Lecture 5:

Emittances & Phase Space

$n\beta$

OR

1 particle, many, or none?

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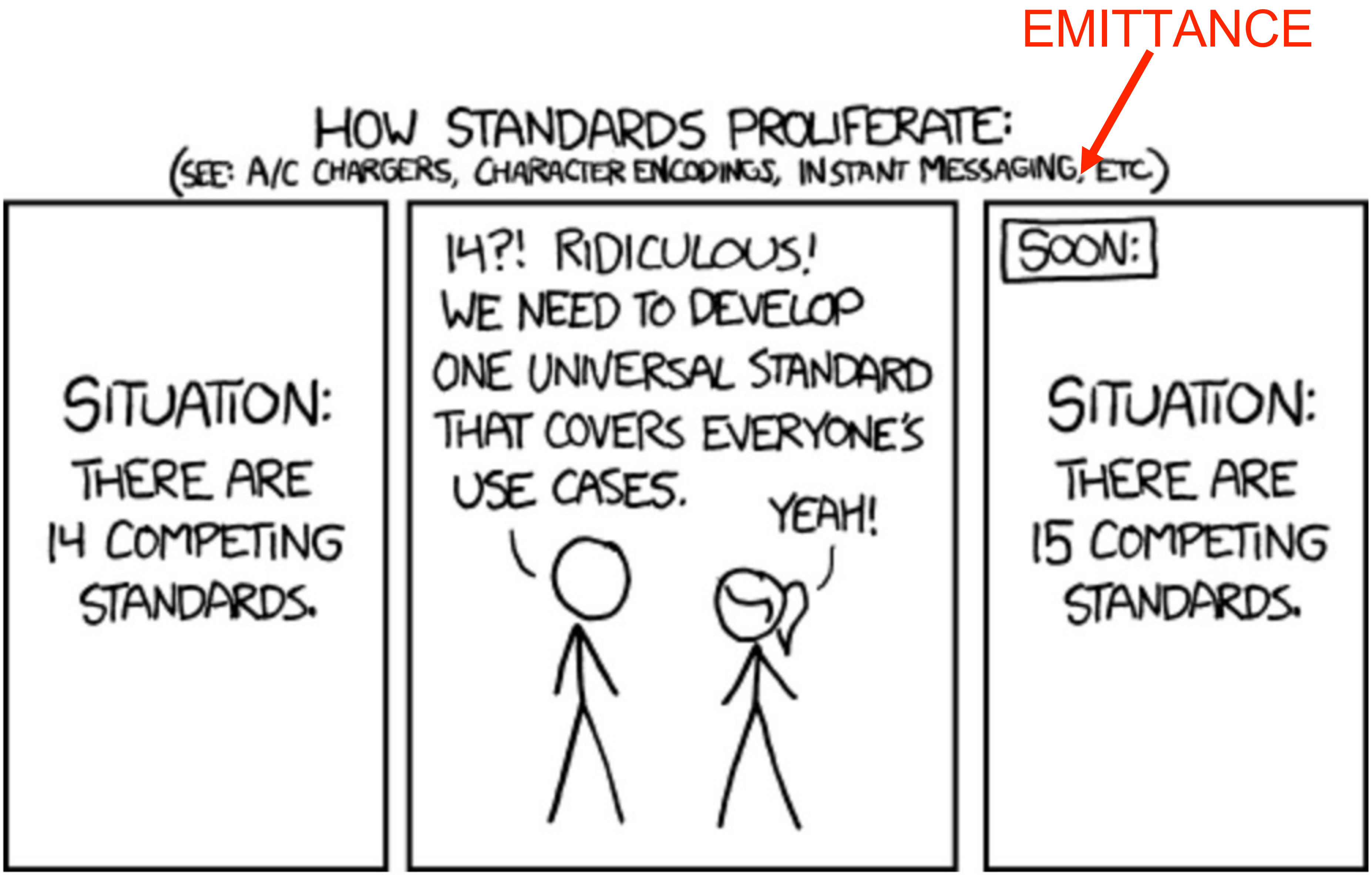


