Lecture 2:

Linear Motion & Stability

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“If all you have is a hammer, everything looks like a nail.”

A. Maslow, “The Psychology of Science”, 1966.

ANALYTIC TOOLS (pre-computer) make dynamic problems look like DIFFERENTIAL EQNS.

NUMERICAL TOOLS make problems look like DIFFERENCE EQNS.

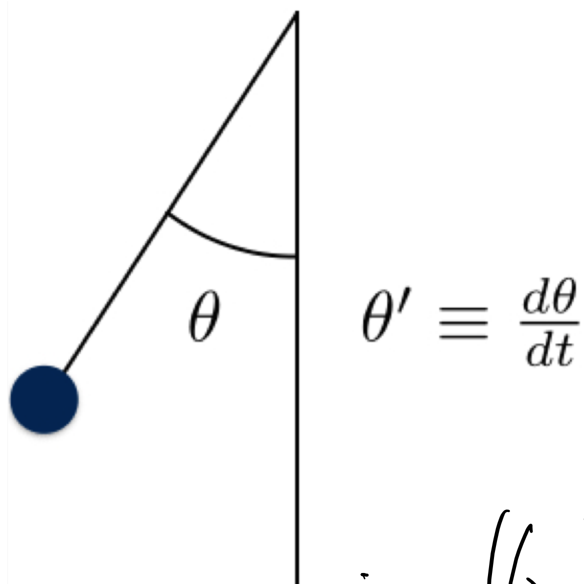
Q1: Which way is right?

Q2: Is time continuous?

A: IT DEPENDS!

EXAMPLE: PENDULUM

1.1 Phase space co-ordinates for a gravity pendulum.



Small?

$$\theta'' = \left[\frac{d^2\theta}{dt^2}\right] = -g\sin(\theta) \simeq -g.\theta$$

How to SIMULATE? until finished {

Small?

$$\theta = \theta + \theta'.\Delta t \to 0$$

$$\theta' = \theta' - g\sin(\theta).\Delta t$$

} End

This is v. much like an accelerator!

STANDARD MAP

Circular accelerator (inherently discrete)

: gravity pulsed once per Δt/period

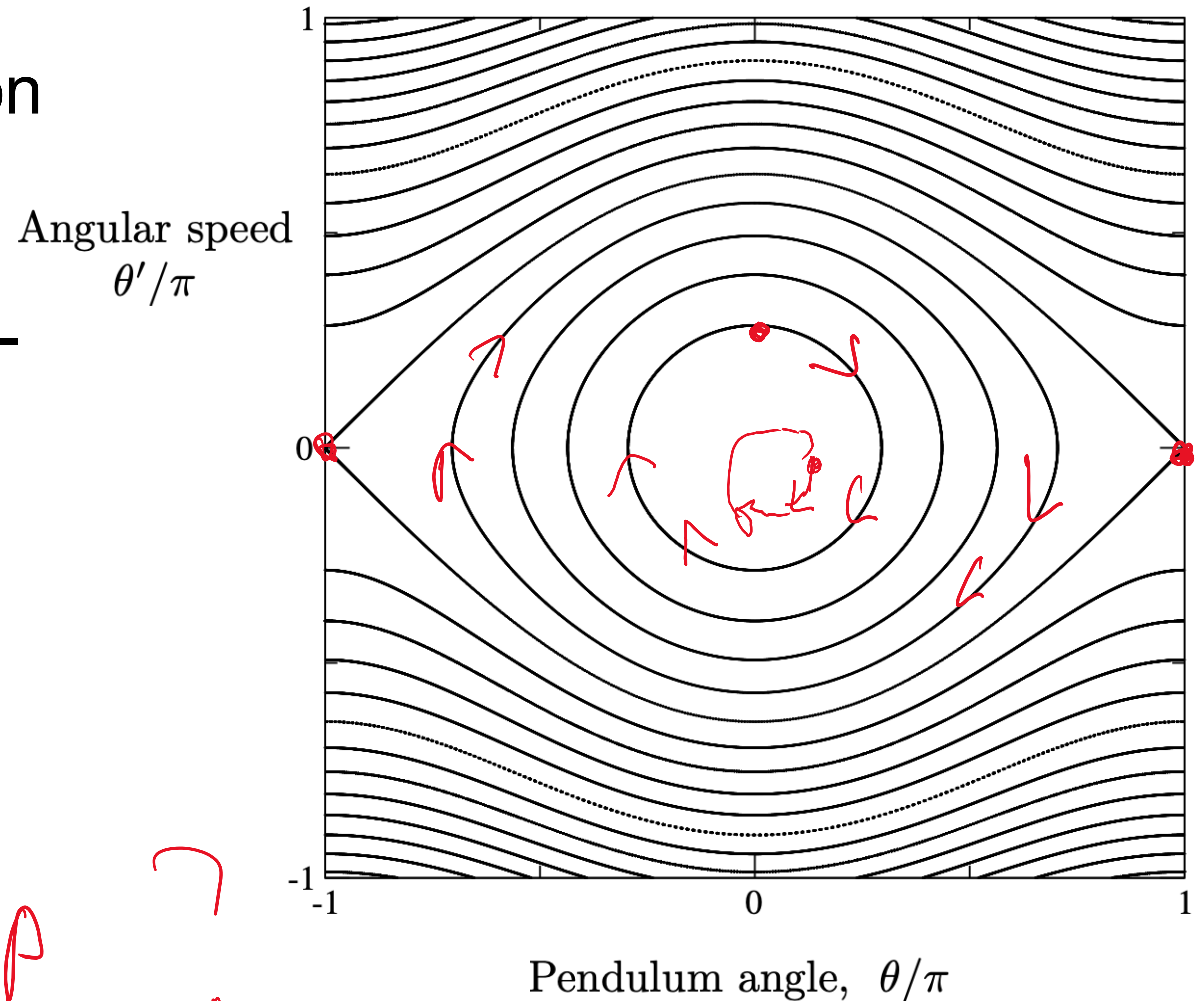
$$\ddot{\theta} = -\sum_{n=1}^{\infty} \delta(t - n.\Delta t) \, . \, g \, . \, \sin\theta \, . \, \Delta t$$

