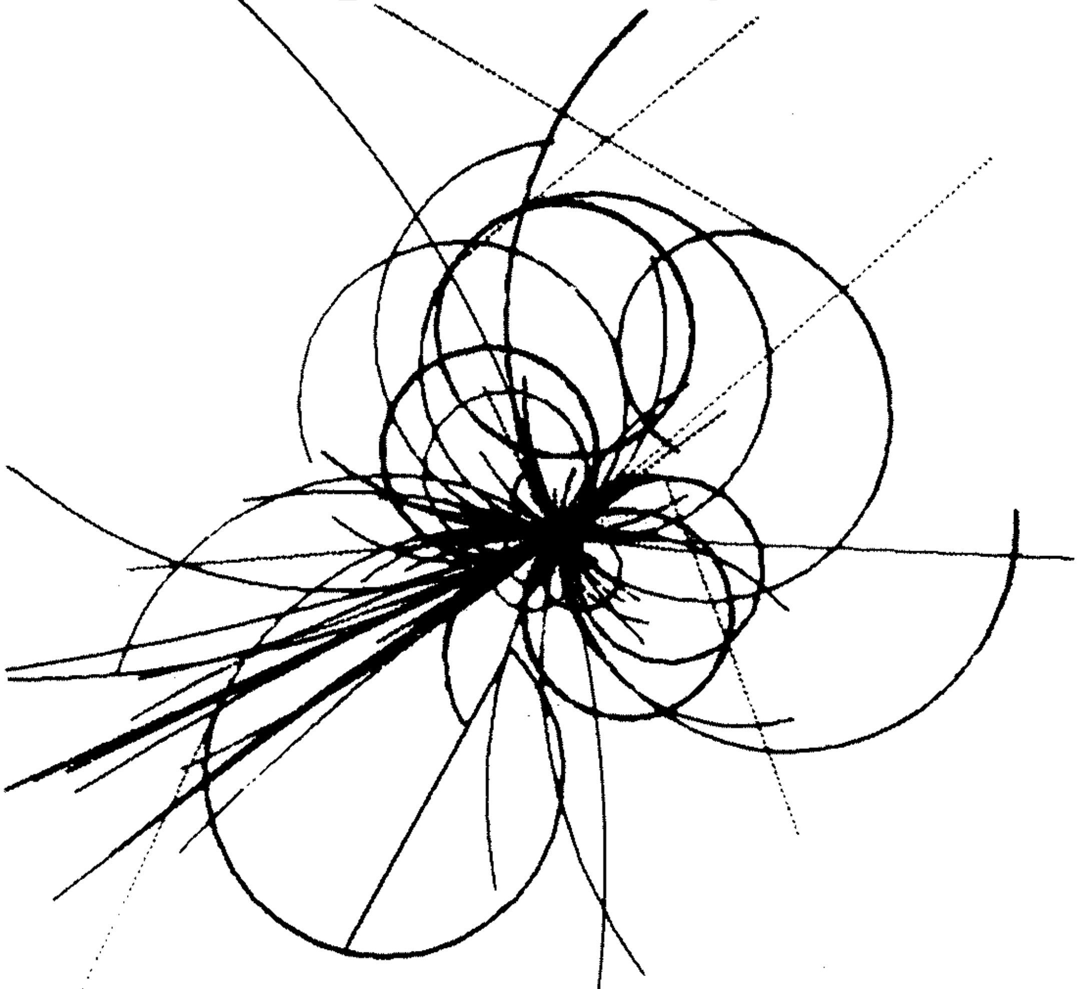

Hamiltonian Theory of the E778 Nonlinear Dynamics Experiment

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ABSTRACT

Short, medium, and long time scale Hamiltonians describing the E778 experiment are presented, corresponding to smear, capture fraction, and tune modulation types of measurements. A one-turn "discrete" Hamiltonian representing motion in the presence of thin multipoles is derived from nonlinear projection maps, leading to expressions for distortion functions, Fourier spectra, normalized covariances, and smear. An N-turn Hamiltonian is derived representing motion at a tune near a rational fraction I/N, leading to expressions for detuning, resonance island width, resonance island tune, and persistent capture fraction. Generating functions appropriate to slow and fast tune modulation are presented, leading to four conditions which partition the tune modulation plane into four distinct "phases" of dynamical behavior.

1. INTRODUCTION

E778 is an accelerator physics experiment that has been performed in the Tevatron proton-antiproton collider, at Fermilab. The original motivation for the experiment was to check that tracking programs and reality agree on the variation of smear and tune shift with amplitude, and to ensure that a real storage ring performs well enough even when these quantities reach the maximum tolerances specified for the "linear aperture" in the Conceptual Design Report of the SSC[1]. Results from the analysis of data taken in May 1987, and preliminary results from the February 1988 data, are presented elsewhere in these proceedings[2], and in other publications[3]. The experiment investigated the behavior of the Tevatron in the presence of strong nonlinearities introduced by 16 special sextupoles. Most of these investigations focussed on the information provided by two neighboring beam position monitors (BPMs), after horizontal betatron oscillations with amplitudes of 2 to 6 millimeters were excited by a one turn kicker. Turn-by-turn information was read out, digitized, and written to magnetic tape, on each of (typically) 64k successive turns - about 1.4 seconds. Data analysis falls naturally into three different time scales - about 50 turns, about 500 turns, and about 50,000 turns. Fifty turns of data are usually sufficient to adequately measure the smear (defined below), and the tune at the amplitude of the kick. These measurements have been successfully completed.