R. Munroe, XKCD comic strip #927, 2015.

One turn matrix

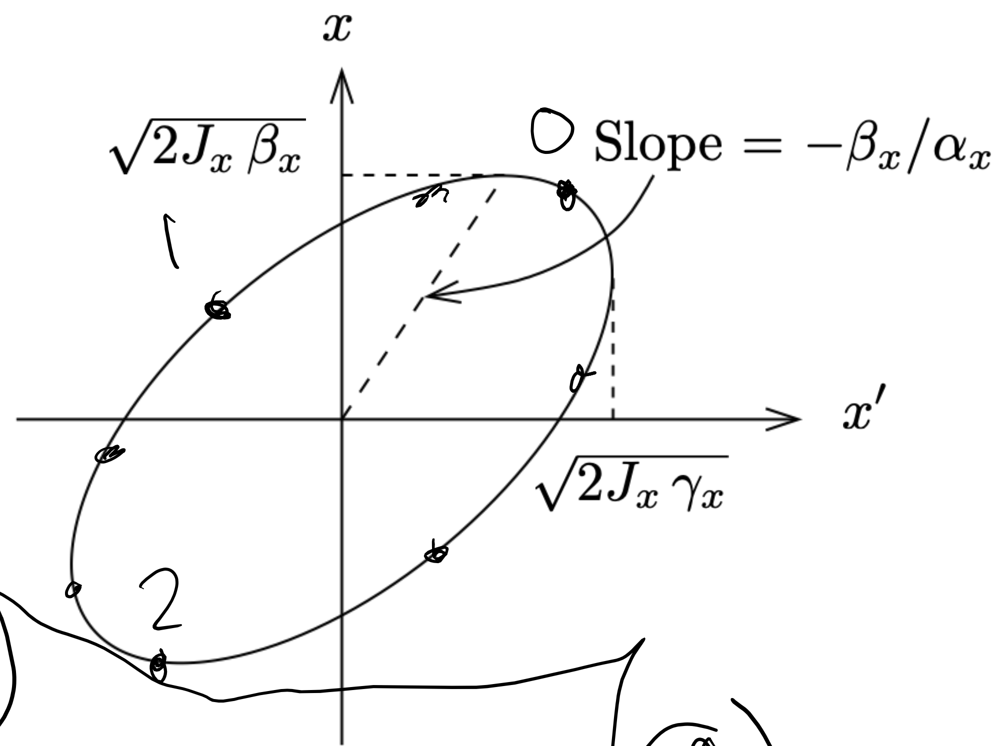
$$M = \begin{pmatrix} \cos(\mu) + \alpha \sin(\mu) \,, & \beta \sin(\mu) \\ -\gamma \sin(\mu) \,, & \cos(\mu) - \alpha \sin(\mu) \end{pmatrix}$$

$\boxed{\text{NO PARTICLE}}$ define Twiss params. β, α, γ

$\Rightarrow$ optical properties of the lattice.

(- But what about single pass systems, e.g. linac or transfer line?)

5.1 The phase-space ellipse described by the Twiss parameters and the action J_x



ONE PARTICLE

$$\boxed{\begin{aligned} x_n &= \sqrt{2J_x\beta_x}\,\sin(\phi_x) \\ x'_n &= \sqrt{2J_x/\beta_x}\,[\cos(\phi_x) - \alpha\sin(\phi_x)] \end{aligned}}$$

(A)