Why bother to force a DIFF. EQN. ?
ERENTIA

DIFFERENCE MAPS ARE
LEGITIMATELY WELL MATCHED
TO ACCELERATORS ... ← LINEAR or NONLINEAR

1.4 Phase space motion of a gravity pendulum.

The angle ranges from -pi to +pi, but the pendulum can wind forwards or backwards.



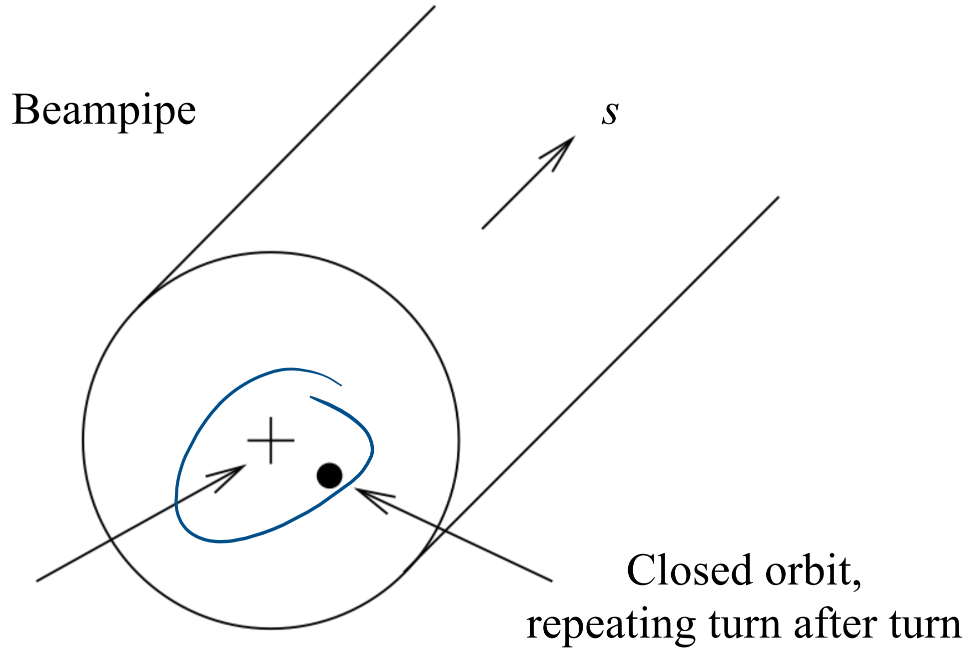
LOGISTIC MAP ?