The focus of E778 analysis has now turned to the phenomenon of resonance trapping, in which a persistent signal is seen on the BPM data, due to some fraction of the kicked beam being trapped on resonance islands. These signals often lasted from kick time until the Tevatron ended its two minute cycle. Untrapped beam decoheres in a time corresponding to the inverse of the tune spread - approximately 100 turns, as shown in Figure 1. Five hundred turns of data are sufficient to accurately measure the "capture fraction" - the ratio of persistent amplitude to initial amplitude - and to measure the size and locations of the resonance islands. The analysis of long time scale (1 second) behavior will examine how the persistent response depends on tune modulation of the form

$$Q = Q_0 + q \sin(2\pi Q_M t) \tag{1}$$

where t is the turn number. Tune modulation was externally introduced into the Tevatron by exciting fast response quadrupoles which are normally used to feedback on the tune during slow extraction. Different kinds of behavior are expected in different regions of the (q, Q_M) plane.

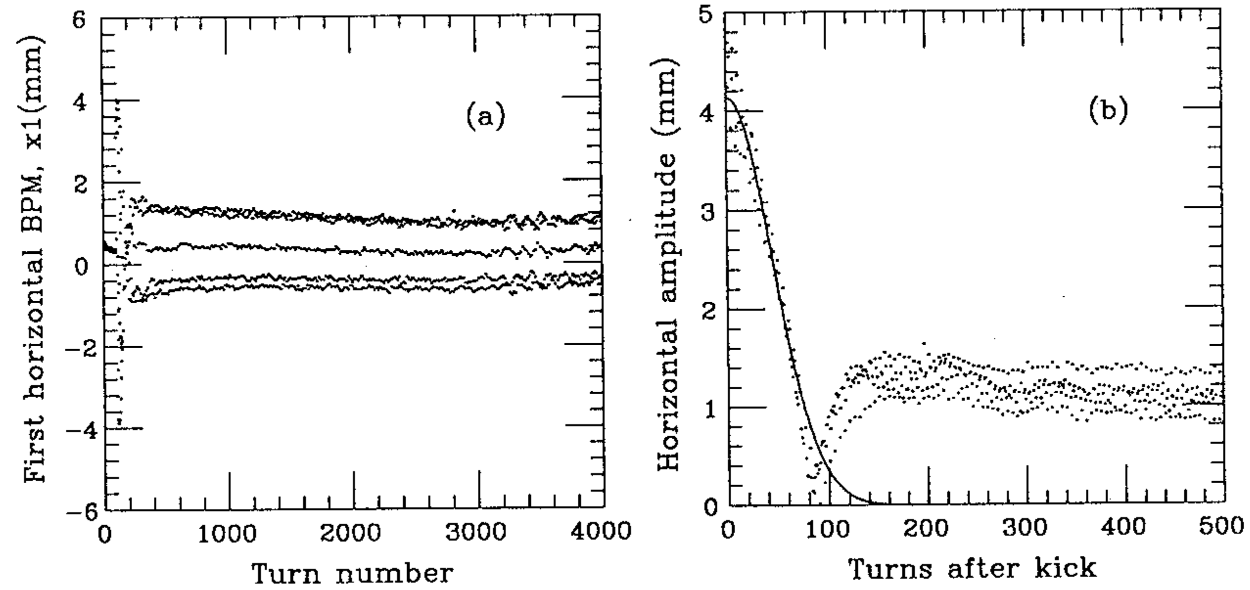

Figure 1. Raw BPM output, (a) for 4000 turns, and (b) for 500 turns after the kick, showing decoherence and a persistent signal. The smooth curve is a Gaussian fit, as expected theoretically

2. SHORT TIME SCALE - NONLINEAR DISTORTIONS

2.1 Projection maps, and the discrete one turn Hamiltonian H_1

Consider the general problem of transverse motion around an accelerator with many thin multipole nonlinearities. Although it is convenient (and appropriate for E778) to concentrate on normal sextupoles in what follows, it is straightforward to extend the results to include any and all multipoles, normal or skew. The angular impulse on a particle passing through a thin sextupole is

$$\Delta X' = g(X^2 - Z^2), \qquad \Delta Z' = -2\,g\,XZ \tag{2}$$

where X and Z are horizontal and vertical displacements, a prime denotes differentiation with respect to the azimuthal coordinate, and g is the sextupole strength. In normalized coordinates, x and z, the perturbation is

$$\Delta x' = g_{xx}\,x^2 - g_{zz}\,z^2, \qquad \Delta z' = g_{xz}\,xz$$

where (3)

$$g_{xx} \equiv g\,\beta_x^{3/2}, \qquad g_{zz} \equiv g\,\beta_x^{1/2}\beta_z, \qquad g_{xz} \equiv -2\,g\,\beta_x^{1/2}\beta_z = -2\,g_{zz}$$

Linear motion from a fixed reference point at the origin of accelerator phases, $\psi_x = \psi_z = 0$, to a given sextupole, is given in this coordinate system by a rotation matrix,

$$R(\psi_x,\psi_z) = \begin{pmatrix} c_x & s_x & 0 & 0 \\ -s_x & c_x & 0 & 0 \\ 0 & 0 & c_z & s_z \\ 0 & 0 & -s_z & c_z \end{pmatrix} \tag{4}$$

where

$$c_x = \cos(\psi_x), \quad s_x = \sin(\psi_x), \quad \text{et cetera.}$$

The "projection" map P is defined as linear motion R from the reference point to a given sextupole, followed by the nonlinear kick, finally followed by inverse linear motion R^{-1} back to the reference point.

