Lecture 7: RF cavities

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“.. [Wideroe] developed .. [the method of] resonating particles with a radio frequency electric field to add energy to each traversal of the field. This experiment … was studied by Ernest Lawrence … and used as the basis for his creation of the cyclotron in 1929. Wideroe began collaborating with the Nazi German government … [In] 1943 he introduced the concept of colliding particles head-on to increase interaction energy and a storage ring device. His Norwegian citizenship was ultimately revoked for working with the Nazi government. In 1946 he filed a patent in Norway for an accelerator based on synchronous acceleration.”

Wikipedia, “Rolf Wideroe”

A) Waveguides

B) Transverse modes

C) Pill boxes

D) Performance limits

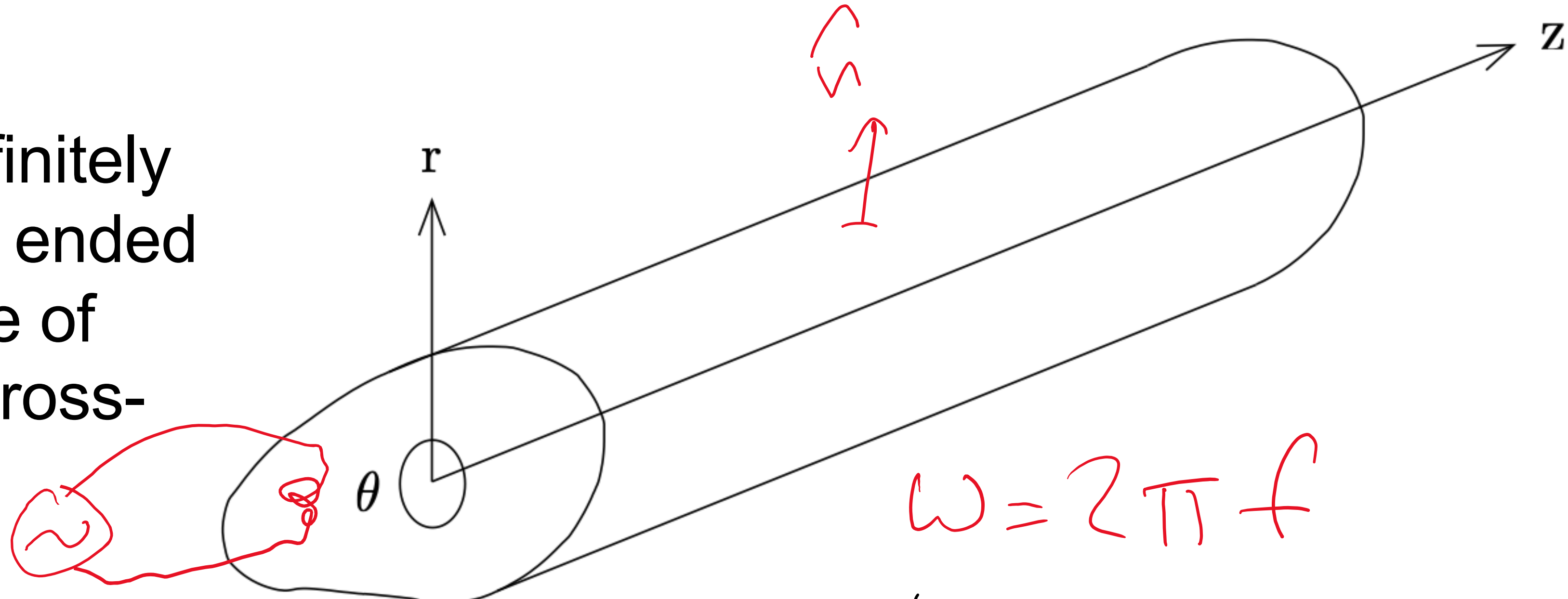
Transit time factor

Kilpatrick criterion

A) WAVEGUIDES

(1)

7.1 An infinitely long (2) open ended waveguide of arbitrary cross-section.



(3) Perfectly conducting walls

Transmitting (?) fields oscillating a $\omega=2\pi f$

Boundary conditions at wall:

$\hat{n} \times \bar{E} = 0$ no parallel E-field

$\hat{n} \cdot \bar{B} = 0$ no perpendicular B-field

$\hat{n}$: unit vector perpendicular to wall.