$$X_{n+1} = \alpha X_n (1 - X_n)$$

$0 < \alpha < 4$

$0 < X_0 < 1$

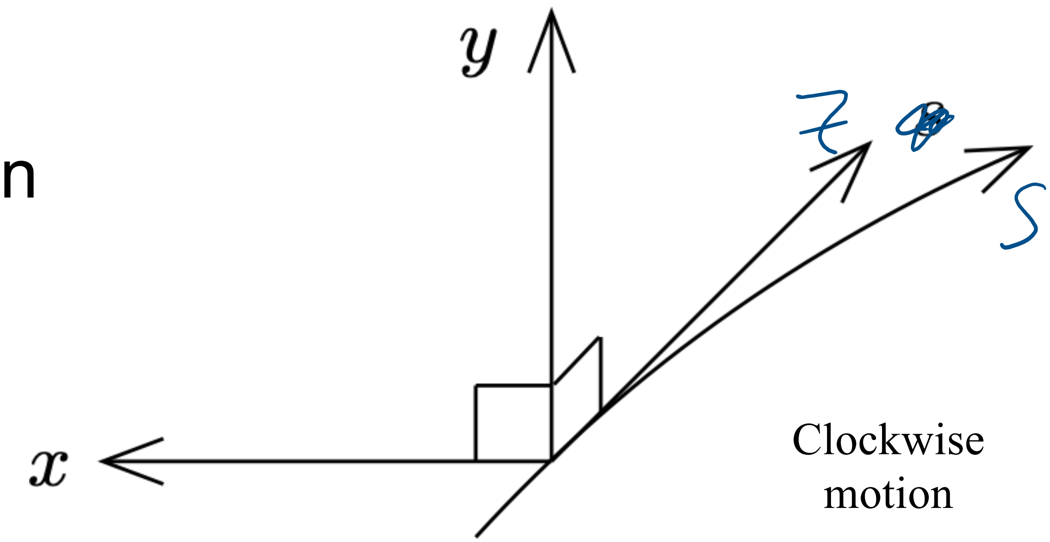
2.1 The closed orbit, displaced from the design orbit by inevitable errors of various sorts.



IT CAN BE SHOWN: if fields are static $\frac{dB}{dt} = \frac{dE}{dt} = 0$ then there is 1 orbit that EXACTLY REPEATS ITSELF: CLOSED ORBIT

MEASURE TEST PARTICLES OFFSETS RELATIVE TO THAT CLOSED ORBIT. (NO ERRORS)

2.2 The right-handed co-ordinate system (x,y,s) often used for clockwise motion.



BETATRON OSCILLATIONS

(x, y, z) rotates, tangential to that location

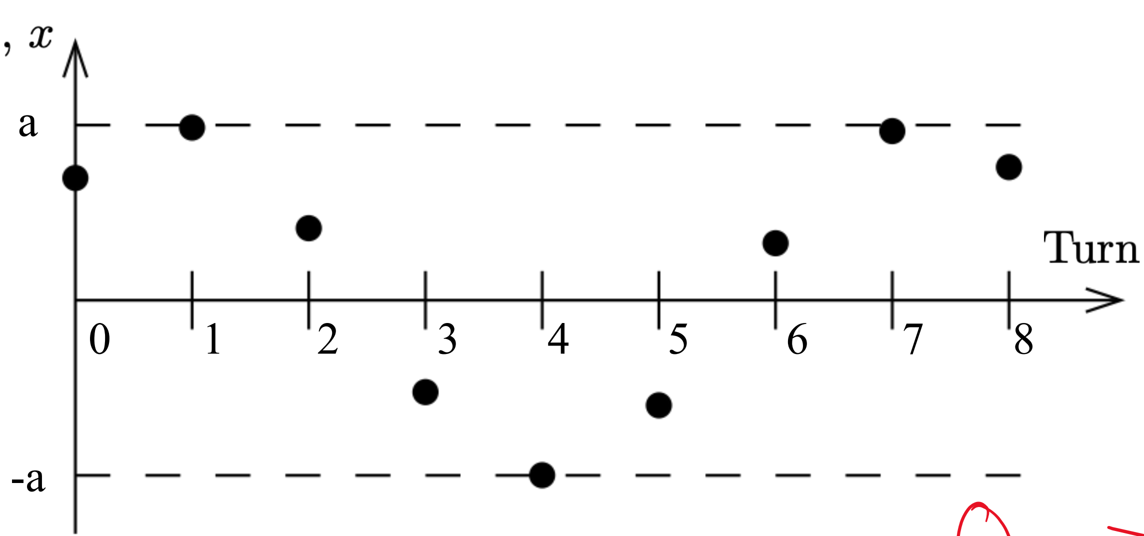
WANT (LINEAR) STABILITY:

$$\boxed{x(n) = a \cdot \cos(2\pi Q_x \cdot n + \phi_0)}$$

AT A REFERENCE POINT

~1mm   BETATRON TUNE   turn number   initial phase

2.3 Horizontal betatron oscillations about the closed orbit.



$Q_x >> 1$

ALSO WANT LONGITUDINAL STABILITY

$$\boxed{z_n = a_n \cos(2\pi Q_s \cdot n)}$$

displacement

~ 100 mm

SYNCHROTRON TUNE

$\approx 0 < Q_s \ll 1$

$\sim .003$?

