Lecture 9: Sextupoles & Chromaticity

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Catch-22: … a concern for one's own safety in the face of dangers that were real and immediate was the process of a rational mind. Orr was crazy and could be grounded. All he had to do was ask; and as soon as he did, he would no longer be crazy and would have to fly more missions.”

Joseph Heller, “Catch-22”.

A) Chromaticity in a FODO lattice

B) Chromaticity correction

C) The Henon map

D) Taxonomy of 1-D motion

E) Dynamic Aperture

A) Chromaticity in a FODO lattice

- SEXTUPOLES are (usually) necessary to avoid NON-linear resonances ... that are most often caused by sextupoles.
- Betatron motion of a particle with $\delta = \frac{\Delta p}{p}$ const.

$$x'' + \frac{k}{(1+\delta)} \cdot x = 0$$
$$y'' - \frac{k}{(1+\delta)} \cdot y = 0$$

} Hill's equations with quads that weaken with δ

- CHROMATICITY measures the rate of change of tune with momentum:

$$\boxed{\chi = \frac{dQ}{d\delta}}$$

Both Q_x + Q_y naturally decrease with increasing δ

EG1 FODO LATTICE WITH $\underline{\underline{N}}$ CELLS

Each cell has ϕ phase advance, so

$$Q = \frac{N}{2\pi} \cdot \phi$$

so NATURAL (uncorrected) chromaticity is

$$\chi_{NAT} = \frac{N}{2\pi} \frac{d\phi}{d\delta}$$

& following Eqn 3-48

$$\frac{d\phi}{d\delta} = -2 \tan\left(\frac{\phi}{2}\right)$$

so

$$\boxed{\chi_{x,NAT} = \chi_{y,NAT} = \frac{-Q \tan(\phi/2)}{\phi/2}} \approx -Q$$

THIS RULE OF THUMB often holds.

EG2 RHIC UNCORRECTED

$$\chi_{NAT} \simeq -50$$

RMS momentum spread $\frac{\sigma_p}{p} = \langle \delta^2 \rangle^{1/2} \simeq 2\times10^{-3}$