Real parts of complex $\bar{B}$ & $\bar{E}$ are physical

CONSIDER a mode labeled by wave number k

(A) $$\boxed{\begin{aligned} \bar{E} &= E(r,\theta) \cdot e^{i(kz-\omega t)} \\ \bar{B} &= B(r,\theta) \cdot e^{i(kz-\omega t)} \end{aligned}}$$

where the phase velocity in z direction is $v_p = \omega / k$

IF k is imaginary then wave is damped
- DOES NOT PROPAGATE

Q1: HOW does k vary ω ?

(Q2: WHAT happens when (perfectly conducting) walls are added at ends to make a CAVITY?)

B) TRANSVERSE MODES

Three categories of modes solve A

1 TRANSVERSE MAGNETIC (TM)

$B_z = 0$ everywhere, with $E_z = 0$ at walls

2 TRANSVERSE ELECTRIC TE

$E_z = 0$ everywhere, with $\frac{\partial B_z}{\partial n} = 0$ at walls

TM modes are most useful: they accelerate, decelerate, & confine

(BUT crab cavities are TE !!)

3) TRANSVERSE ELECTROMAGNETIC TEM

No longitudinal fields : FREE-SPACE waves

$$\boxed{k = \sqrt{\mu_R \epsilon_R}\,\frac{\omega}{c}}$$

In a vacuum $\mu_R = \epsilon_R = 1$ $\quad k = \frac{\omega}{c}$

$\lambda = 2\pi / k$

Wavelength is MUCH smaller than WG width

Frequency . . . . larger - - - -

SOLVE FOR THE TM (or TE) MODES

of a particular geometry.

This identifies a FAMILY of modes, each with a cut-off frequency ω_n with $n = 0, 1, 2, \ldots \infty$

where

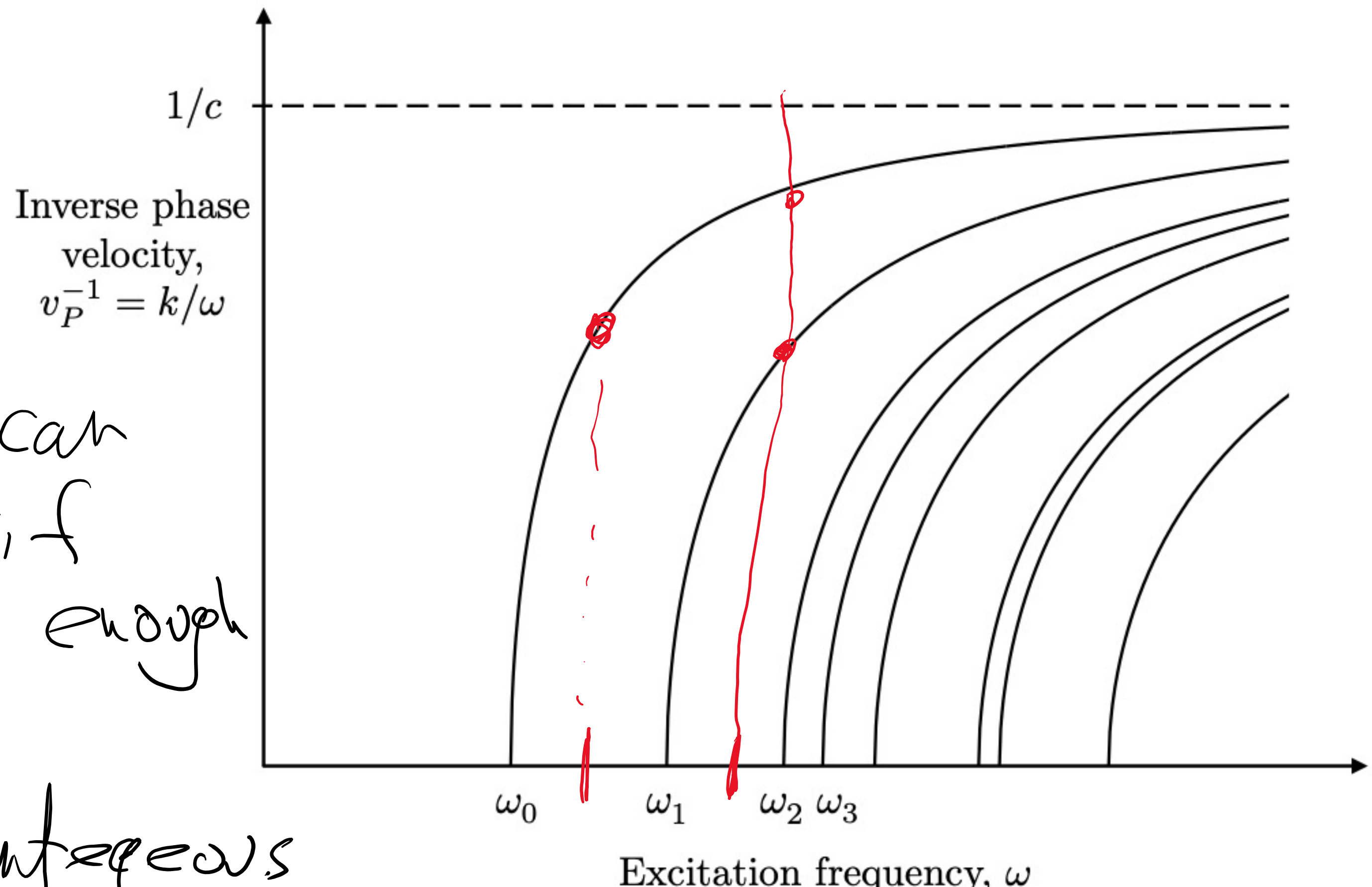
$$\boxed{k = \frac{\pm 1}{c}\sqrt{\omega^2 - \omega_n^2}}$$