MOTION THROUGH A MAGNET

Assume: magnet MUCH larger than its bore

$\therefore$ 2D fields $B_x = B_x(x,y)$, $B_y = B_y(x,y)$, $B_s = 0$

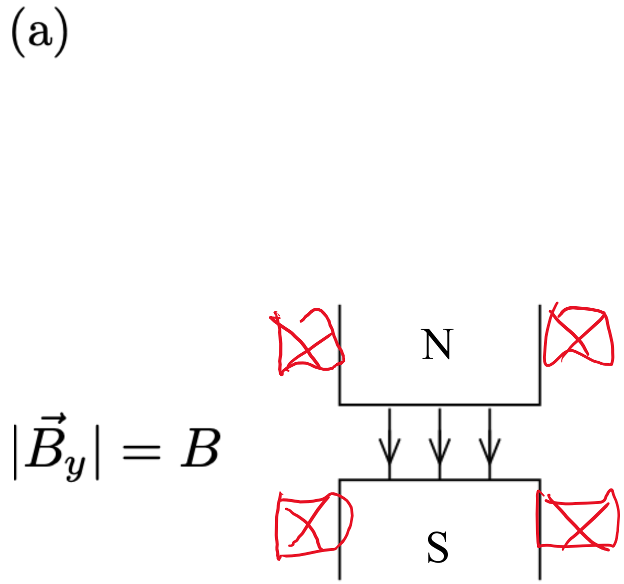
Maxwell's EQNS become 2-D

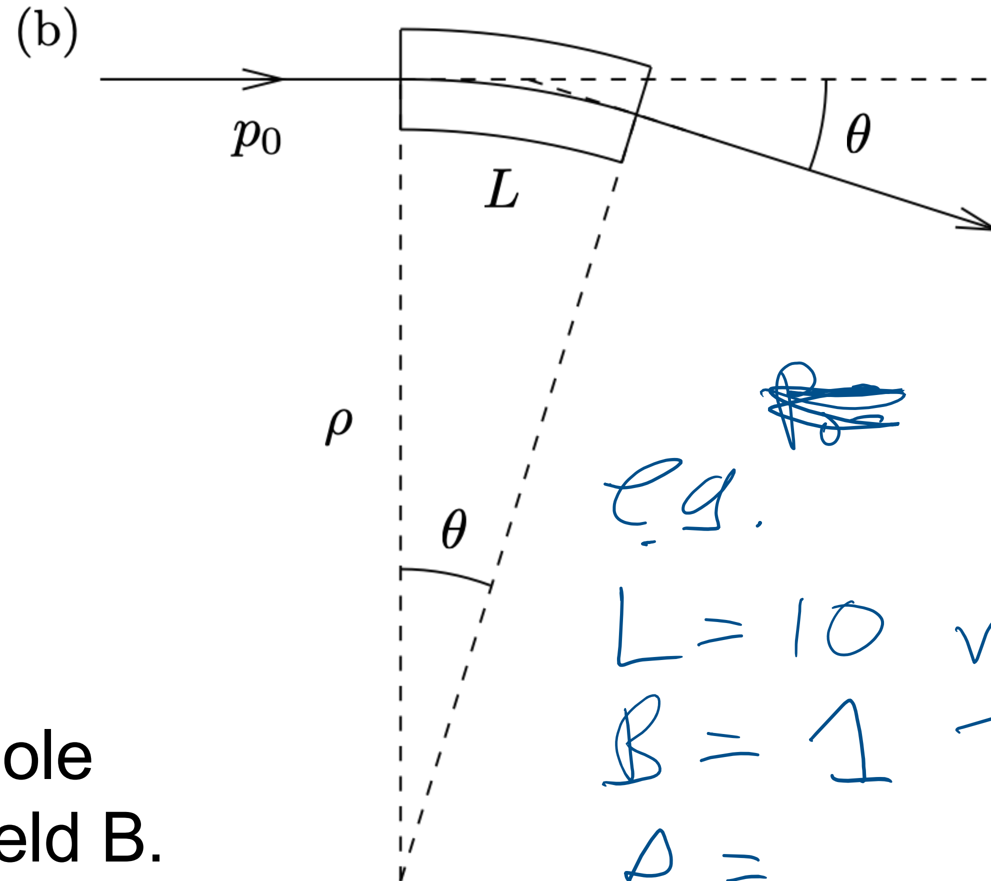
$$\nabla \cdot \bar{B} = 0 \quad , \quad \nabla \times \bar{B} = \mu_0 J = 0$$

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0} \qquad \boxed{\frac{\partial B_x}{\partial B_y} - \frac{\partial B_y}{\partial x} = 0}$$

SIMPLEST SOLUTION:

$B_y = \text{constant}$, $B_x = 0$ $\boxed{\text{DIPOLE}}$





2.5 Motion through a dipole with a constant vertical field B.

e.g.

