Lecture 4:

Longitudinal & Off-Momentum Motion

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"[We] found … a 'transition energy' at which the stable and metastable phase equilibrium points that give phase stability exchange roles …"

E.D. Courant, "Accelerators, Colliders, and Snakes", 2003.

Q1 : stability ✓

Q2 : F/D problem ✓

Q3 : Beam size control - in progress

Q4 : Why do we need nonlinear sextupoles?

START

Real world is 3D :

OFF-MMTM PARAMETER

$$\delta = \frac{p - p_0}{p_0} \neq 0 \qquad \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix}$$

1) δ = CONSTANT
2) δ OSCILLATING

① δ = CONSTANT

Switch to differential equation.

HILLS EQUATIONS $\delta = 0$

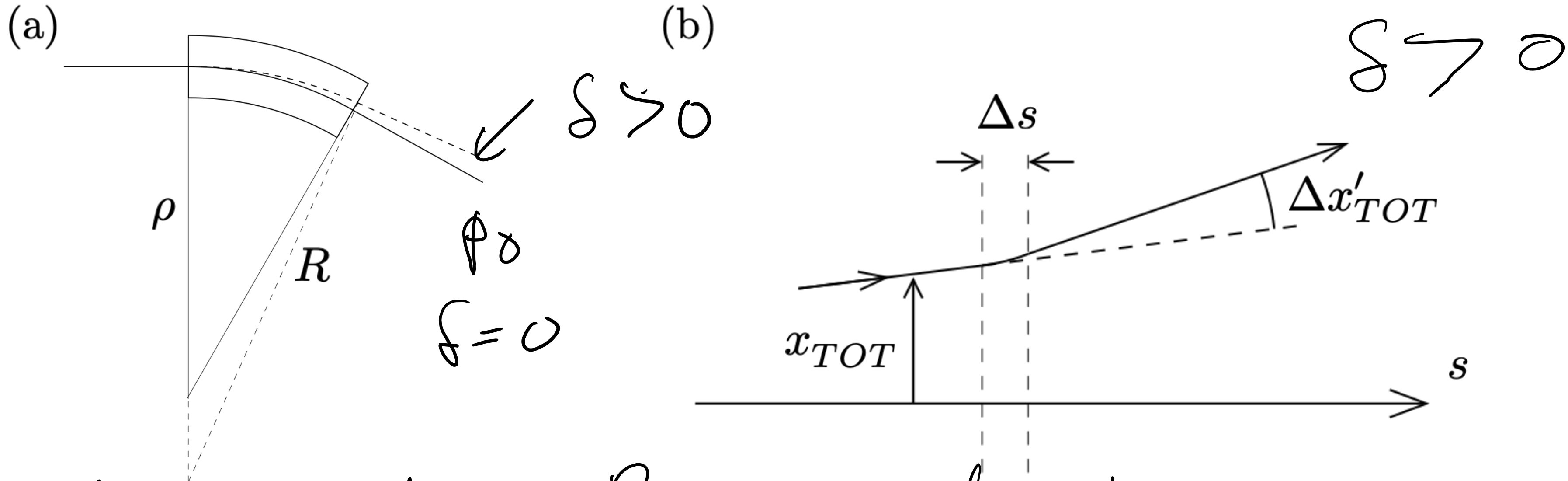
$$x'' + K(s) \cdot x = 0$$

$$y'' - K(s) \cdot y = 0$$

Quad strength K is periodic in s

$$0 \leq s < C$$

4.1 A particle with a slightly large momentum acquires a horizontal angle Delta x'TOT in a thin dipole slice.



Bending radius R in a dipole is

$$\frac{1}{R} = \frac{1}{\rho} \cdot \frac{1}{(1+\delta)} = G \cdot \frac{1}{(1+\delta)}$$