where J_x (action) is a constant &

$\phi_x = 2\pi Q_x \cdot n + \phi_0$

n is turn number,

(J, ϕ) are "Action-angle" co-ordinates

Recall the TWISS IDENTITY

$$\boxed{\beta\gamma = 1 + \alpha^2}$$ so that $\det(M) = 1$

Invert A to give for ANY n value

$$\boxed{2J_x = \beta_x x'^2 + 2\alpha_x x x' + \gamma_x x^2}$$

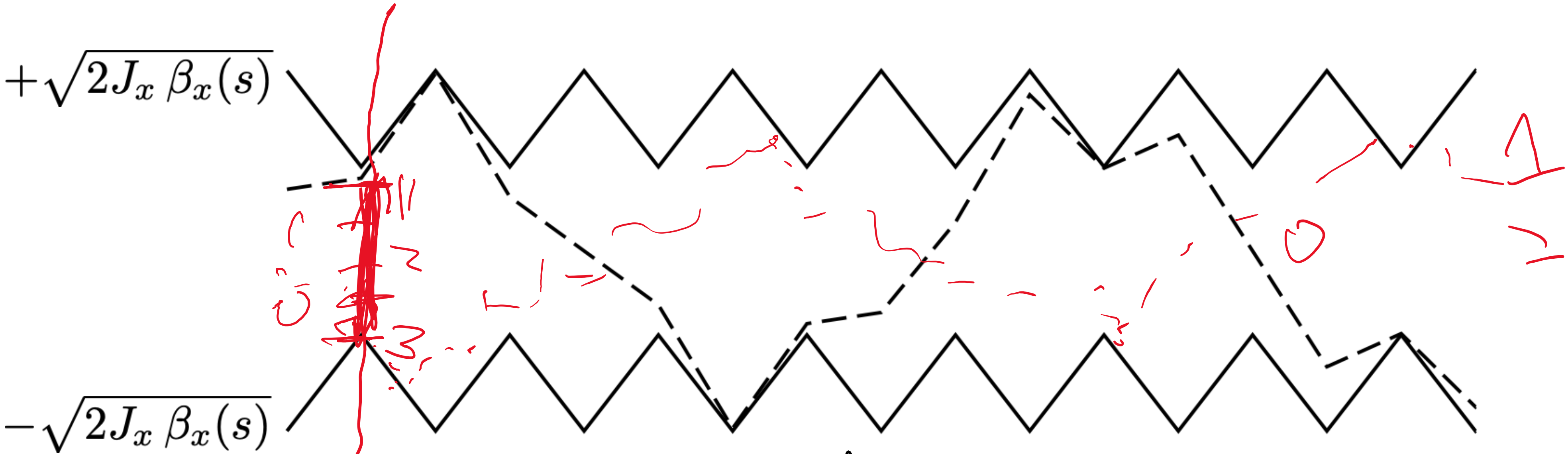
Length

How to calculate action

Area of the ellipse is $\boxed{A_x = 2\pi J_x}$

EVERYWHERE in ring, despite ellipse stretching & tilting with $\beta(s), \alpha(s), \gamma(s)$

5.2 Horizontal oscillations within a betatron envelope, for a FODO channel of 80 degrees per cell.



RMS displacement of $\boxed{\text{ONE PARTICLE}}$ is

$$\sigma_{x,1} \equiv \langle x^2 \rangle^{1/2} = 2\sqrt{J\beta}\,\langle \sin^2(\phi) \rangle$$

$$\boxed{\sigma_{x,1} = \sqrt{J_x \beta_x}}$$

SIMPLE!

$\boxed{\text{MANY PARTICLES}}$ with an action distribution of $\rho(J)$, with

$$N = \int_0^\infty \rho(J) \cdot dJ \quad \text{particles per bunch}$$

RMS size of the bunch is

$$\sigma_x^2 = \frac{1}{N} \int_0^\infty \sigma_{x,1}^2 \, \rho(J_x) \cdot dJ_x$$

$$= \beta_x \cdot \frac{1}{N} \int J_x \, \rho(J) \cdot dJ$$

$$= \beta_x \langle J_x \rangle$$

$$\boxed{\sigma_x^2 = \beta_x \, \epsilon_{x,u}}$$

$\epsilon_{x,u}$ unnormalised emittance is JUST mean action!!

SIMPLE... so long as we stick to RMS measures....

More generally

$$\begin{pmatrix} \langle x^2 \rangle \\ \langle x x' \rangle \\ \langle x'^2 \rangle \end{pmatrix} = \epsilon_{x,u} \begin{pmatrix} \beta_x \\ -\alpha_x \\ \gamma_x \end{pmatrix}$$

Shows how Twiss parameters connect to beam distributions... NOT necessarily Gaussian !!