cut-off frequency

- CLEARLY mode n does not propagate if $\omega < \omega_n$

→ $\pm k$ modes propagate forwards or backwards

7.2 Dependence of wave number *k* on excitation frequency ω for a family of TE or TM waveguide modes.



- Many modes can co-exist if ω is large enough

- It is advantageous
  1. transmit at $\omega_0 < \omega < \omega_1$
  2. use a geometry with $\omega_0 \ll \omega_1$

- The beam itself can drive many unwanted HIGHER ORDER MODES (HOM) up to frequencies of

$$\boxed{\omega_{MAX} \simeq 2\pi \frac{c}{\sigma_z}}$$

where, σ_z is RMS bunch length, which may be very short

- HOM's sometimes need explicit damping ... How?

C) PILL-BOX CAVITIES (TUNA CAN)

- CYLINDRICAL RESONANT CAVITY

ADD flat ends at $z=0$, & $z=L$

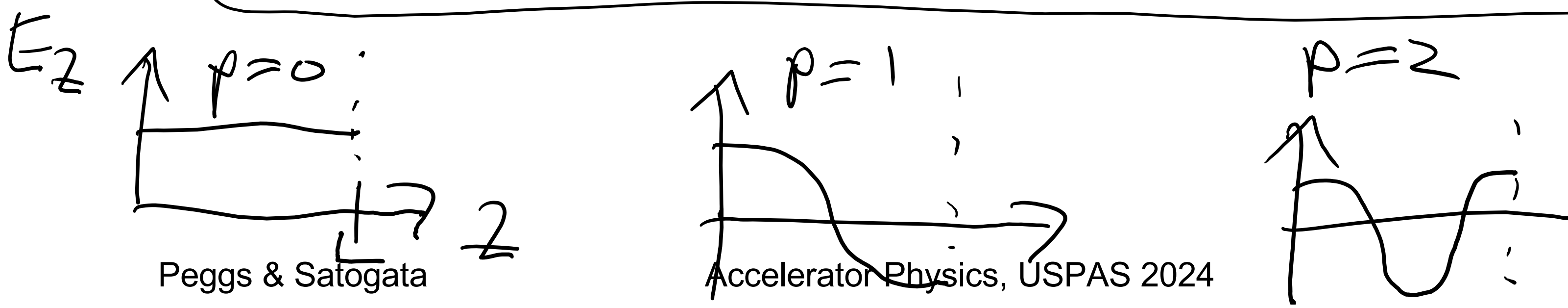
Take a pair of waveguide modes at ω

$$k = \pm k_p = \pm p \cdot \frac{\pi}{L}, \quad p = 0, 1, \ldots \infty$$

add them together to form a resonant TM mode

$$e^{\pm ikz} = \cos(kz) \pm i\sin(kz)$$

$$\boxed{E_z = \psi(r,\theta) \cdot \cos\left(p\pi \frac{z}{L}\right) \cdot e^{-i\omega_{REST}}}$$



NEXT SOLVE $\psi(r,\theta)$ for a circle

- this adds 2 more INDICES: m, n

where the natural RESONANT frequency is

$$\boxed{\omega_{mnp} = c\sqrt{\left(\frac{u_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}}$$