dipole geometric

The TOTAL bend accumulated in slice Δs is

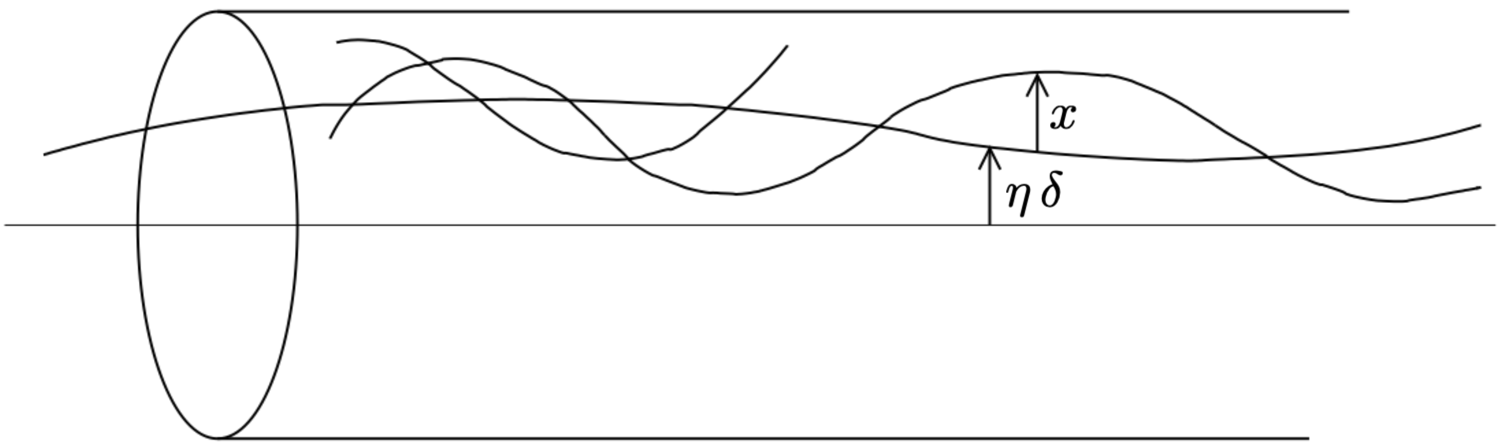
$$\boxed{\Delta x'_{TOT} = G\left(1 - \frac{1}{1+\delta}\right)\Delta s}$$

thanks to the rotating co-ordinate frame

Introduce TOT: eg

$$\boxed{x_{TOT} = x_{\substack{\text{CLOSED}\\\text{ORBIT}}}(s) + x_{\substack{\text{BETATRON}\\\text{OSCILLATIONS}}}}$$

4.2 Total horizontal displacement is 1) a constant closed orbit offset plus 2) a varying betatron oscillation.



Adding in quads of geometric strength $\frac{K}{1+\delta}$

GENERALISED HILL'S EQNS.

$$\boxed{\begin{aligned} x''_{TOT} + \frac{K}{1+\delta}\, x_{TOT} &= G\left(1 - \frac{1}{1+\delta}\right) \\ y''_{TOT} - \frac{K}{1+\delta}\, y_{TOT} &= 0 \end{aligned}}$$

a) Assume no vertical dipoles

b) These Equs. are EXACT in δ

c) $K(s)$ & $G(s)$ are periodic in s

To first order in δ:

(A)
$$\boxed{\begin{aligned} x''_{TOT} + K(1-\delta)\,x_{TOT} &\cong G\cdot\delta \\ y''_{TOT} - K(1-\delta)\,y_{TOT} &\cong 0 \end{aligned}}$$

DISPERSION FUNCTION η (hadrons: $D(s)$)

(B)
$$\boxed{x_{TOT} = \underbrace{\eta(s)\cdot\delta}_{\text{closed orbit}} + \underbrace{x}_{\text{betatron oscillation}}}$$

Closed orbit (dispersion function) respects PBC (periodic boundary condx.)

Strictly speaking dispersion is a polynomial:

$$\eta(s) = \eta_0(s) + \eta_1(s)\delta + \eta_2\delta^2 + \dots$$

but usually just consider the constant piece.

Substitute (B) into (A), get

$$\boxed{\eta'' + K\cdot\eta = G}$$

FIRST ORDER DISP. WITH PBC

AND

$$\boxed{\begin{aligned} x'' + \frac{K}{(1+\delta)}x &= 0 \\ y'' - \frac{K}{(1+\delta)}y &= 0 \end{aligned}}$$

BETATRON OSCILLATIONS.