In general we ignore coupling between x, y, z planes, but what about DISPERSION?

$$x_{TOT,n} = \sqrt{2J_x\beta_x}\sin(\phi_{x,n}) + \eta\cdot\delta_n$$

$$y_n = \sqrt{J_y\beta_y}\sin(\phi_{y,n})$$

where

$$\delta_n = d_\delta \sin(\phi_{s,n})$$

<u>IF</u> $\phi_{x,n}$ & $\phi_{s,n}$ are uncorrelated

<u>THEN</u> $$\boxed{\sigma^2_{x,TOT} = \beta_x\epsilon_{x,n} + \eta^2\left(\frac{\sigma_p}{p_0}\right)^2}$$

MEAN SQUARE SUM !!

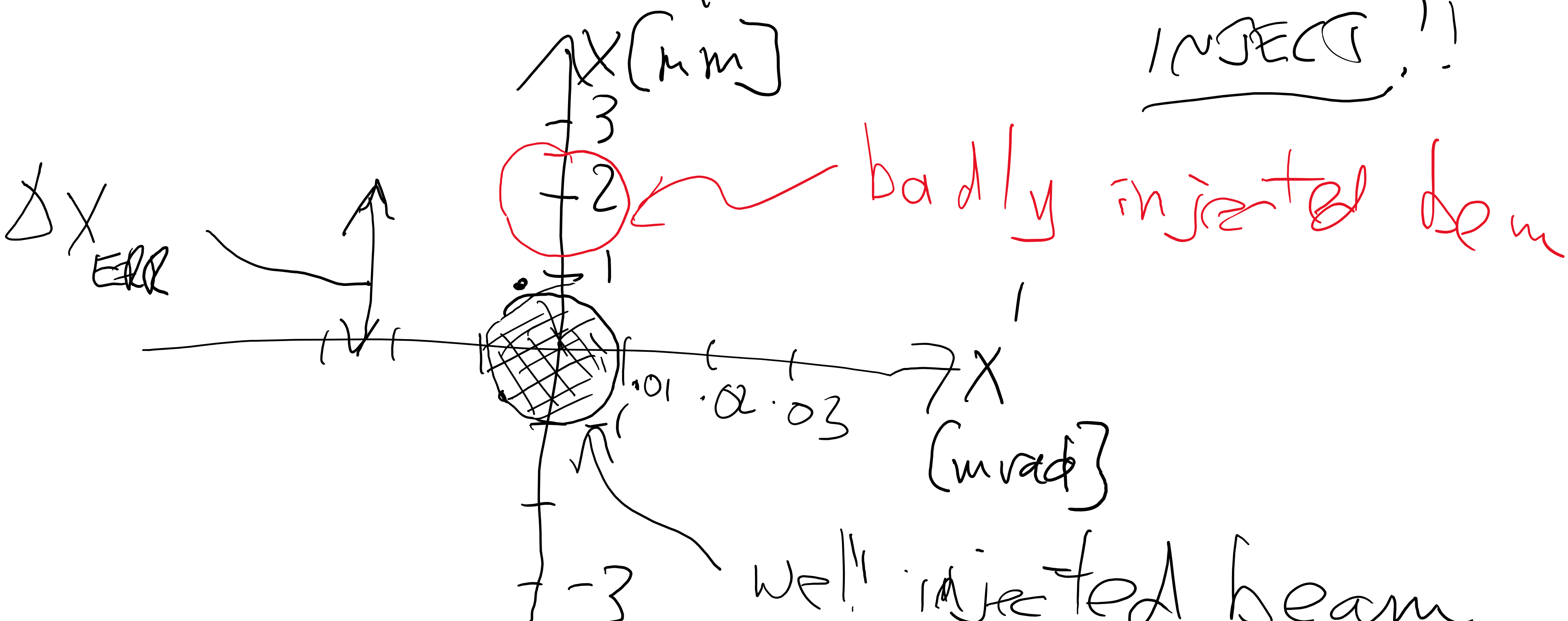
RMS measures are simple ... in a quasi-static storage rings.