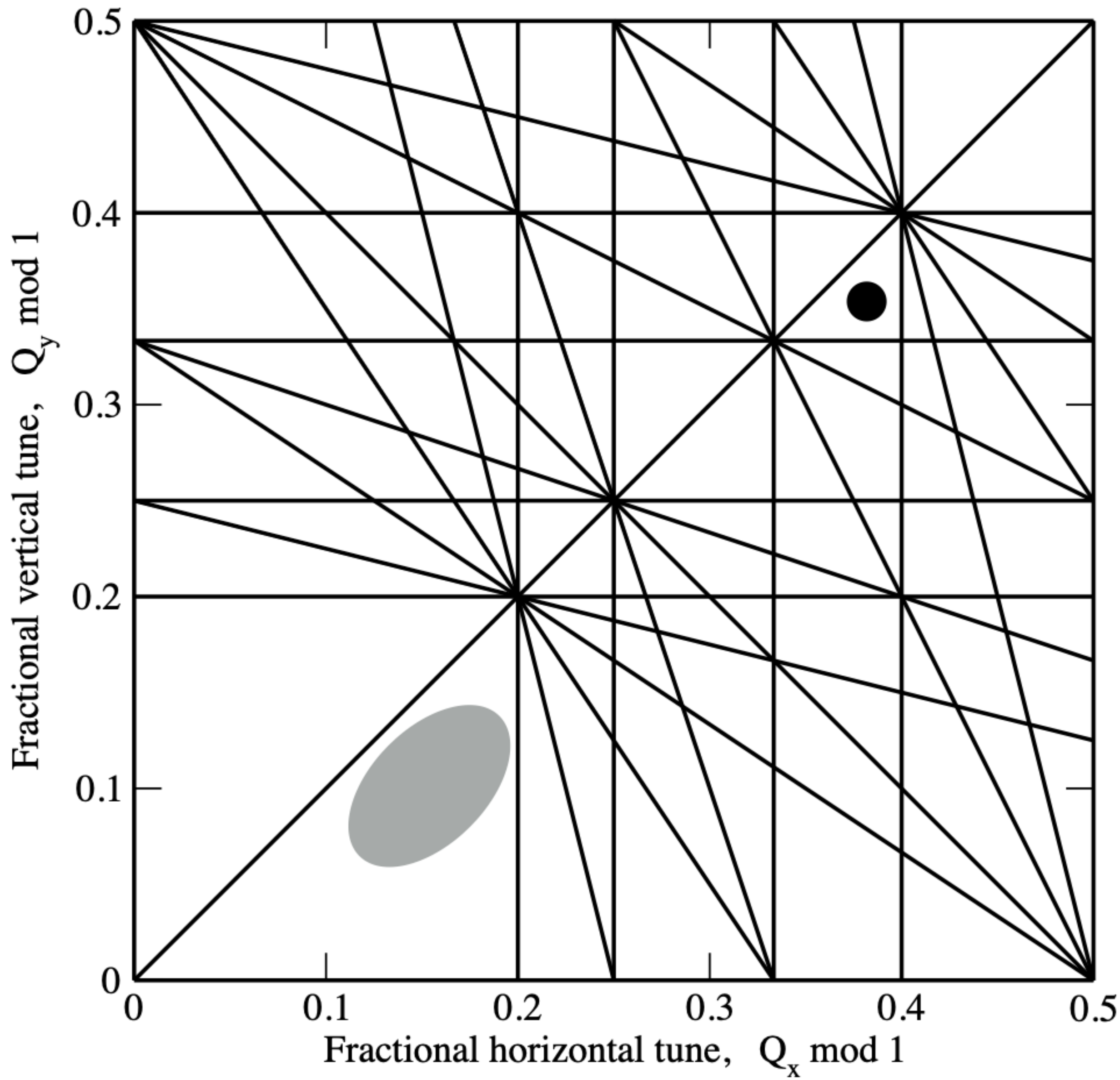
- UNCORRECTED TUNE SPREAD: $\boxed{\sigma_Q = |\chi_{NAT}| \frac{\sigma_p}{p} \lesssim 0.1}$!!

- UNACCEPTABLY limits the available tune place locations

$\Rightarrow$ MUST DECREASE $|\chi|$

to ~ 1 or ~ 2

9.2 Tune plane lines $p = q\,Q_x + r\,Q_y$ up to order 5

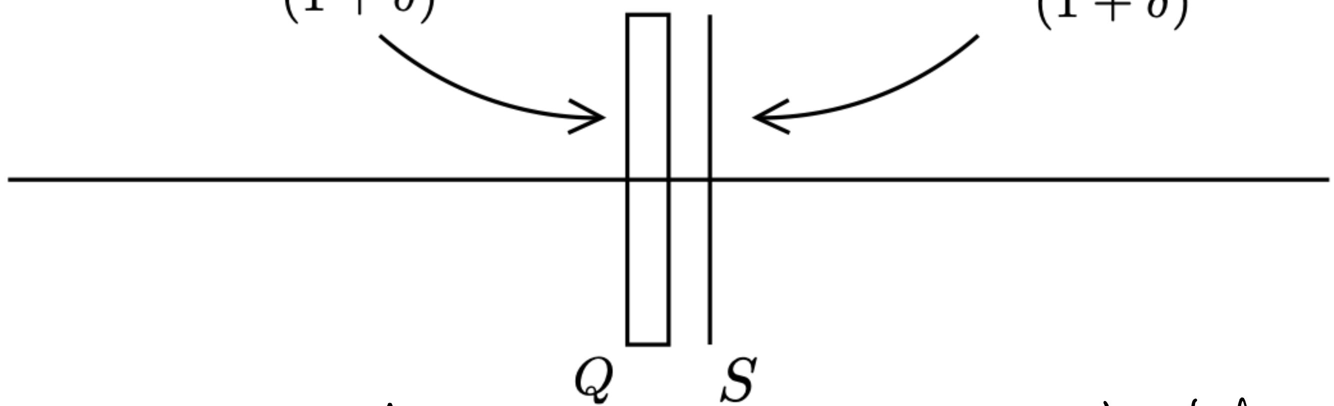


B) Chromaticity correction

EG3

9.1 A sextupole strength S next to EVERY quad strength q

$$\Delta x'_{TOT} = \frac{-q}{(1+\delta)}\, x_{TOT} \qquad\qquad \Delta x'_{TOT} = \frac{-S}{(1+\delta)}\, x_{TOT}^2$$