The net effect of a projection map P is found by combining equations (3) and (4), to give

$$\begin{pmatrix} \Delta x \\ \Delta x' \\ \Delta z \\ \Delta z' \end{pmatrix} = \begin{pmatrix} -s_x[g_{xx}(c_x x+s_x x')^2 - g_{xx}(c_x x+s_x x')^2] \\ c_x[g_{xx}(c_x x+s_x x')^2 - g_{xx}(c_x x+s_x x')^2] \\ -s_z\, g_{xz}(c_x x+s_x x')(c_z z+s_z z') \\ c_z\, g_{xz}(c_x x+s_x x')(c_z z+s_z z') \end{pmatrix} \tag{5}$$

which has the remarkable property of leading directly to a "discrete projection Hamiltonian",

$$H_P = -\frac{g_{xx}}{3}(c_x x+s_x x')^3 + g_{zz}\,(c_x x+s_x x')(c_z z+s_z z')^2 \tag{6}$$

that exactly reproduces the map (5) under partial differentiation

$$\begin{pmatrix} \Delta x \\ \Delta x' \\ \Delta z \\ \Delta z' \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial H_P}{\partial x'} \\ -\frac{\partial H_P}{\partial x} \\ \frac{\partial H_P}{\partial z'} \\ -\frac{\partial H_P}{\partial z} \end{pmatrix} \tag{7}$$

It is important to note that these are DIFFERENCE, and NOT DIFFERENTIAL, equations - explaining what is meant by a "discrete" Hamiltonian. If it is assumed that the difference vector is small, and if action-angle variables J and ϕ are introduced through

$$\begin{pmatrix} x \\ x' \\ z \\ z' \end{pmatrix} \equiv \begin{pmatrix} (2J_x)^{1/2}\sin(\phi_x) \\ (2J_x)^{1/2}\cos(\phi_x) \\ (2J_z)^{1/2}\sin(\phi_z) \\ (2J_z)^{1/2}\cos(\phi_z) \end{pmatrix} \tag{8}$$

then the projection Hamiltonian becomes

$$H_P = \frac{g_{xx}}{3\,2^{1/2}} J_x^{3/2} [\sin(3\alpha_x) - 3\sin(\alpha_x)] + \frac{g_{zz}}{2^{1/2}} J_x^{1/2} J_z [2\sin(\alpha_z) - \sin(\alpha_x + 2\alpha_z) - \sin(\alpha_x - 2\alpha_z)] \qquad (9)$$

where

$$\alpha_x = \psi_x + \phi_x, \qquad \alpha_z = \psi_z + \phi_z \qquad (10)$$

and powers of trigonometric functions have been expanded into multiple angle form.One-turn motion around the Tevatron is given by following map P_1 with P_2, et cetera, up to P_{16}, and finally by applying $R(2\pi Q_{x0}, 2\pi Q_{z0})$, where Q_{x0} and Q_{z0} are the small amplitude linear tunes. The nonlinear part of the motion is described to first order in sextupole strengths – and not at all to higher order – by summing H_P for each sextupole, so that the discrete one-turn Hamiltonian is

$$H_1 = 2\pi Q_{x0} J_x + 2\pi Q_{z0} J_z + \sum_{\text{sextupoles}} H_P \qquad (11)$$

H_1 is shorthand for a set of difference equations, NOT differential equations, which are

$$\begin{pmatrix} J_x \\ \phi_x \\ J_z \\ \phi_z \end{pmatrix}_{t+1} = \begin{pmatrix} J_x \\ \phi_x \\ J_z \\ \phi_z \end{pmatrix}_t + \begin{pmatrix} -\dfrac{\partial H_1}{\partial \phi_x} \\ \dfrac{\partial H_1}{\partial J_x} \\ -\dfrac{\partial H_1}{\partial \phi_z} \\ \dfrac{\partial H_1}{\partial J_z} \end{pmatrix}_t \qquad (12)$$

The linear contribution on the right hand side – $\Delta\phi_x = 2\pi\, Q_{x0}$, $\Delta\phi_z = 2\pi\, Q_{z0}$ – is constant and (usually) large. Consequently, the value of H_1 is not a constant of the motion, and the motion cannot be graphically understood, (in one dimension) by plotting its contours. Projection maps appear to have been first used in nonlinear accelerator applications by Kobayashi in 1970[4], although they were also independently developed for application to linear coupling problems[5]. It remains to be shown that this formal structure is more than just academically interesting.

2.2 Distortion functions

It is conceptually natural and practically straightforward to rewrite the one-turn Hamiltonian (11) as

$$H_1 = 2\pi Q_{x0} J_x + 2\pi Q_{z0} J_z + \sum_{\{ijkl\}} V_{ijkl} J_x^{i/2} J_z^{j/2} \sin(k\phi_x + l\phi_z + \phi_{ijkl})$$

where the sum is over (13)

$$\{ijkl\} = \{3030, 3010, 1210, 1212, 121\text{-}2\}$$

The first two indices of the constants V_{ijkl} and ϕ_{ijkl} refer to the powers of $J_x^{1/2}$ and $J_z^{1/2}$, while the last two identify a particular harmonic. It is trivial to solve for the phase space "distortion functions", $J_x(\phi_x, \phi_z)$ and $J_z(\phi_x, \phi_z)$, after substituting (13) into the equations

$$J_x(\phi_x+2\pi Q_{x0}, \phi_z+2\pi Q_{z0}) - J_x(\phi_x, \phi_z) = -\frac{\partial H_1}{\partial \phi_x}$$

and (14)

$$J_z(\phi_x+2\pi Q_{x0}, \phi_z+2\pi Q_{z0}) - J_z(\phi_x, \phi_z) = -\frac{\partial H_1}{\partial \phi_z}$$