WHAT HAPPENS IN THE RADICAL TRANSIENTS AT INJECTION?

FILAMENTATION & TUNE SPREAD

SCENARIO: Assume $\beta_x = 100$ m, $\alpha_x = 0$

<u>INJECT!!</u>



Badly injected ~~beam~~ particles have a typical action of $J \sim \frac{\Delta x_{ERR}^2}{\beta_x} \pm \ldots$

.... with a spread in tunes because

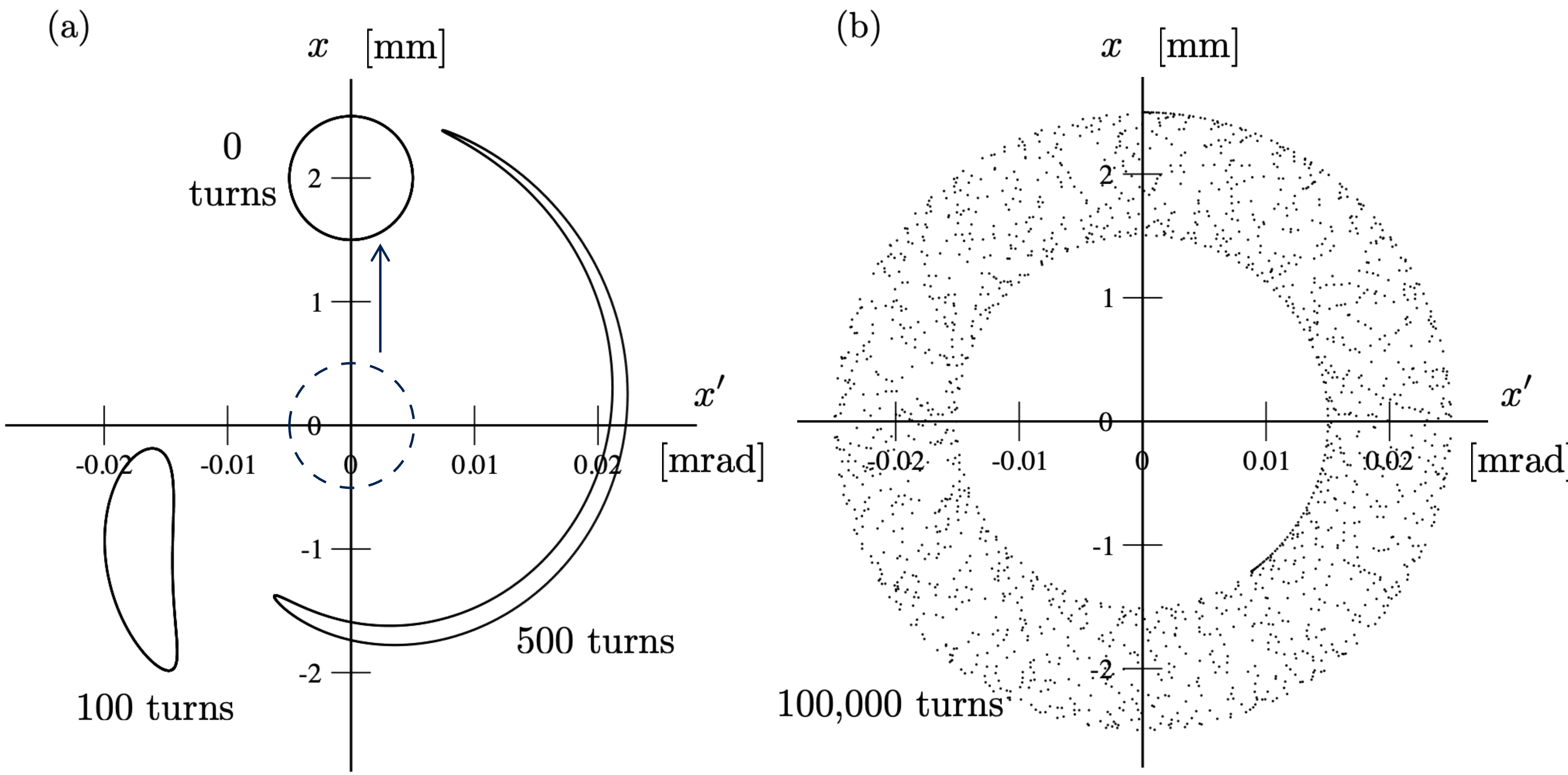
$$\boxed{Q(J) = Q(0) + \frac{dQ}{dJ} \cdot J}$$

e.g. 28.15

approximately constant (it turns out)