- FIRST separate closed orbit & betatron oscillations

$$x_{TOT} = \underbrace{\eta\delta}_{\text{CONSTANT}} + \underbrace{x}_{\text{OSCILLATION}}$$

- THEN expand composite quad-sext kick:

$$\Delta x'_{TOT} = -q(1-\delta)(\eta\delta + x) - S(1-\delta)(\eta\delta + x)^2 + \ldots$$

- AND collect terms in $x^n \delta^m$

COLLECTING TERMS in $x^n \delta^m$

$$
\begin{aligned}
(\Delta\eta')\delta + \Delta x' \;=\; & -q.\eta\delta \\
& -q.x \;-\; S.x^2 \\
& +q.x\delta - S.2\eta.x\delta + S.x^2\delta \\
& +\mathcal{O}(\delta^2)
\end{aligned}
$$

$x^0\delta^1$

$x^m\delta^0$ GEOMETRIC TERMS

- TO ORDER $x^1\delta^1$, dropping constants

$$\boxed{\Delta x' = -q \cdot x + (q - 2\eta S)\delta \cdot x}$$

- IF $\boxed{S = \dfrac{q}{2\eta}}$ at every quad-sext pair

THEN "effective" quad strength is independent of δ !!

$$\boxed{S = \frac{q}{2\mu}}$$

In a FODO lattice $\mu_F \simeq 2\mu_D$

$$S_F \simeq -\frac{1}{2} S_D$$

Q1: What happens if η is small?

[Q2: What do the geometric terms do?]

See later: Hénon map

EG4: CHROMATICITY CORRECTION, GENERAL

- Previously (Eqn 8.29) if a quad changes strength by Δq, tune changes by

$$\Delta Q = \frac{\beta_0}{4\pi} \cdot \Delta q$$

- Quad strength weakens with δ like

$$\frac{dq}{d\delta} = -q$$

- Sextupolar chromaticities are

$$\begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix} = -\frac{1}{4\pi} \begin{pmatrix} \Sigma\, q \beta_x \\ \Sigma -q \beta_y \end{pmatrix}$$

where sums Σ are over all (thin) quads

- STRENGTH q is usually +ve(−ve) when β_x (β_y) is large

$\Rightarrow$ NATURAL CHROMATICITIES ARE NEGATIVE

- SEXTUPOLES contribute a QUAD component

$$\frac{dq}{d\delta} = 2\eta S$$

- So $\underline{\text{ONE FAMILY}}$ of sextupoles changes chroms.

$$\Delta \begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix} = \frac{S_1}{2\pi} \begin{pmatrix} \Sigma \eta \beta_x \\ \Sigma -\eta \beta_y \end{pmatrix}$$

η_x is H

η_y is zero!

in different directions!!

- $\underline{\text{TWO FAMILIES}}$ placed near F & D arc quads:

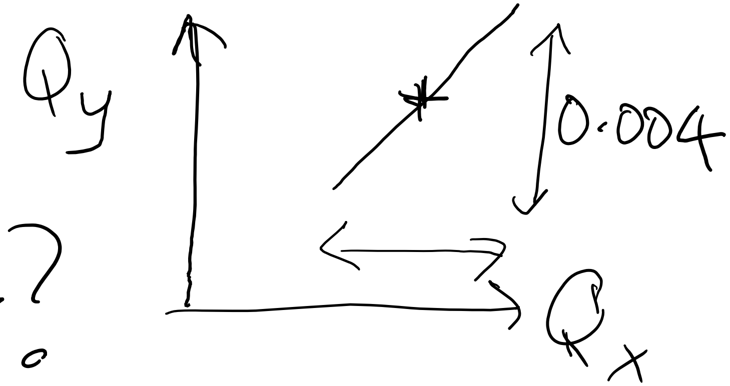
EQN 9-17

$$\boxed{\begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix}_{GOAL} = \frac{\cancel{2}}{4\pi} \begin{pmatrix} \sum_F \eta\beta_x & \sum_D \eta\beta_x \\ \sum_F -\eta\beta_y & \sum_D -\eta\beta_y \end{pmatrix} \begin{pmatrix} S_F \\ S_D \end{pmatrix} + \begin{pmatrix} \chi_x \\ \chi_y \end{pmatrix}_{NAT}}$$

SOLVE for S_F & S_D by inverting the matrix!!