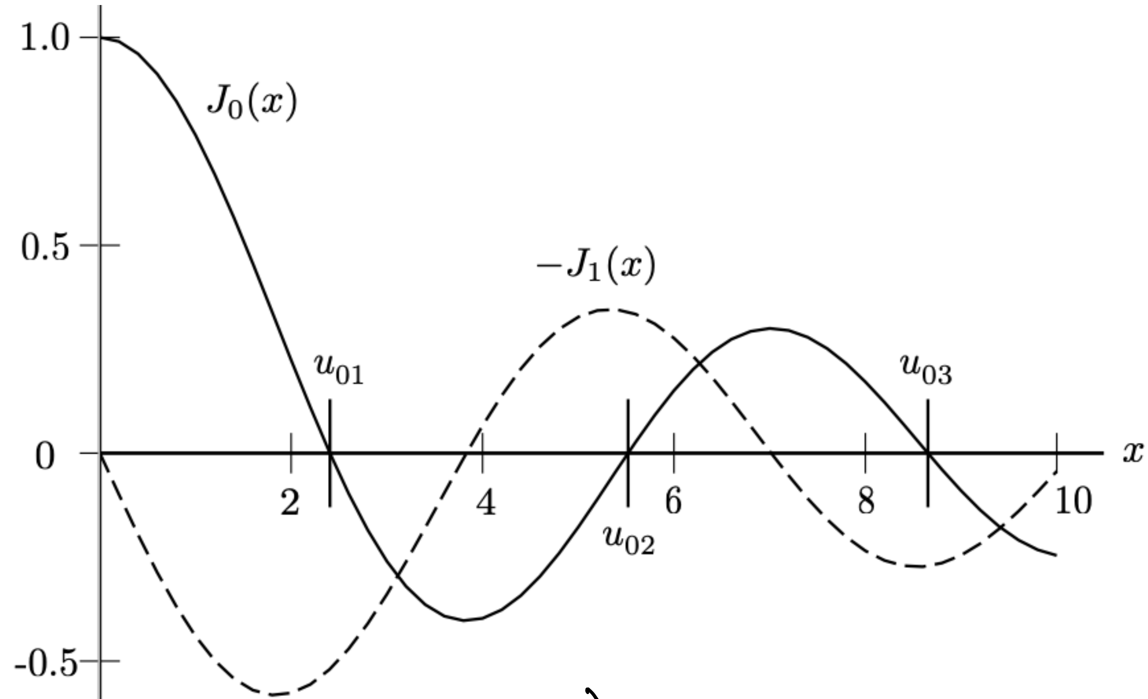
and $u_{mn} \geq 1$ is n'th root of Bessel function J_m

$$\boxed{J_m(u_{nm}) = 0}$$

SIMPLIFY

- Use "lowest" resonant frequency $p=0$
- Assume $m=0$ (no azimuthal structure)

7.3 Bessel function J_0, its slope, and its first 3 roots at $J_0 = 0$



$(u_{01}, u_{02}, u_{03}) = (2.4, 5.52, 8.65)$

n labels ~~counts~~ the number of zero crossings

E7.10 TM modes in a pill-box cavity, labeled by *m, n, p*

$$
\begin{aligned}
E_z &= E_0 \cos\left(p\pi\frac{z}{L}\right)\cdot J_m\left(u_{mn}\frac{r}{R}\right)\cdot\cos(m\theta)\cdot e^{-i\omega_{mnp}t} \\
E_r &= -E_0\,\frac{p\pi R}{u_{mn}L} \sin\left(p\pi\frac{z}{L}\right)\cdot J'_m\left(u_{mn}\frac{r}{R}\right)\cdot\cos(m\theta)\cdot e^{-i\omega_{mnp}t} \\
E_\theta &= E_0\,\frac{mp\pi R}{u_{mn}^2 L} \sin\left(p\pi\frac{z}{L}\right)\cdot\frac{R}{r}J_m\left(u_{mn}\frac{r}{R}\right)\cdot\sin(m\theta)\cdot e^{-i\omega_{mnp}t} \\
B_z &= 0 \\
B_r &= B_0\,\frac{m\omega_{mnp}R}{u_{mn}^2 c} \cos\left(p\pi\frac{z}{L}\right)\cdot\frac{R}{r}J_m\left(u_{mn}\frac{r}{R}\right)\cdot\sin(m\theta)\cdot e^{-i\omega_{mnp}t} \\
B_\theta &= B_0\,\frac{\omega_{mnp}R}{u_{mn}c} \cos\left(p\pi\frac{z}{L}\right)\cdot J'_m\left(u_{mn}\frac{r}{R}\right)\cdot\cos(m\theta)\cdot e^{-i\omega_{mnp}t}
\end{aligned}
$$