This gives, to lowest order in sextupole strength,

$$J_x(\phi_x, \phi_z) = J_{x0} - \sum_{\{ijkl\}} \frac{k\, V_{ijkl}}{2\sin[\pi Q_{kl}]} J_{x0}^{i/2} J_{z0}^{j/2} \sin(k\phi_x + l\phi_z + \phi_{ijkl} - \pi Q_{kl})$$

and (15)

$$J_z(\phi_x, \phi_z) = J_{z0} - \sum_{\{ijkl\}} \frac{l\, V_{ijkl}}{2\sin[\pi Q_{kl}]} J_{x0}^{i/2} J_{z0}^{j/2} \sin(k\phi_x + l\phi_z + \phi_{ijkl} - \pi Q_{kl})$$

where it is convenient to define the harmonic tune

$$Q_{kl} \equiv kQ_{x0} + lQ_{z0} \tag{16}$$

Note that the vertical sum in (15) only includes the three terms with l non-zero, $\{ijkl\} = \{1210, 1212, 121\text{-}2\}$, while the horizontal sum continues to include all five terms. Note also the presence of Q_{kl} in the resonance denominators. Although expressions for distortion functions have already been found by many other authors[6], their derivation in the formalism of discrete Hamiltonians is especially economical and conceptually clear. The extension of this description to include other multipoles is straightforward, and is left as an exercise for the reader.

2.3 Fourier spectra, normalized covariances, and smear

The lowest order solution for $J_x(t)$ and $J_z(t)$ on turn t is given by substituting

$$\phi_x = \phi_{x0} + 2\pi Q_{x0} t, \qquad \phi_z = \phi_{z0} + 2\pi Q_{z0} t \tag{17}$$

into equation (15), giving

$$J_x(t) = J_{x0} - \sum_{\{ijkl\}} \frac{k\, V_{ijkl}}{2\sin[\pi Q_{kl}]} J_{x0}^{i/2} J_{z0}^{j/2} \sin(2\pi Q_{kl}\, t + \phi_{0ijkl})$$

and (18)

$$J_z(t) = J_{z0} - \sum_{\{ijkl\}} \frac{l\, V_{ijkl}}{2\sin[\pi Q_{kl}]} J_{x0}^{i/2} J_{z0}^{j/2} \sin(2\pi Q_{kl}\, t + \phi_{0ijkl})$$

with

$$\phi_{0ijkl} \equiv k\phi_{x0} + l\phi_{z0} + \phi_{ijkl} - \pi Q_{kl}$$

Rewriting (18) in terms of amplitudes, rather than actions, gives

$$a_x(t) = a_{x0} - \sum_{\{ijkl\}} \frac{k\, V_{ijkl}}{2^{(i+j+2)/2} \sin[\pi Q_{kl}]} a_{x0}^{i-1} a_{z0}^{j} \sin(2\pi Q_{kl} t + \phi_{0ijkl})$$

and (19)

$$a_z(t) = a_{z0} - \sum_{\{ijkl\}} \frac{l\, V_{ijkl}}{2^{(i+j+2)/2} \sin[\pi Q_{kl}]} a_{x0}^{i} a_{z0}^{j-1} \sin(2\pi Q_{kl} t + \phi_{0ijkl})$$

Each term in the horizontal or vertical sum in (19) corresponds to one line in a discrete Fourier analysis of the amplitudes (not the displacements). While the lines are ideally narrow, in practice their width is proportional to the tune spread of the beam. It is nonetheless possible to reconstruct the single particle motion by properly summing the power and the phase of the bins under the broadened peaks - assuming that the peaks can be resolved. This summarizes the situation in terms of a small set $\{ijkl\}$ of physically important and theoretically predictable parameters, V_{ijkl} and ϕ_{0ijkl}.

The motion is further summarized by calculating three statistics, the "normalized covariances",

$$\sigma_{xx} \equiv \frac{\langle a_x a_x \rangle}{\langle a_x \rangle \langle a_x \rangle} - 1 = \sum_{\{ijkl\}} \frac{k^2 V_{ijkl}^2}{2^{(i+j+3)} \sin^2[\pi Q_{kl}]} a_{x0}^{2i-4} a_{z0}^{2j}$$

$$\sigma_{zz} \equiv \frac{\langle a_z a_z \rangle}{\langle a_z \rangle \langle a_z \rangle} - 1 = \sum_{\{ijkl\}} \frac{l^2 V_{ijkl}^2}{2^{(i+j+3)} \sin^2[\pi Q_{kl}]} a_{x0}^{2i} a_{z0}^{2j-4} \qquad (20)$$

$$\sigma_{xz} \equiv \frac{\langle a_x a_z \rangle}{\langle a_x \rangle \langle a_z \rangle} - 1 = \sum_{\{ijkl\}} \frac{kl\, V_{ijkl}^2}{2^{(i+j+3)} \sin^2[\pi Q_{kl}]} a_{x0}^{2i-2} a_{z0}^{2j-2}$$

where angle brackets $\langle\,\rangle$ imply a time average. (These equations are incorrect if two members of $\{ijkl\}$ have identical kl values.) The covariances are "normalized" in the sense that they are dimensionless, and are zero for linear motion. Two of them, σ_{xx} and σ_{zz}, are positive-definite, but the cross term σ_{xz} can be negative, with

$$\sigma_{xz}^2 \leq \sigma_{xx}\, \sigma_{zz} \qquad (21)$$

If one of the harmonics in the sum dominates - if V_{ijkl} is very large or Q_{kl} is very close to an integer for some $ijkl$ - then there is a simple invariant of the motion,

$$l\, a_x - k\, a_z = \text{constant} \qquad (22)$$

and the equality holds in (21). In the most common E778 experimental conditions, motion was induced in the horizontal plane, with the tune Q_{x0} in the range from 19.37 to 19.42. The horizontal rms "smear" is then just

$$s \equiv \sigma_{xx}^{1/2} = \left(\frac{3\, V_{3030}^2}{2^6 \sin^2(3\pi Q_{x0})} + \frac{V_{3010}^2}{2^6 \sin^2(\pi Q_{x0})} \right)^{\frac{1}{2}} a_{x0} \qquad (23)$$

and is linear in the initial amplitude. Notice that $\sin(3\pi Q_{x0}) = 0.729$ at the upper end of the tune range, and the resonance denominator in the first term in (23) is not small, showing that the E778 smear was not (necessarily) dominated by the $3Q_{x0}$ harmonic.