Q: What do weaker (stronger) quads to $\beta(s)$? To Q_x, Q_y? Stability? ... (Nonlinearities & resonances)

REVERT TO MATRICES !

In a long dipole with $L = \rho\theta$ ($ek = 0$)

$$y'' = G \left(= \frac{1}{\rho} \right)$$

so

$$\begin{pmatrix} y \\ y' \\ 1 \end{pmatrix}_{OUT} = \begin{pmatrix} 1 & L & \rho(1-\cos(\theta)) \\ 0 & 1 & \sin(\theta) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ 1 \end{pmatrix}_{IN}$$

OR in general, reduce the 6x6 matrix M_{21} to a 3x3 matrix:

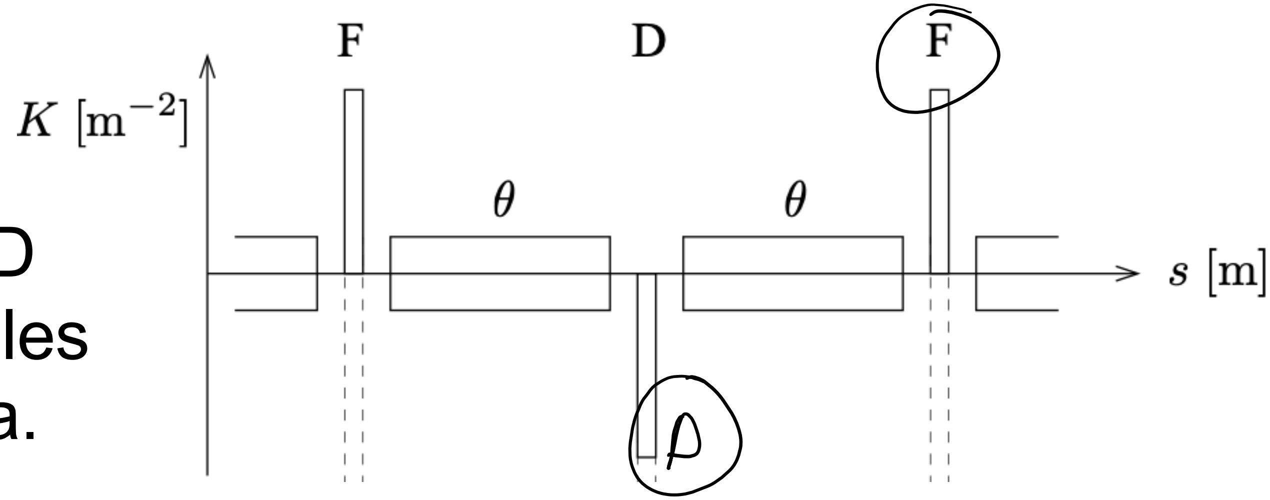
$$\begin{pmatrix} y \\ y' \\ 1 \end{pmatrix}_{OUT} = \begin{pmatrix} m_{11} & m_{12} & m_{16} \\ m_{21} & m_{22} & m_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ 1 \end{pmatrix}_{IN}$$

$\boxed{\text{EG 1}}$ M_3 to one-turn (3x3) matrix

$$\begin{pmatrix} y \\ y' \\ 1 \end{pmatrix} = M_3 \begin{pmatrix} y \\ y' \\ 1 \end{pmatrix}$$

"easily" solved at reference point, r

3.5A FODO cell with equally spaced F and D quads, containing dipoles with bend angles theta.