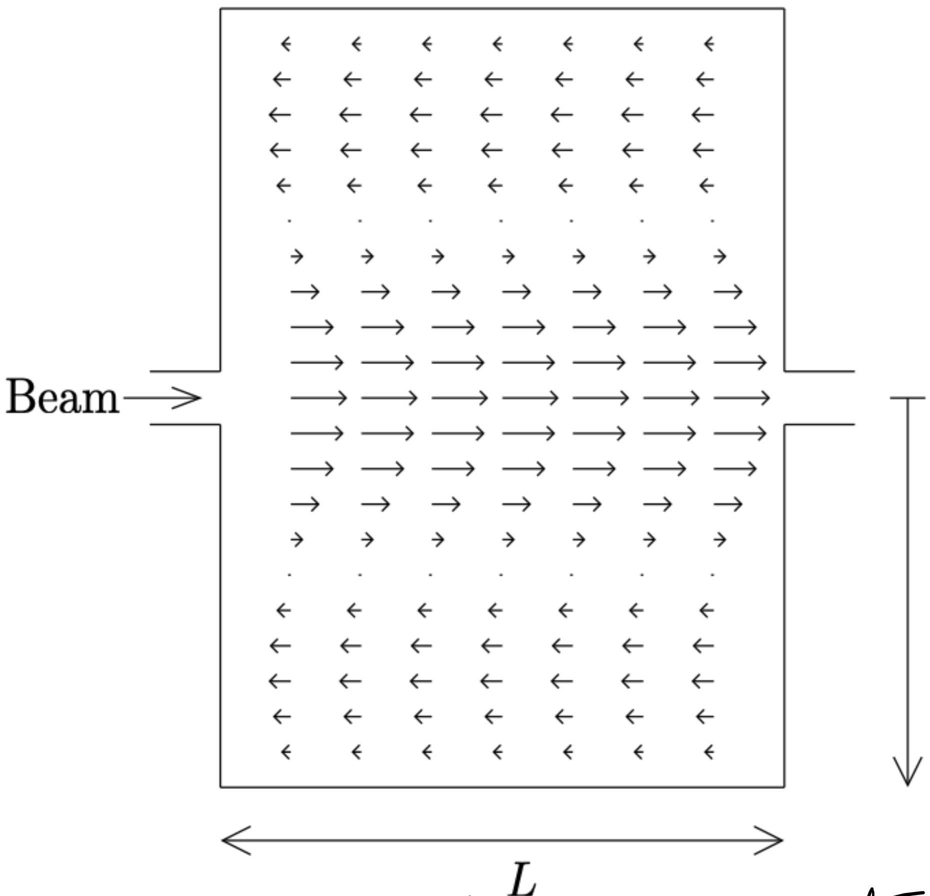
Consider $m = 0$, $p = 0$: ONLY 2 COMPONENTS SURVIVE !!

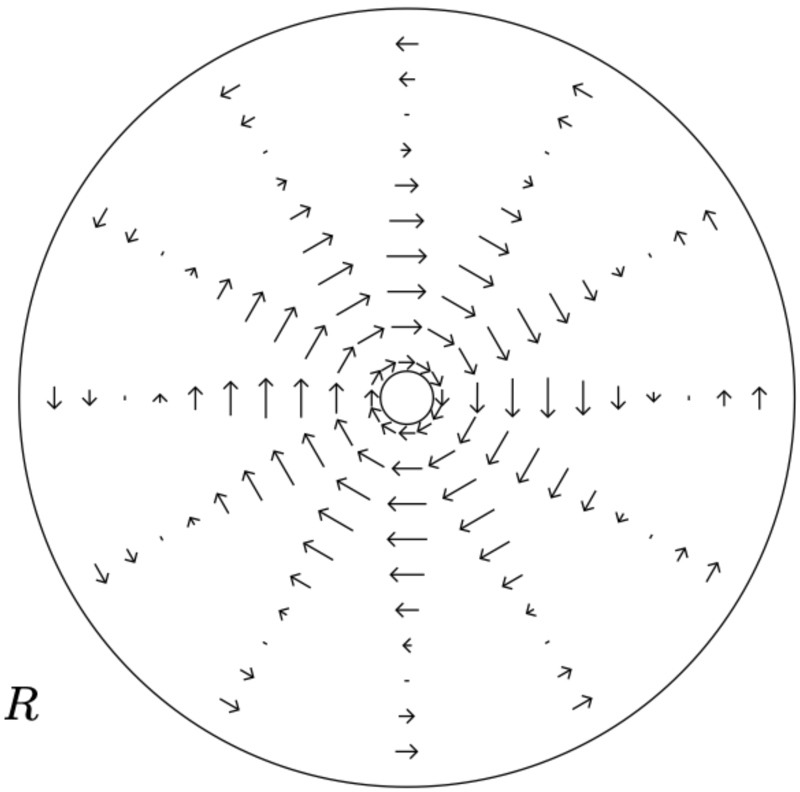
$\underline{TM_{0n0}}$ has (only) 2 components:

$$\boxed{\begin{aligned} E_z &= E_0 \, J_0\left(u_{0n}\frac{r}{R}\right) e^{-i\omega_{0n0}t} \\ B_\theta &= -B_0 \frac{\omega_{0n0}R}{u_{0n}\,c} \cdot J_1\left(u_{0n}\frac{r}{R}\right) e^{-i\omega_{0n0}t} \end{aligned}}$$

7.4 Transverse magnetic mode TM020 in a pill-box cavity.

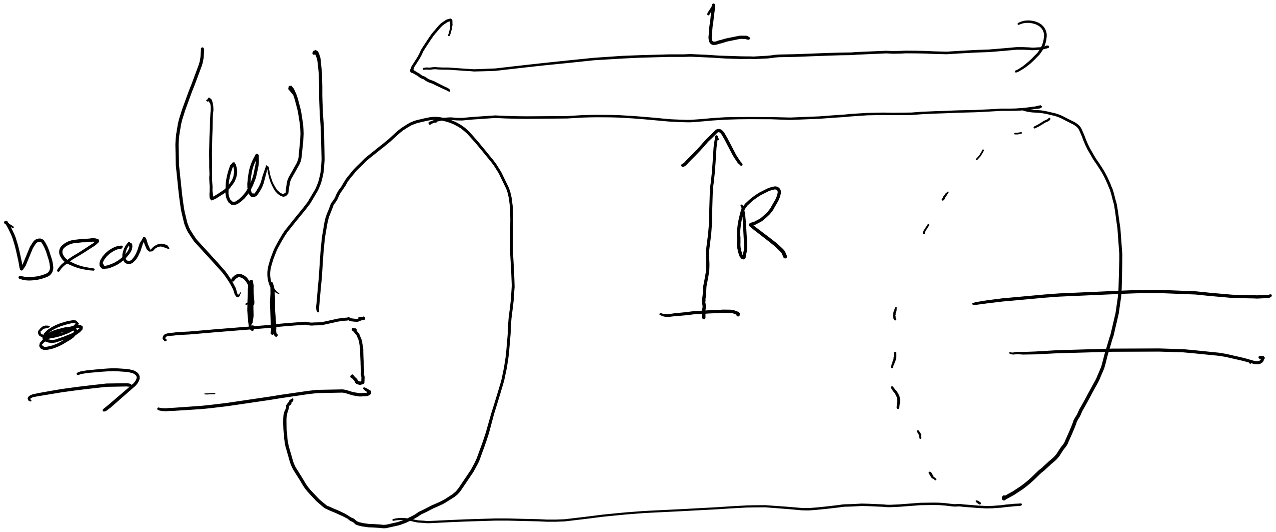
$$E_z \sim J_0\left(u_{02}\frac{r}{R}\right) \qquad B_\theta \sim -J_1\left(u_{02}\frac{r}{R}\right)$$





- n counts the E_z zero crossings between r=0 → R
- NOTE: B_θ is non-zero at r=R !!

D) PERFORMANCE LIMITS



Add holes at each end to allow beam through

- This let's field leak a little into the beampipe,
  but not far if ω_{0n0} is much smaller
  than the pipe cut-off frequency
  ∴ if pipe radius $\ll R$
- (Can put input power couplers (or HOM dampers) there

TRANSIT TIME FACTOR

ADJUST R to get the right frequency

- HOW LONG L should the pill-box be?