3.0 MEDIUM TIME SCALE - PERSISTENT SIGNALS

3.1 The N-turn Hamiltonian H_N

If the base tune of an accelerator is near a rational fraction, $Q_0 \approx I/N$, then the net phase space motion after N turns is comparatively small. For example, the E778 fractional tune was between $2/5 - 0.03$ and $2/5 + 0.02$, so the magnitude of the net phase advance was typically less than one tenth of 2π. It could also be argued that the E778 tune was close to 1/3, although it remains to be seen how close is close enough. Consider, then, the general N-turn case, where the motion is described to lowest order by the N-turn Hamiltonian

$$H_N = 2\pi\,(Q_0 - \frac{I}{N})\,J + \frac{1}{N}\sum_{n=0}^{N-1}\sum_{\{ik\}} V_{ik}\, J^{i/2} \sin[k(\phi + n2\pi Q_0) + \phi_{ik}] \tag{24}$$

All subscripts x are dropped from here on, and the set of indices {ijkl} is contracted to {ik}, since only horizontal motion is treated. The N-turn difference equations of motion are now

$$\begin{pmatrix} J \\ \phi \end{pmatrix}_{t+N} = \begin{pmatrix} J \\ \phi \end{pmatrix}_{t} + N \begin{pmatrix} -\dfrac{\partial H_N}{\partial \phi} \\ \dfrac{\partial H_N}{\partial J} \end{pmatrix}_{t} \tag{25}$$

The crucial difference between H_N and H_1 - see equation (13) - is that now the difference step is small. To a reasonable approximation, H_N is a constant of the motion, and the difference equations can be replaced by the more common Hamiltonian differential equations,

$$\begin{pmatrix} \dfrac{dJ}{dt} \\ \dfrac{d\phi}{dt} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial H_N}{\partial \phi} \\ \dfrac{\partial H_N}{\partial J} \end{pmatrix} \tag{26}$$

Strictly speaking, the motion obtained by "integrating" (26) is only correct for values of t which are exact multiples of N . The outer sum in (24) is easily removed by using the trigonometric identity

$$\sum_{n=0}^{N-1} \sin(A + nB) \equiv \frac{\sin(NB/2)}{\sin(B/2)} \sin(A + \frac{N-1}{2}B) \tag{27}$$

so that

$$H_N = 2\pi\,(Q_0 - \frac{I}{N})\,J + \sum_{\{ik\}} V_{Nik}\, J^{i/2} \sin(k\phi + \phi_{Nik}) \tag{28}$$

where

$$V_{Nik} \equiv \frac{\sin(NkQ_0\pi)}{N\sin(kQ_0\pi)}\, V_{ik}, \qquad \phi_{Nik} \equiv \phi_{ik} + (N-1)k\pi Q_0 \tag{29}$$

A remarkable and important property of V_{Nik} is that

$$V_{Nik} \approx V_{ik} \quad \text{if } \mathrm{mod}(k,N) = 0$$
$$V_{Nik} << V_{ik} \quad \text{if } \mathrm{mod}(k,N) \neq 0$$

provided that (30)

$$|Q_0 - \frac{I}{N}| << \frac{1}{k\pi}$$

This defines when the tune is "close enough" - when most of the V_{Nik} terms are negligible. For example, if the maximum value of k is 3, then it reasonable to drop most of the $\{ik\}$ terms in (28) if the base tune is within, say, 0.03 of I/N. This shows that E778 conditions were "close" to the 2/5 resonance, but not to the 1/3.

3.2 The three turn motion, and octupolar detuning

Suppose that the tune is between 0.33 and 0.36 (not true for E778), and that $\{ik\}$ is $\{33, 31, 44, 42, 40\}$, including both sextupolar and octupolar terms. If the extra terms come from true octupoles, their V_{ik} and ϕ_{ik} values are easily calculated. However, if, as in the E778 case, they come from cross terms between sextupoles, they are not calculable without resorting to second order perturbation theory. Combining (28) and (30),

$$H_3 = 2\pi\left(Q_0 - \frac{1}{3}\right) J + V_{33} J^{3/2} \sin(3\phi + \phi_{33}) + V_{40} J^2 \qquad (31)$$

Since H_3 is a constant of the motion in this approximation, the distortion function is just

$$J(\phi) = J_0 - \frac{1}{2\pi\left(Q_0 - \frac{1}{3}\right)} \left(V_{33} J_0^{3/2} \sin(3\phi + \phi_{33}) + V_{40} J_0^2 \right) \qquad (32)$$

in agreement with (15), if the constant term proportional to V_{40} is (legitimately) dropped. Equation (32) describes the classic (normalized phase space) topology, of small amplitude circles becoming more and more distorted at larger and larger amplitudes, out to a separatrix in the shape of an equilateral triangle. This description is accurately confirmed by tracking. The perturbed tune at an average action of J_0 is given by

$$Q(J_0) = \frac{1}{3} + \frac{1}{T} \qquad (33)$$

where, using (26),

$$T = \int dt = \int_0^{2\pi} \left(\frac{\partial H_3}{\partial J}\right)^{-1} d\phi \qquad (34)$$

After equation (31) is differentiated and substituted into (34), the integrand depends explicitly on J and ϕ. Next, the integrand is expanded in a Taylor series up to order J^1, and then (32) is used to make the integrand depend solely on ϕ, allowing T to be evaluated. This gives

$$Q(J_0) = Q_0 - \frac{3}{8\pi^2} \frac{V_{33}^2}{\left(Q_0 - \frac{1}{3}\right)} J_0 - \frac{V_{40}}{\pi} J_0 \qquad (35)$$

which is correct to first order in J_0. The presence of the resonance denominator in the V_{33}^2 term shows that it is unnecessary to consider the cross terms between sextupoles, if the tune is close enough to 1/3.