S

5.3 Phase space filamentation of a badly injected beam. No significant azimuthal structure is left after 100,000 turns.

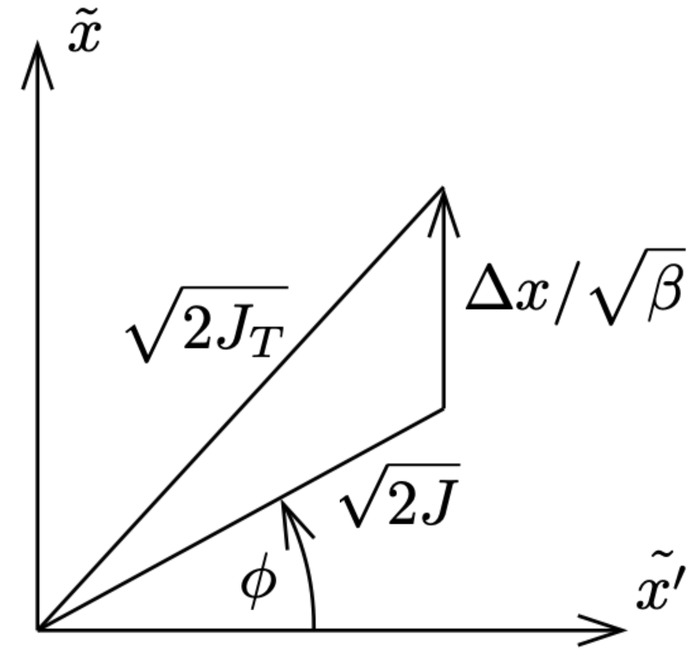


CONCLUDE: When transients die down (filamentation is complete) the AREA occupied by the bunch is FORMALLY the same as initially, but WHO CARES (for practical purposes)!!

DISPLACEMENT ERRORS

$\boxed{\text{ONE PARTICLE}}$

5.4 A particle injected with a pure displacement error of Delta x that increases its action to J_T



$$\boxed{J_T = J + \Delta x\sqrt{\frac{2J}{\beta}}\cos(\phi) + \frac{\Delta x^2}{2\beta}}$$

MANY PARTICLES Average over ϕ,

$$\langle \cos(\phi) \rangle = 0$$

$$\boxed{\epsilon_{v,T} = \epsilon_{v,0} + \frac{\Delta x^2}{2\beta_x} = \epsilon_{v,0}\left[1 + \frac{\Delta x^2}{2\sigma^2}\right]}$$