$\boxed{\text{EG2}}$ Go from thin quad D to F in a FODO cell

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_F = \begin{pmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ +q & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_D$$

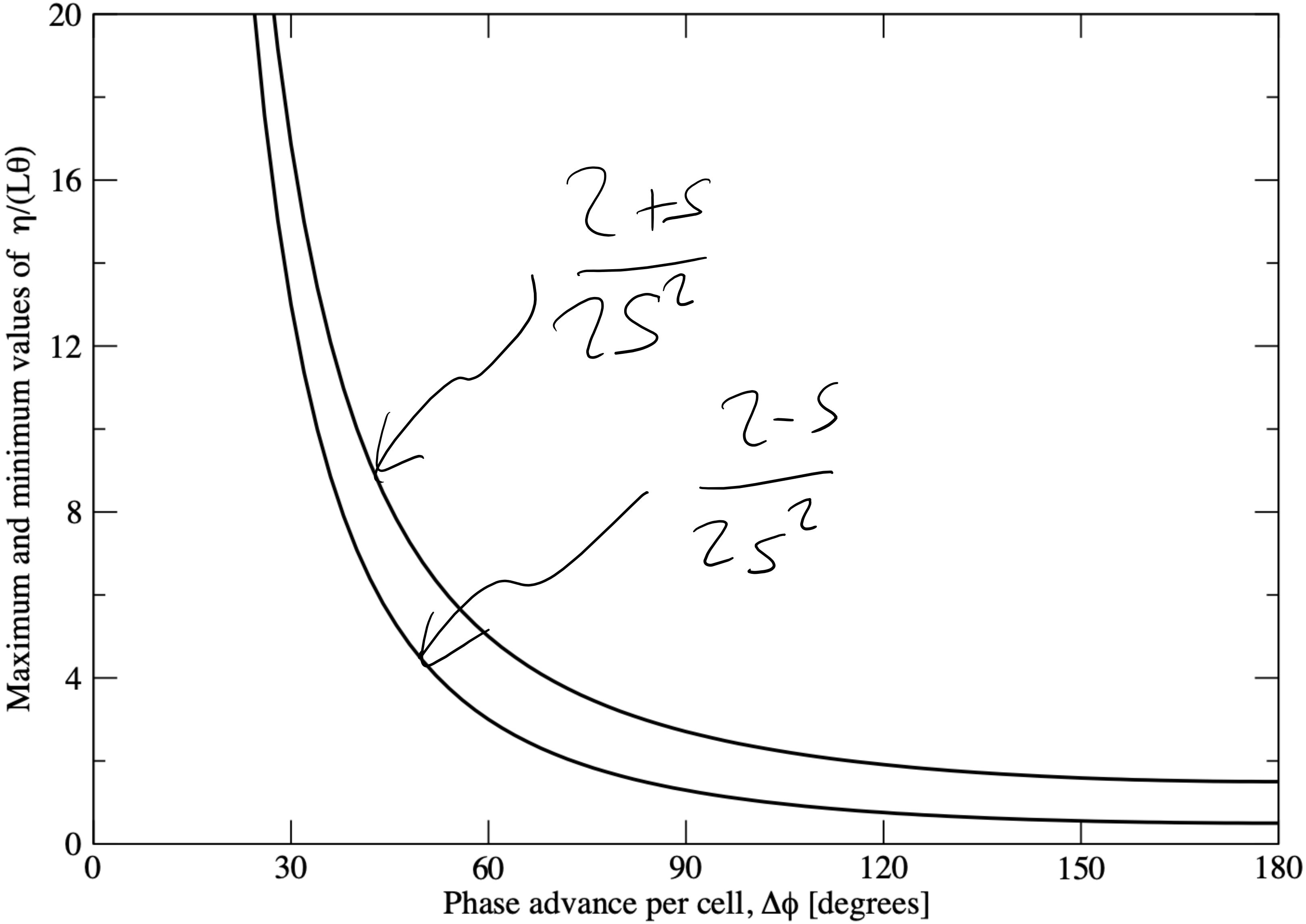
with $\eta'_F = \eta'_D = 0$ (matched PBC) and so $\boxed{\begin{aligned} \eta_F &= L\theta\left(\frac{2+s}{s^2}\right) \\ \eta_D &= L\theta\left(\frac{2-s}{s^2}\right) \end{aligned}}$ $\begin{aligned} &\sim \frac{L^2}{\rho} \\ &\sim \frac{L^2}{\rho}. \end{aligned}$

cf $\hat{\beta} \sim L$, $\check{\beta} \sim L$

$s = \sin\left(\frac{\Delta\phi}{2}\right) = |qL|$

and $\Delta\phi$ = is phase advance per FODO cell

4.3 Min & max dispersion values in a FODO cell as a function of the phase advance per cell.



② OSCILLATING MMTM (δ) or Z $\delta \neq$ const

Q: A particle with $\delta > 0$ goes faster, but travels further. Which wins? Electrons?? (Clue: is $\gamma \gg 1$?)