$L = 10$ m

$B = 1$ T

$\rho =$

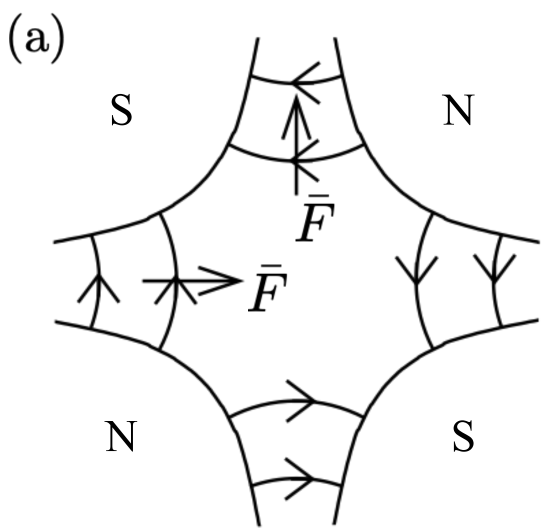
$$\boxed{\text{RIGIDITY } (B\rho)\,[\text{Tm}] = 3.34\, p_0\,[\text{GeV/c}]}$$

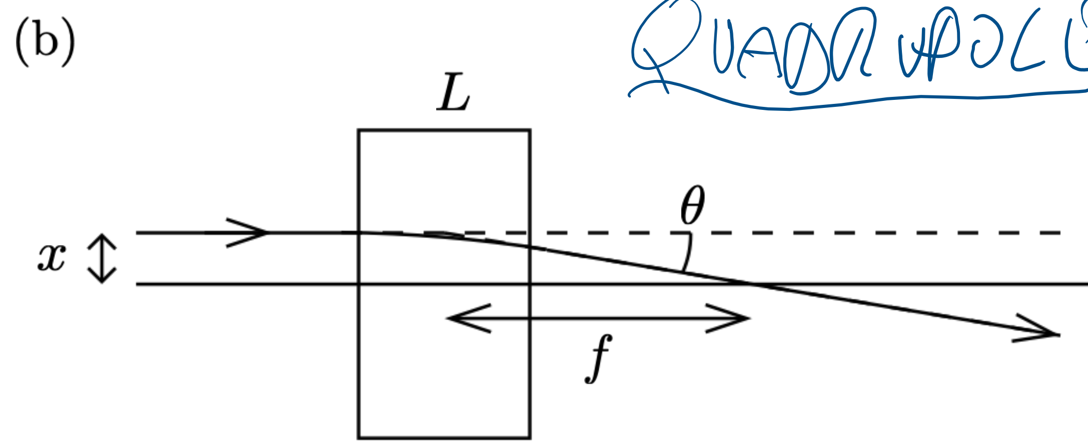
singly charged

1

10 m

3.4 GeV/c





QUADRUPOLE

$$\boxed{\begin{aligned} B_x &= B' \cdot y \\ B_y &= B' \cdot x \end{aligned}}$$

2.6 Motion through an iron quadrupole, focusing in the horizontal but defocusing in the vertical.

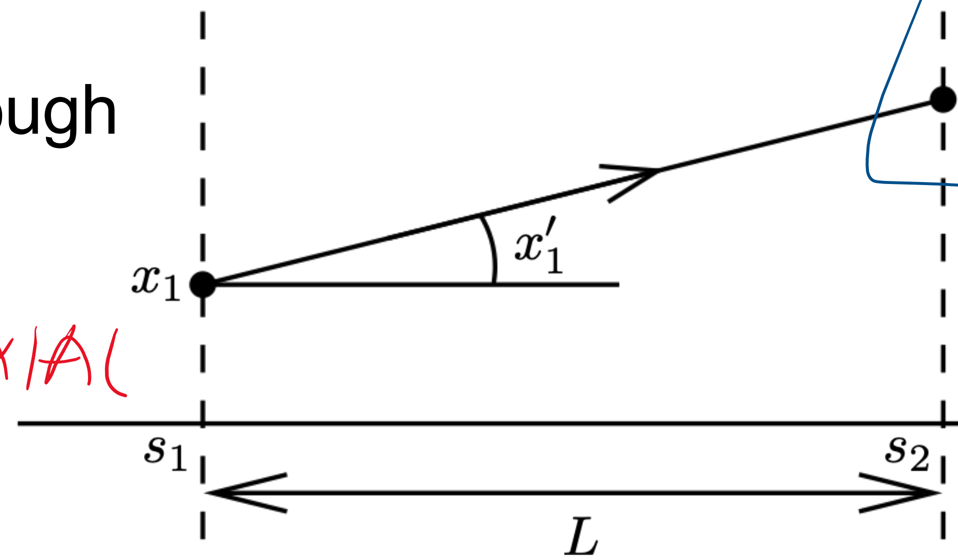
$$\boxed{\theta_x = \frac{qB_y}{p_0} L = \frac{B'}{(B\rho)} \cdot L\, x}$$