$\boxed{\text{MOMENTUM ERROR}}$ δ_{ERR}

If incoming beam is slightly off-momentum then since

$$x_{TOT} = x + \eta \cdot \delta$$

analogous to

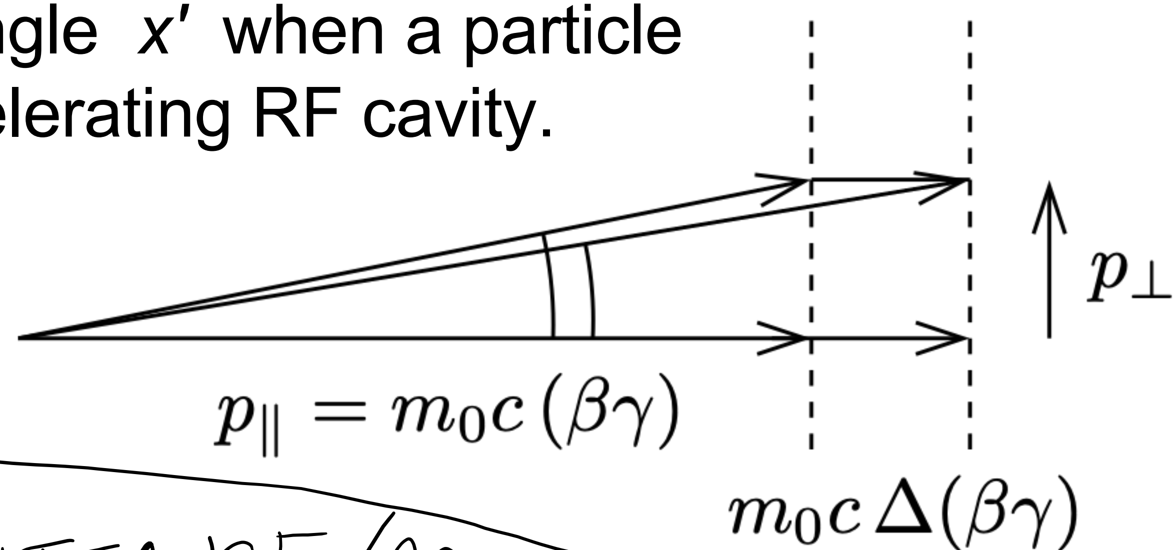
$$\Delta x = -\eta\, \delta_{ERR}$$

and

$$\boxed{\epsilon_{v,T} = \epsilon_{v,0} + \frac{\eta^2}{\beta}\, \delta_{ERR}^2}$$

shows why $\eta = \eta' = 0$ is a GOOD THING at injection location! DISPERSION SUPRESSORS!!

5.5 Shrinkage of the angle x' when a particle passes through an accelerating RF cavity.



NORMALISED EMITTANCE/ADIABATIC DAMPING

When RF accelerates the beam by $\Delta(\beta\gamma)$ the angle of the particle $x' = p_\perp / p_{||}$ decreases

$$x' \to x' \Big/ \left(1 + \frac{\Delta(\beta\gamma)}{\beta\gamma}\right)$$

Averaging over MANY particles with the same J

$$\langle J \rangle_{NEW} = \frac{\langle J \rangle}{1 + \Delta(\beta\gamma)/\beta\gamma}$$

and the UNnormalised emittance shrinks

$$\boxed{\epsilon_U = \frac{\epsilon_N}{\beta\gamma}}$$

NORMALISED EMITTANCE IS "CONSTANT" IN HADRON machine

$$\boxed{\sigma^2 = \frac{\beta \epsilon_N}{(\beta\gamma)}}$$

ELECTRONS? SYNCH. RAD?

ELECTRONS or PROTONS?

Electrons radiate MANY photons per turn, both damping (H + V) and exciting (H) betatron oscillations, rapidly coming to equilibrium. (See later lecture.)

1) PROTONS/IONS

$$\epsilon_{u,x} = \frac{\epsilon_{N,x}}{\beta\gamma}$$

$$\epsilon_{u,y} = \frac{\epsilon_{N,y}}{\beta\gamma} \simeq \epsilon_{u,x}$$

"ROUND" BEAMS

ϵ_N is (quite) constant

2) ELECTRONS

$$\epsilon_{\nu,x} = \epsilon_{\nu,x}(\gamma, \text{OPTICS})$$

$$\epsilon_{\nu,y} \ll \epsilon_{\nu,x}$$

FLAT BEAMS

ϵ_N is not very useful!

SINGLE PASS SYSTEMS (LINACS, transfer lines)