$\boxed{\text{PATH LENGTH}}$ (1 turn closed orbit) increases

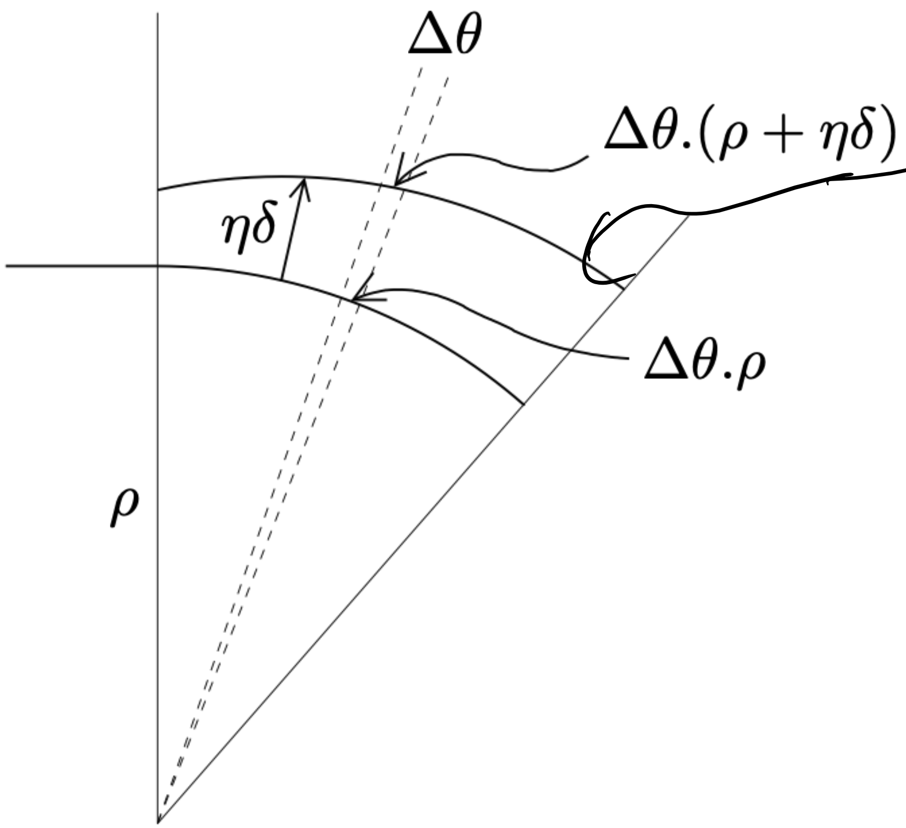
by $\Delta C_{PATH} = \delta \oint \mu \cdot d\theta$

and since $\frac{d\theta}{ds} = \frac{1}{\rho}$

$$\boxed{\Delta C_{PATH} = \delta \oint \frac{\mu}{\rho} \cdot ds}$$

Moving to the BACK of the bunch

4.4 Additional path length of an off-momentum particle passing through a thin dipole slice.



displaced cl. orb.

$\boxed{\text{SPEED}}$ MOVES PARTICLE FORWARD

per turn by

$$\boxed{\Delta C_{\text{SPEED}} = C\frac{\Delta\beta}{\beta} = C\frac{\delta}{\gamma^2}}$$

Beware notation: β, C, η/D, ...