3.3 The five turn motion, and experimental observables

Most of the persistent signals observed in E778 were due to the 2/5 resonance, and so the five turn motion is very relevant. After expanding the {ik} set even further, to be {33, 31, 44, 42, 40, 55, 53, 51}, but keeping only terms with k=0 and 5 according to (30), then the 5-turn Hamiltonian is written down as

$$H_5 \;=\; 2\pi\,(Q_0 - \tfrac{2}{5})\,J \;+\; V_{40}\,J^2 \;+\; V_{55}\,J^{5/2}\sin(5\phi + \phi_{55}) \tag{36}$$

The decapole terms with i=5 coming from cross terms between sextupoles can also, in principle, be calculated in second order perturbation theory. However, as Taf says, "Beyond first-order results I know of no useful result from perturbation theory in (celestial) mechanics... "[7]. What is more, tracking results show that (36) is not an accurate description even at moderate amplitudes - V_{33} terms are still important in practice. Without minimizing these difficulties, it is possible to proceed with a general description by rewriting H_5 as

$$H_5 \;=\; 2\pi\,(Q_0 - \tfrac{2}{5})\,J \;+\; U(J) \;-\; V_5(J)\cos(5\phi) \tag{37}$$

where

$$Q(J) \;\approx\; \frac{2}{5} \;+\; (Q_0 - \tfrac{2}{5}) \;+\; \frac{U'(J)}{2\pi} \tag{38}$$

and a prime now indicates differentiation with respect to J. This Hamiltonian exhibits island structure, with five stable and five unstable fixed points at local minima and maxima. They occur close to an action J_I which makes the tune exactly 2/5, and which is found by solving (38).

It is illuminating to rewrite (37) in a Taylor expansion in I, the action displacement from the fixed points,

$$H_5(I,\phi) \;=\; \frac{1}{2}U''_I\,I^2 \;-\; V_{5I}\cos(5\phi) \tag{39}$$

where

$$I \;=\; J - J_I, \qquad U''_I \;\equiv\; U''(J_I), \qquad V_{5I} \;\equiv\; V_5(J_I) \tag{40}$$

Thus the stable and unstable fixed point phases are, respectively, even and odd integer multiples of $\pi/5$ (assuming U''_I and V_{5I} have the same sign). The island half width is found by solving $H_5(I_W, 0) = H_5(0,\pi/5)$,

$$I_W \;=\; 2\left(\frac{V_{5I}}{U''_I}\right)^{1/2} \tag{41}$$

Small oscillations about the stable fixed point at the origin are described by

$$\frac{dI}{dt} \;=\; -\,5^2\,V_{5I}\,\phi, \qquad \frac{d\phi}{dt} \;=\; U''_I\,I \tag{42}$$

so that motion around the center of an island is characterized by the island tune

$$Q_I = \frac{5}{2\pi} (U''_I V_{5I})^{1/2} \tag{43}$$

These are the theoretical variables: what can be measured in E778?

The detuning function $Q(a)$ already measured in E778 is in good agreement with tracking at small and moderate amplitudes[2,3]. This leads to a measurement of U as a function of the action. The fraction of particles captured on fifth order islands is expected to be roughly proportional to the island half width in amplitude, a_W. Accepting the parameterisation in (36) for a moment,

$$a_W = 2^{-\frac{1}{4}} \left(\frac{V_{55}}{V_{40}}\right)^{\frac{1}{2}} a_I^{\frac{3}{2}} \tag{44}$$

in which case the capture fraction rises slightly faster than linear with the resonance amplitude, a_I. In any case, observation of the capture fraction leads to measurement of $\frac{V_{5I}}{U''_I}$ as a function of the action, after correction for systematic effects by comparison with multi-particle simulation. This measurement is being actively pursued. These two sets of observations are sufficient to measure the primary theoretical functions $U(J)$ and $V_{5I}(J)$. Knowledge of U and V_{5I} leads to a prediction for the island tune which, if Q_I can be measured independently, imposes a redundant test on the simple theory. When a single particle is captured close to the center of an island in a tracking program, a Fourier transform of its phase reveals a peak at Q_I, the island tune. Although the real signal caused by a beam of particles with finite size is weaker, it is hoped (with some justification) that Q_I can be measured as a function of a_I in the E778 data.

4. LONG TIME SCALE - TUNE MODULATION

The long time scale behavior of the Tevatron was probed, in E778, by observing the response of persistent signals under the tune modulation described in equation (1). Data were usually taken for 64k turns, but some were taken over one megaturn, at the limit of the instrumentation[8]. This is only two orders of magnitude short of the SSC storage time, about $3*10^8$ turns in one day. The following description is broken into slow and fast regimes, where the modulation tune Q_M is much less, or much greater, than Q_I, the island tune.