Can't use one-turn matrices with PBC to define Twiss parameters

PARADIGM SHIFT: Beam distribution defines β, α, γ

$$\begin{pmatrix} \beta_x \\ -\alpha_x \\ \gamma_x \end{pmatrix} = \frac{1}{\epsilon_{x,A}} \begin{pmatrix} \langle x^2 \rangle \\ \langle x x' \rangle \\ \langle x'^2 \rangle \end{pmatrix}$$

where AREA EMITTANCE:

$$\boxed{\epsilon_{x,A} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}}$$

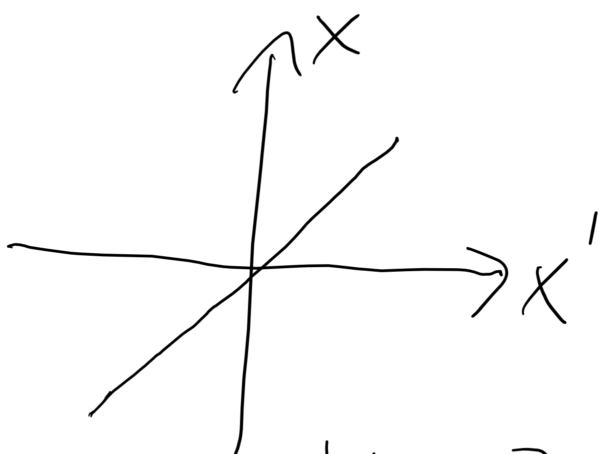
(c.f. Twiss identity)

$\epsilon_{x,A}$ may have little to do with **BEAM** area in phase space!!

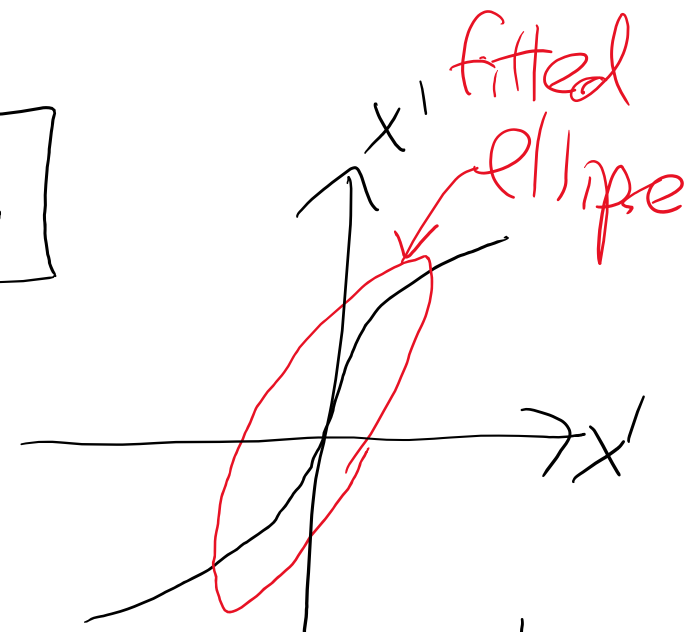
CONSIDER 2 "WANGLER" DISTRIBUTIONS

$$x' = \underset{\text{constant}}{C}\, x^{m} \quad \text{integer, eg } 1 \text{ or } 3$$

$\boxed{m=1}$



$\boxed{m=3}$



In both cases the BUNCH area is zero, but ...

... while

$$\epsilon_{x,A} = C\sqrt{\langle x^2\rangle\langle x^{2m}\rangle - \langle x^{m+1}\rangle^2}$$

$= 0$ for $m = 1$ $\quad > 0$ for $m \geq 3$

TWISTED LINES CAN STRAIGHTEN $\Rightarrow$

$\epsilon_{x,A}$ MAY DECREASE !!

LONGITUDINAL PARAMETERS

RMS bunch length σ_z & momentum spread are "constant" around a ring.

$$\sigma_p = \frac{\langle \Delta p^2 \rangle^{1/2}}{p_0}$$

$\boxed{\text{ONE}}$ convenient longitudinal normalised emittance is ϵ_s, where

$$\sigma_s = \sqrt{\frac{\beta_s \epsilon_s}{(\beta\gamma)}}, \quad \sigma_p = \sqrt{\frac{\epsilon_s}{\beta_s(\beta\gamma)}}$$

REMARKABLY

$$\boxed{\beta_s = \frac{\sigma_s}{\sigma_p} = \frac{C}{2\pi} \cdot \frac{|\eta_s|}{Q_s}}$$

$$\frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}$$

Exciting things happen when $\gamma \to \gamma_T$ and $\eta_s \to 0$!!