Adding the 2 effects together gives

(A) $$\boxed{z_{n+1} = z_n - \eta_s \cdot \delta_n}$$

where

SLIP FACTOR $$\boxed{\eta_s \equiv \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}}$$

lattice property

relativistic factor (beam)

$$\boxed{\frac{1}{\gamma_T^2} \equiv \frac{1}{C}\oint \frac{\eta}{\rho}\,ds}$$

lattice property

circum-ference

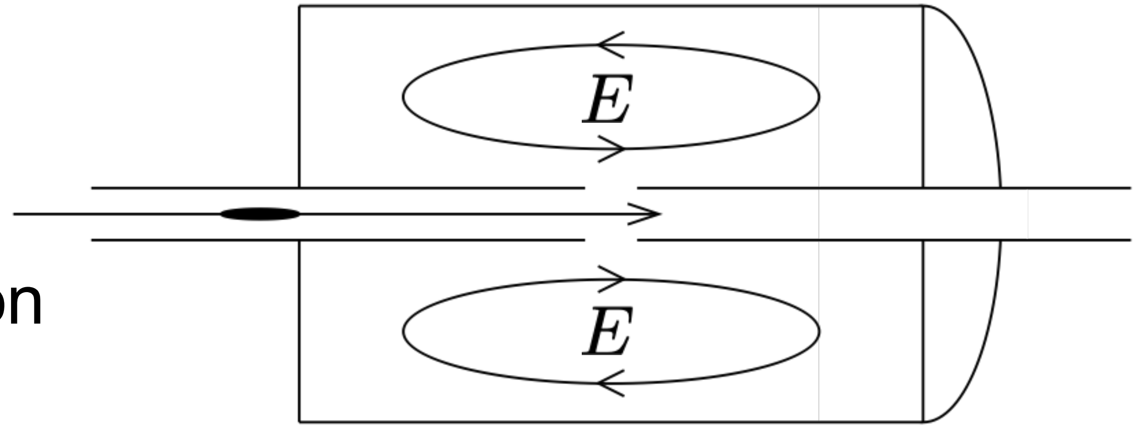
nominal bend radius

dispersion

Q: Is $\gamma > \gamma_T$?? (of electrons)

HOW DOES δ_n CHANGE FROM TURN TO TURN?

4.5 A particle bunch passing through an RF cavity oscillating at a harmonic of the revolution frequency.



Going through an RF cavity ENERGY changes:

$$E_{n+1} = E_n + q V_{RF} \cdot \sin\left(2\pi \frac{z_n}{\lambda_{RF}}\right)$$

where

$$f_{RF} = \frac{c}{\lambda_{RF}} = h \cdot f_{REV}$$

INTEGER HARMONIC NUMBER

$$\frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E}$$

relativistic

(B)

$$\boxed{\delta_{n+1} = \delta_n + \left(\frac{qV_{RF}}{\beta^2 E_0}\right)\cdot \sin\left(2\pi \frac{z_n}{\lambda_{RF}}\right)}$$

FOR SMALL OSCILLATIONS $(z \ll \lambda_{RF})$

$$z_{n+1} = z_n - (\eta_s C)\cdot \delta_n$$

$$\delta_{n+1} \simeq \delta_n + \left(\frac{qV_{RF}}{\beta^2 E_0}\right)\left(\frac{2\pi}{\lambda_{RF}}\right)\cdot z_{n+1}$$

Solved by

$$\boxed{\begin{aligned} z_n &= a_z \cdot \sin\left(2\pi Q_s \cdot n + \phi_0\right) \\ \delta_n &= a_\delta \cdot \cos\left(\qquad\qquad \right) \end{aligned}}$$

so long as the SYNCHROTRON TUNE

$$Q_s = \sqrt{\frac{|\eta_s|}{2\pi} \cdot \frac{C}{\lambda_{RF}} \cdot \frac{qV_{RF}}{\beta^2 E_0}}$$

IS MUCH LESS THAN ONE.