$$\frac{1}{f} = KL \,,$$

where

$$\boxed{K = \operatorname{sgn}(q) \cdot \frac{B'}{B\rho}}$$

2.7 Motion through a field-free drift



$$\boxed{\begin{aligned} x_2 &= x_1 + L x'_1 \\ x'_2 &= x'_1 \end{aligned}}$$

$$x' = \frac{dx}{ds}$$

PARAXIAL

$$\boxed{x' = \tan(\theta) \simeq \theta}$$

MATRICES: DRIFT

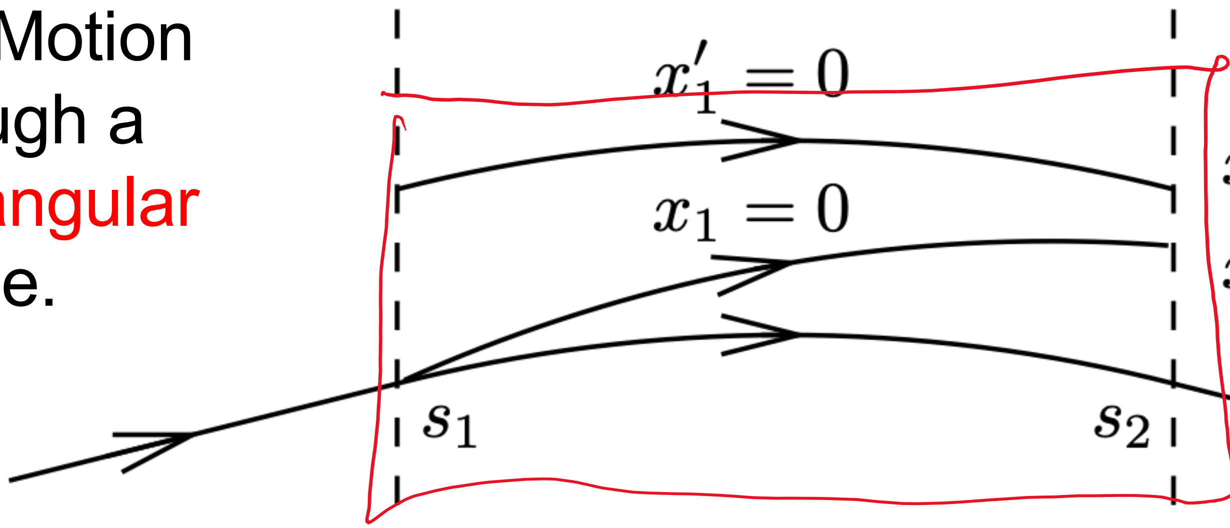
$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = M_{DRIFT} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1$$