- A particle with speed βc passes cavity center at $t=0$ acquiring a voltage

$$V_A = \int_{-L/2}^{L/2} E_z . dz = \beta c . E_0 \int_{-L/2\beta c}^{L/2\beta c} e^{i\omega t} . dt$$

from a TM_{010} mode, so

$$V_A = E_0 L . T_1$$

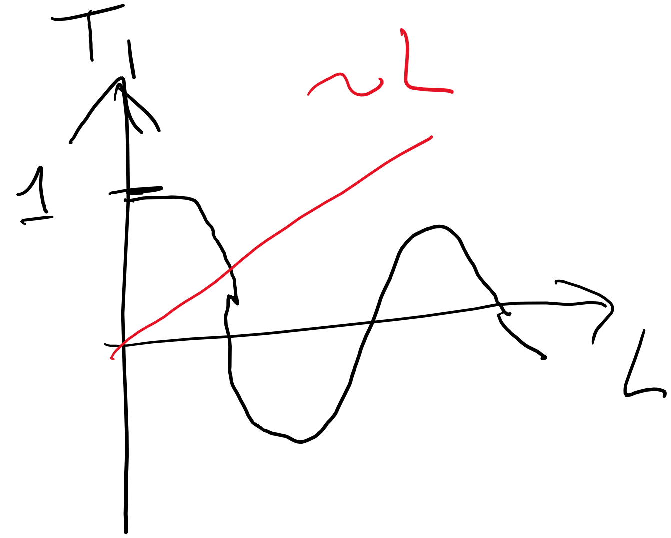
Transit time factor

where ....

$$V_A = E_0 L \cdot T_1$$

where

$$\boxed{T_1(L) = \frac{\sin(\omega L/2\beta c)}{\omega L/2\beta c}}$$



V_A has a maximum when $T_1 = 2/\pi$

when $\boxed{L_{opt} = \pi \frac{\beta c}{\omega}}$

E.G. in the TM_{010} mode $\quad L_{opt} = \frac{\pi}{2.41} \beta R$

VERY INEFFICIENT when $\beta \ll 1$, e.g. protons or ions

NON-RELATIVISTIC particles (say $\beta \lesssim 0.5$)

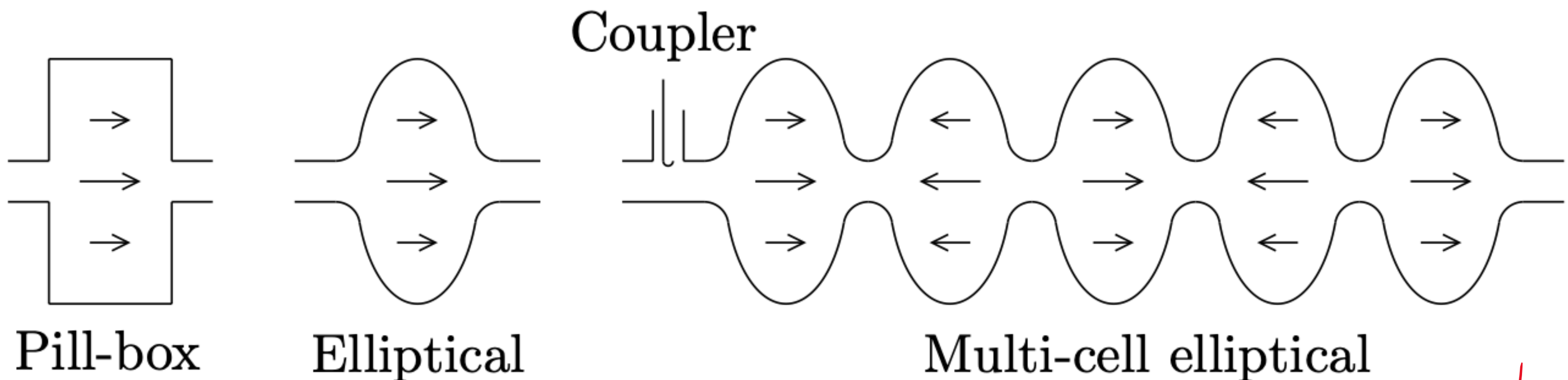
need different cavity geometries.

Zoo:

- spoke
- split ring
- inter-digital
- 1/4 wave
- 1/2 wave

⋮

7.5 Single-cell and multi-cell relativistic topologies

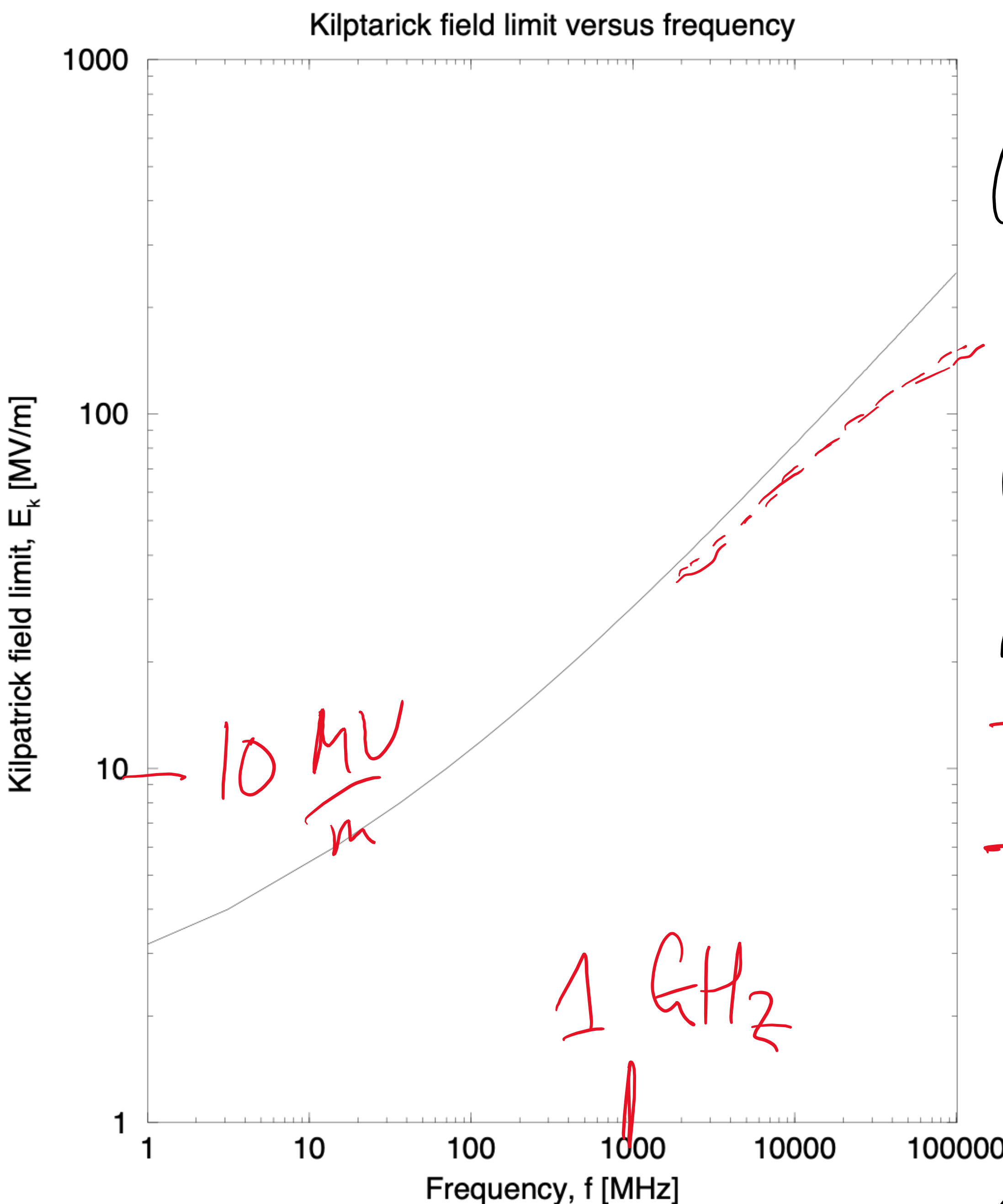


with a π phase shift

- The TM_{mnp} "language" survives if rotational symmetry is assumed.

Q1: What is the optimum number of cells per cavity? (see later)

Q2: How big can E gradient be? Copper? Superconducting?



KILPATRICK CRITERION

(1950's)

Empirical:

$$\boxed{f = 1.64\, E_0^2\, e^{-\left(\frac{8.5}{E_0}\right)}}$$

MHz MV/m

- Somewhat conservative
- Suggests (???) that copper cavities at $f \gg 10$ GHz can out-perform SC cavities ($E_{SC} > 10$ MV/m)

SUPERCONDUCTING CAVITIES

- Many geometries are possible despite more difficult manufacturing
- Lower operating cost: "no" heat dissipation in walls
- Higher capital costs: complexity, material treatment, cryogenics, ...
- Very high Q values mean very small bandwidths

$$f_{BW} = \frac{f}{Q} \sim \frac{10^9}{10^{10}} \text{ Hz}$$

$\Rightarrow$ sensitivity to mechanical distortion:

1) microphonics (any noise, e.g. cryogenics)
2) Lorentz force detuning (pulsed operation)

$\Rightarrow$ QUIET CW OPERATION indicates SC technology e.g. LCLS II