4.1 Adiabatic tune modulation, and trapping

If the tune is changing slowly at a constant rate of $\dot{Q}$, then (39) is modified to become

$$H_5(I,\phi) = 2\pi\dot{Q}\,t\,I + \tfrac{1}{2}U''_I\,I^2 - V_{5I}\cos(5\phi) \tag{45}$$

This shows that the fixed point action I_{FP}, where $\frac{\partial H_5}{\partial I} = 0$, moves according to

$$I_{FP} = -\frac{2\pi\,\dot{Q}}{U''_I}\,t \equiv -\varepsilon\,t \tag{46}$$

The explicit time dependence in the Hamiltonian is reduced to second order in the small quantity ε, defined above, by performing a canonical transformation from (I,ϕ) to $(\bar{I},\bar{\phi})$, using the generating function

$$W(I,\bar{\phi},t) = I\bar{\phi} + \varepsilon t \bar{\phi} \tag{47}$$

so that

$$\bar{I} \equiv \frac{\partial W}{\partial \bar{\phi}} = I + \varepsilon t, \qquad \phi \equiv \frac{\partial W}{\partial I} = \bar{\phi}, \qquad \bar{H}_5 \equiv H_5 + \frac{\partial W}{\partial t} = H_5 + \varepsilon \bar{\phi} \tag{48}$$

The new action variable $\bar{I}$ is the action displacement from the moving fixed point, while the angle variable is unchanged. The new Hamiltonian is no longer periodic in the angle variable $\bar{\phi}$,

$$\bar{H}_5 = \frac{1}{2} U''_I \bar{I}^2 - V_{5I} \cos(5\bar{\phi}) + \varepsilon \bar{\phi} - \frac{1}{2} U'' \varepsilon^2 t^2 \tag{49}$$

and only has stable fixed points if there is a solution to $\frac{\partial \bar{H}_5}{\partial \bar{\phi}} = 0$, that is, if

$$2\pi \left| \frac{\dot{Q}}{U''} \right| < 5 \left| V_{5I} \right| \tag{50}$$

This is analogous to the well known problem of radio frequency acceleration, in which the stable buckets shrink, to become shaped like tear drops, or even to disappear, when $\frac{dE}{dt} = \frac{dI_{FP}}{dt}$ is non zero. If the tune modulation is sinusoidal, as in (1), then the maximum value of $\dot{Q}$ is $2\pi q Q_M$, and, comparing equations (43) and (50), particles are only adiabatically trapped on resonance islands if

$$q\, Q_M < \frac{Q_I^2}{N} \tag{51}$$

(after generalization to the case of an N'th order resonance). This condition is a factor of two more stringent than the one originally proposed by Chao and Month, which was based on a more heuristic model[9].

4.2 Rapid modulation, and synchrobetatron sidebands

When the sinusoidal nature of the tune modulation is explicitly included in the time independent Hamiltonian (39), the time dependent five-turn Hamiltonian is described, not by (45), but by

$$H_5 = 2\pi q \sin(2\pi Q_M t) I + \frac{1}{2} U''_I I^2 - V_{51} \cos(5\phi) \tag{52}$$

This is canonically transformed by a generating function different from (47), namely

$$W(I,\bar{\phi},t) = \bar{\phi} I + \frac{q}{Q_M} \cos(2\pi Q_M t)\, I \tag{53}$$

to give

$$\bar{I} = I, \qquad \phi = \bar{\phi} + \frac{q}{Q_M}\cos(2\pi Q_M t), \qquad \bar{H}_5 = H_5 - 2\pi q \sin(2\pi Q_M t) \tag{54}$$

The new action is unchanged, while the new angle is sinusoidally modulated with respect to the old angle, and the tune modulation in the new five-turn Hamiltonian is shifted inside the cosine

$$\bar{H}_5 = \frac{1}{2}U''_I \bar{I}^2 - V_{5I}\cos[5\bar{\phi} + \frac{5q}{Q_M}\cos(2\pi Q_M t)] \tag{55}$$

This is rewritten, without overbars, as

$$H_5 = \frac{1}{2}U''_I I^2 - V_{5I}\sum_i J_i(\frac{5q}{Q_M})\cos(5\phi + i2\pi Q_M t) \tag{56}$$

using the identity

$$\cos(A + B\cos(C)) \equiv \sum_{-\infty}^{\infty} J_i(B)\cos(A + iC) \tag{57}$$

where the J_i are Bessel functions. Now, if the action of a particular test particle is close to

$$I_k = k\frac{2\pi Q_M}{5 U''_I} \tag{58}$$

then its tune is close to

$$Q_k = \frac{2}{5} + k\frac{Q_M}{5} \tag{59}$$

and after five modulation periods, 5M turns, the net phase advance is small. All of the harmonic terms except one disappear in going to the 5M-turn Hamiltonian,

$$H_{5Mk} = \frac{1}{2}U''_I (I - I_k)^2 + V_{5I} J_{-k}(\frac{5q}{Q_M})\cos(5\phi) \tag{60}$$

due to the same averaging which made most terms disappear in going from the one turn (difference) map to the five turn (differential) map, equations (24) to (30). Note that this averaging is only valid if not much happens in 5M turns - if $Q_M >> Q_I$. Just as the N-turn Hamiltonian motion was only correct every N turns, the motion found by "integrating" H_{5Mk} is only strictly correct at integral multiples of 5M.

Equation (60) has stable fixed points and resonance island chains for every integer k, at a family of tunes with a spacing of $\frac{Q_M}{N}$, where N is the general resonance order. These "synchrobetatron" sidebands are strongly suppressed by small Bessel functions at large values of k, since

$$J_k(A) \approx \left(\frac{2}{\pi A}\right)^{\frac{1}{2}}\cos(A - \frac{k\pi}{2} - \frac{\pi}{4}) \qquad \text{if } A > k > 0$$

$$J_k(A) \approx 0 \qquad \text{if } A < k \tag{61}$$

Physically, this means that the test particle is hardly perturbed if its tune modulation amplitude q is insufficient to carry it the $\frac{kQ_M}{N}$ distance to the tune $\frac{I}{N}$ of the resonance. Only the fundamental $k = 0$ is present if

$$q < \frac{Q_M}{N} \tag{62}$$

The half width of a significant synchrobetatron island is given, in comparison with (41), by

$$I_{Wk} = 2\left(J_{-k}\left(\frac{Nq}{Q_M}\right)\frac{V_{5I}}{U''_I}\right)^{\frac{1}{2}} \approx 2\left(\frac{Q_M}{\pi Nq}\right)^{\frac{1}{4}}\left(\frac{V_{5I}}{U''_I}\right)^{\frac{1}{2}} \tag{63}$$

where the value of a Bessel function is approximated by its rms size. The action separating neighboring sidebands is given by (58), so the condition for synchrobetatron sideband overlap is just

$$I_{separation} = \frac{2\pi\, Q_M}{N\, U''_I} < 2\, I_{Wk} \tag{64}$$

or, using (43),

$$Q_M^{\frac{3}{4}}\,(Nq)^{\frac{1}{4}} < \frac{4}{\pi^{1/4}}\, Q_I \tag{65}$$

Large scale chaos is expected when this condition is satisfied.