where

$$\boxed{M_{DRIFT} = \begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

BLOCK DIAGONAL

2.8 Motion through a rectangular dipole.



$$x_2 = x_1 + L x_1'$$
$$x_2' = x_1'$$

SBEND

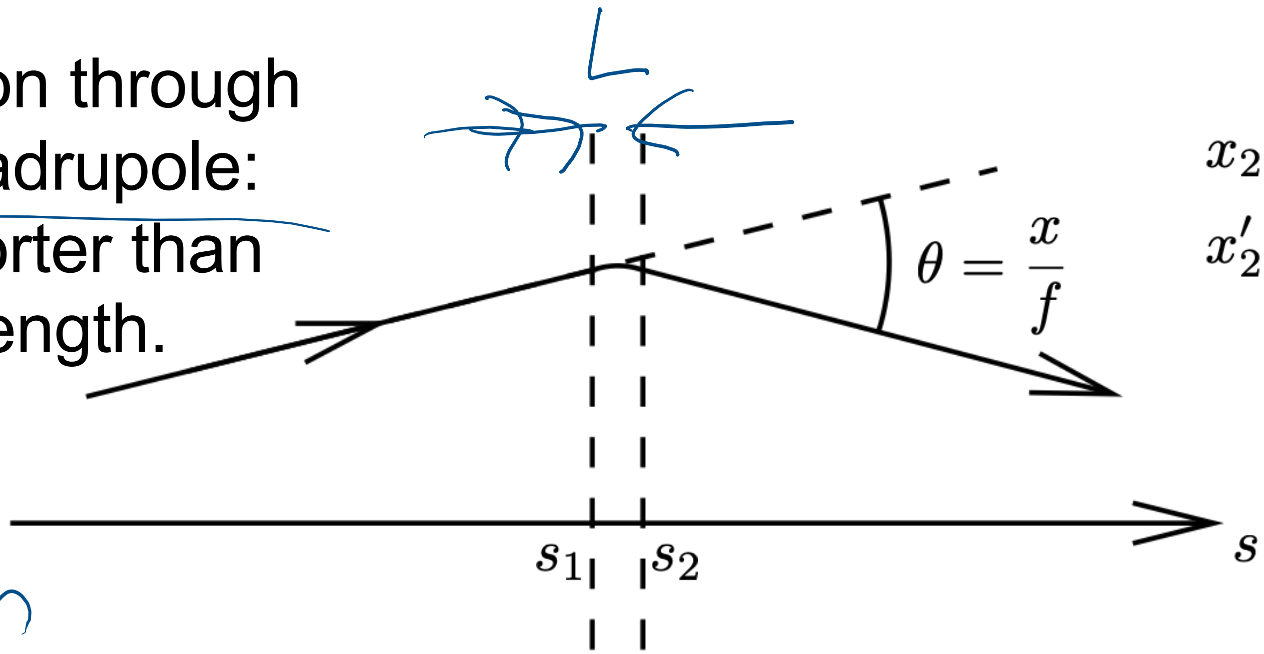
DESIGN ORBIT

Bend angle is independent of (x, x') RBEND

$$\boxed{M_{RBEND} = M_{DRIFT}}$$

(co-ordinate frame rotates)

2.9 Motion through a thin quadrupole: much shorter than its focal length.



$$x_2 = x_1$$
$$x_2' = x_1' - \frac{x}{f}$$

$L \ll f$

$$x_2 = x_1$$
$$x_2' = x_1' - \frac{L}{f}$$

$$M_{\text{THIN QUAD}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & +\frac{1}{f} & 1 \end{pmatrix}$$

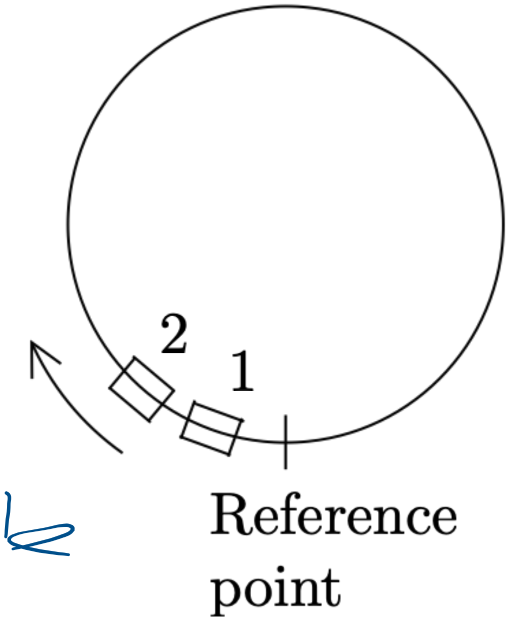
Focus / Defocus

THICK QUADRUPOLE

$$M_{QUAD} = \begin{pmatrix} \cos(kL) & \frac{1}{k}\sin(kL) & 0 & 0 \\ -k\sin(kL) & \cos(kL) & 0 & 0 \\ 0 & 0 & \cosh(kL) & \frac{1}{k}\sinh(\ ) \\ 0 & 0 & k\sinh(\ ) & \cosh(\ ) \end{pmatrix}$$

$$\boxed{k = \sqrt{K}} \qquad m^{-\frac{1}{2}}$$

3.1 Matrices representing lattice elements are multiplied in sequence to derive the one-turn matrix at a reference point.



LEGO lattice of drift, dipole, quads that bend by 2π

Q1: How to test if (transverse) motion is stable?

Q2: How to deal with the F/D problem?

Q3: How to make beam small/large in some places?

Q4: Why (& when) must nonlinear magnets enter? (esp. sextupoles?)

LINEAR STABILITY

ONE-TURN MATRIX

$$M = M_{m,m-1} \cdots M_{32}\, M_{21}\, M_{10}$$

The 4X4 matrices are block diagonal: so is $\underline{M}$

so horizontal stability problem is 2X2

$$\boxed{\bar{X}_n = \begin{pmatrix} x \\ x' \end{pmatrix} = M^n \cdot \bar{X}_0}$$

complex vector

M has 2 complex eigenvectors $\bar{V}_1$, $\bar{V}_2$

such that $\boxed{M\bar{V} = \lambda \bar{V}}$ complex scalar write initial vector as

$$\bar{x}_0 = A\bar{v}_1 + B\bar{v}_2$$

Both sides are real, but $A, B, \bar{v}_1, \bar{v}_2$ are complex

On turn n

$$\bar{x}_n = M^n \bar{x}_0 = A\lambda_1^n \bar{v}_1 + B\lambda_2^n \bar{v}_2$$

IF $\bar{x}_n$ is to be bounded for all n

THEN λ_1^n & λ_2^n must also be bounded

SOLVE THE "CHARACTERISTIC" EQN.

$$\boxed{\det(M - \lambda I) = 0} \quad \text{for } \lambda_1 \text{ \& } \lambda_2$$

Start by writing $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then

$$\underbrace{(ad - bc)}_{\det M = 1} - \underbrace{(a+d)}_{Tr(M)}\lambda + \lambda^2 = 0$$

So $\boxed{\lambda^{-1} + \lambda = Tr(M)}$

By inspection the eigenvalues are reciprocals

$$\lambda_1 = e^{i\mu}, \quad \lambda_2 = e^{-i\mu} = \cos(\mu) - i\sin(\mu)$$

μ may be complex! Find it by solving

$$\boxed{2\cos(\mu) = \mathrm{Tr}(M)}$$

<u>IF</u> μ <u>IS</u> complex, then λ_1^n or $\lambda_2^n \to \infty$

THEREFORE STABILITY CONDITION IS

$$\boxed{-1 \leq \tfrac{1}{2}\mathrm{Tr}(M) \leq 1}$$

[IN BOTH H & V.]