③ STANDARD MAP REPRISE

Equs. Ⓐ + Ⓑ are the standard map

until finished $\left\{ \begin{array}{l} \theta = \theta + \theta' \cdot \Delta t \\ \theta' = \theta' - \sin(\theta) \cdot \Delta t \end{array} \right\}$

with $\boxed{\cos(2\pi \cdot Q_s) = 1 - \frac{\Delta t^2}{2}}$

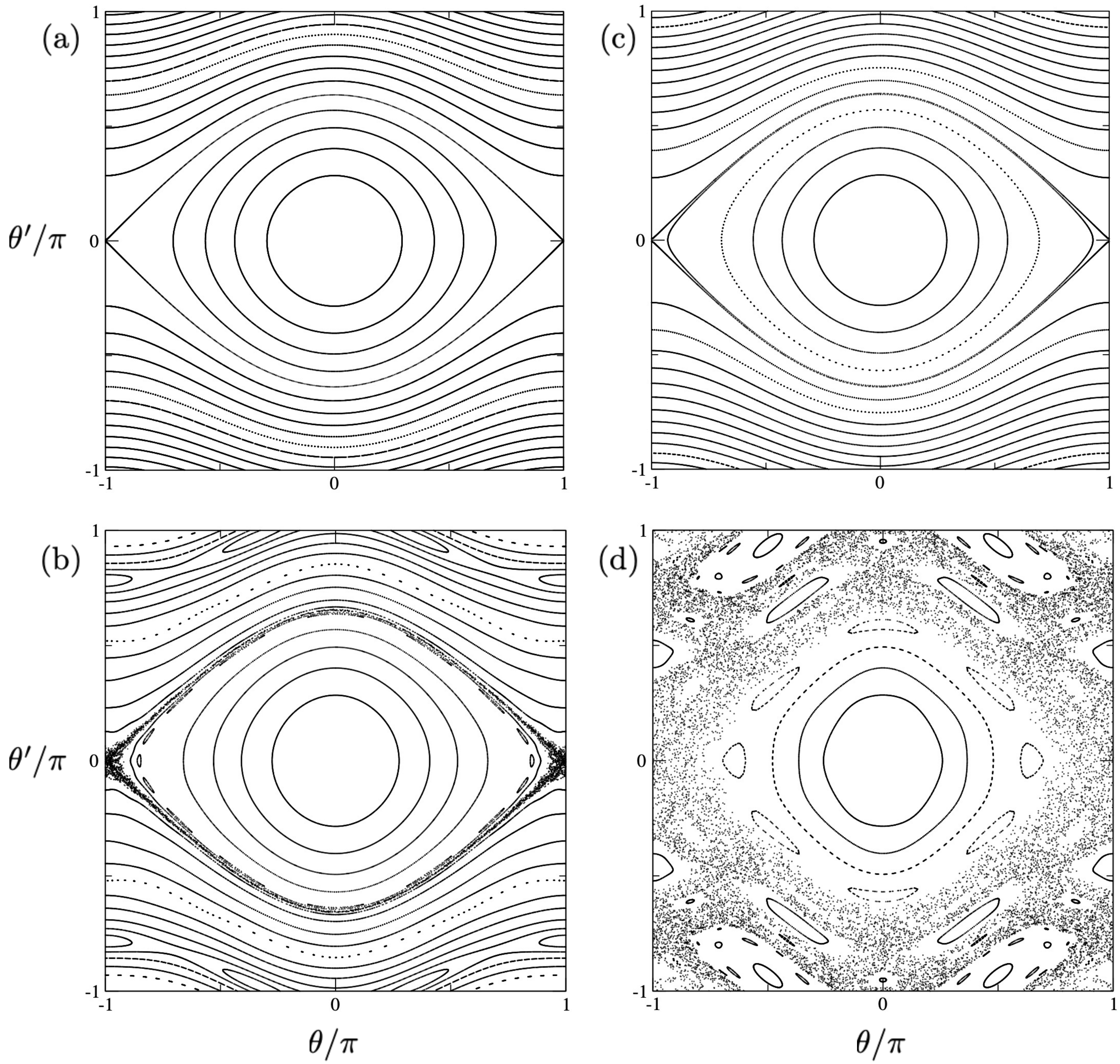
- Clearly, something goes "wrong" when $\Delta t > 2$
- Q: What happens when Q_s becomes large?

Contours

0.06

0.12

0.18



4.6 Standard map behaviour with tunes up to $Q_0 = 0.18$