4.3 Dynamical "phases" in the tune modulation plane, (Q_M,q)

Figure 2a shows how the (Q_M,q) tune modulation plane is broken into different regions by three solid boundaries corresponding to the conditions (51), (62), and (65), drawn here with $N = 5$ and $Q_I = 0.0053$. Assuming that the order of the resonance N is fixed, the only independent variables in these conditions are the three tunes q, Q_M, and Q_I. These three occur only in combinations of two more basic quantities, Q_M/Q_I and q/Q_I, which are the externally controlled tunes in units of the island tune. This shows that Q_I is a fundamental dimensionless measure of the resonance strength. The dashed boundary, $Q_M = Q_I$, roughly separates the zones where the slow and fast generating functions, (47) and (53), are valid.

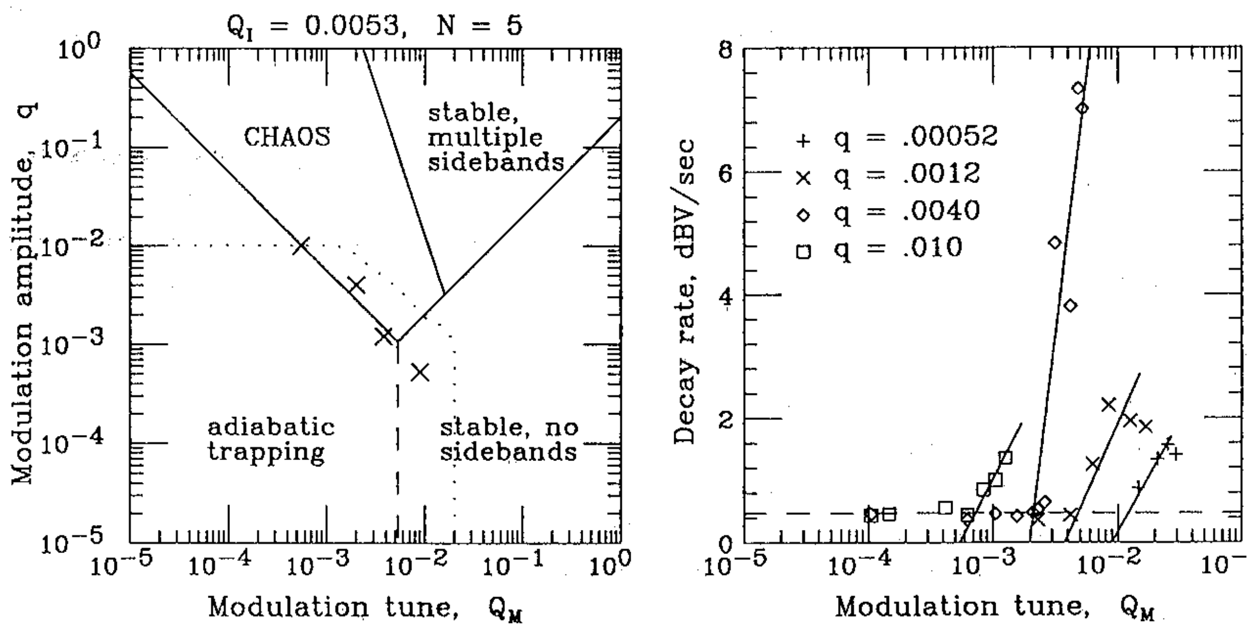

Figure 2 a) Tune modulation plane phases, and b) persistent signal decay rate versus modulation tune, Q_M

In the bottom left corner of this "phase" diagram, particles are trapped in a single fundamental island chain which "breathes" in and out, to larger and smaller amplitudes. Hence the strength of the persistent signal is amplitude modulated, in step with the tune modulation. In the bottom right corner there is still only a fundamental island chain, but trapped particles do not exhibit coherent amplitude modulations. Sideband island chains occur in the top right hand corner, in addition to the fundamental. If the size of the kicked beam in this region is large enough, then more than one sideband is populated at a time, and a Fourier transform of the persistent signal reveals peaks separated in tune by Q_M/N, without amplitude modulation. The fourth region, the top left, is chaotic. If "persistent" signals are observed there at all, they decay away very rapidly.

The dotted line in Fig. 2a shows the region which was physically accessible in the E778 experiment during the February 1988 run. Only the "adiabatic trapping" and "chaos" regions were accessible at values of Q_I where the persistent signals were significantly strong. The 64k turns of data typically taken per shot could not be analyzed on line (for example, in searching for amplitude modulation) in time for the next shot. Consequently, the main real time observable was the decay rate of the persistent signal. Figure 2b shows how, at a particular base tune Q_0, and hence at a particular island tune Q_I, the decay rate increased dramatically above a critical value of Q_M. The four crosses drawn on Fig. 2a correspond to the four q values in Fig. 2b, showing behavior consistent with crossing the boundary between adiabatic behavior and chaos. Detailed analysis of the hundreds of megabytes of data taken in the tune modulation phase of the E778 experiment is only just beginning.

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