If the motion <u>IS</u> stable, then write the one-turn matrix as

$$\boxed{M(s) \equiv \begin{pmatrix} \cos(\mu)+\alpha(s)\sin(\mu), & \beta(s)\sin(\mu) \\ -\gamma(s)\sin(\mu), & \cos(\mu)-\alpha(s)\sin(\mu) \end{pmatrix}}$$

not a function of s !!

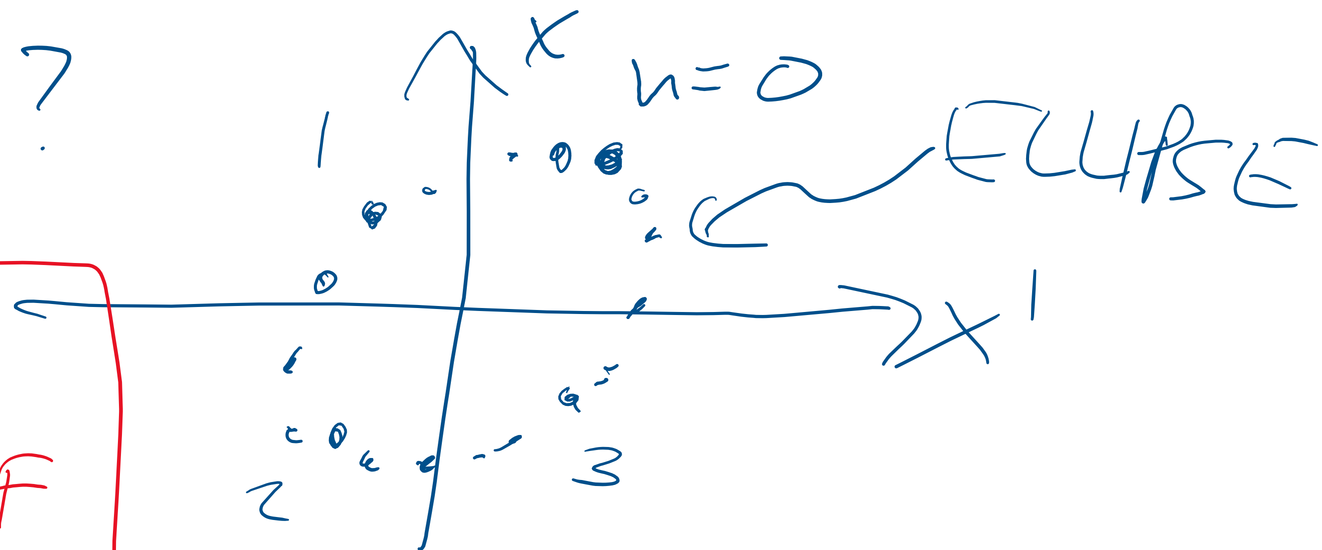
where the "Twiss" or "(Courant-Snyder)" functions have $\boxed{\gamma \equiv \frac{1+\alpha^2}{\beta}}$ to guarantee $\boxed{\det(M) = 1}$

NOTE

① $\mu(s)$, $\beta(s)$, $\alpha(s)$, $\gamma(s)$ all functions of s

② BUT $\mu_x = 2\pi Q_x$ is NOT !!

Q: What does this motion look like in phase space?



POINCARE SURFACE OF SECTION

Can convert ellipses to circles

$$M = \begin{pmatrix} c+\alpha s, & \beta s \\ -\gamma s, & c-\alpha s \end{pmatrix} \qquad \begin{aligned} c &= \cos(\mu) \\ s &= \sin(\mu) \end{aligned}$$

$$= \underbrace{\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}}_{T^{-1}} \underbrace{\begin{pmatrix} c & s \\ -s & c \end{pmatrix}}_{R(\mu)} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}}_{T}$$

Floquet matrix

$$\boxed{M = T^{-1} R T} \qquad R^n(\mu) = R(n\mu)$$

After n turns

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$= (T^{-1} R T)(T^{-1} R T)(T^{-1} R T) \cdots (T^{-1} R T) \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$= T^{-1} R(n\mu) T$$

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strongly suggests that

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix} = T \begin{pmatrix} x \\ x' \end{pmatrix}$$

Is a valuable transformation...