EGS RHIC

$$\chi_{GOAL} = +2 \quad , \quad \frac{\sigma_p}{p} = 2 \times 10^{-3}$$



Q1: Why keep χ slightly +ve?

Q2: Why is TUNE FOOTPRINT usually a blob, not a straight line?

A2: $$Q_x = Q_x(0) + \frac{dQ}{dJ_x} \cdot J_x$$

cf filamentation see later

Q3: What drives the resonance lines? Sextupoles!!

C) The Henon map

HÉNON: "exhibits ALL TYPICAL PROPERTIES of ... dynamical systems"

until finished { (9.19)

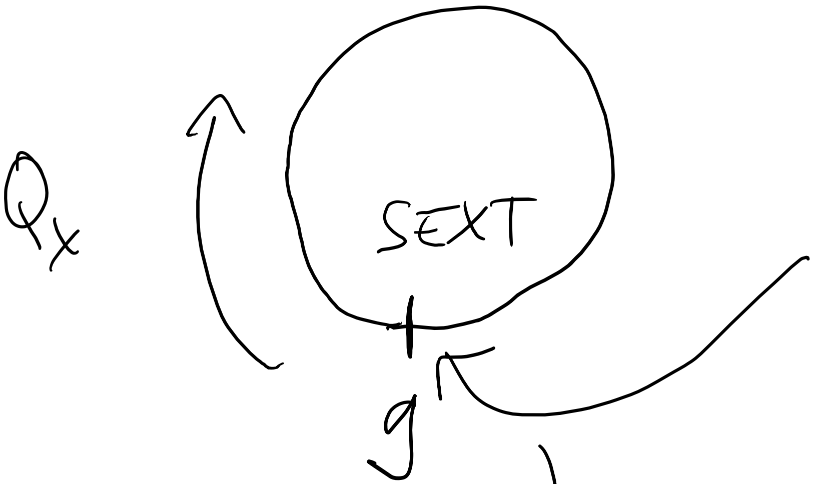
$$\begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix} + \begin{pmatrix} 0 \\ z^2 \end{pmatrix}$$

}

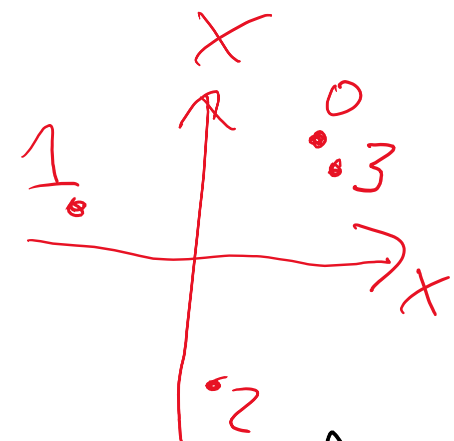
GEOMETRIC TERMS ($\delta = 0$)

- The ONLY control parameter, Q, looks like the horizontal tune
- The z^2 kick looks like a sextupole

ONE SEXTUPOLE, horizontal motion



reference point just BEFORE sextupole



kick: $\begin{pmatrix} x \\ x' \end{pmatrix}_{0+\epsilon} = \begin{pmatrix} x \\ x' - gx^2 \end{pmatrix}$

ROTATE: $\begin{pmatrix} x \\ x' \end{pmatrix} = R(2\pi Q) \begin{pmatrix} x \\ x' \end{pmatrix}_{0+\epsilon}$

EG1 TUNE CLOSE TO 1/3

$$\boxed{Q = \frac{1}{3} + \delta Q}$$

- Net motion after 3 turns is small

$$\begin{pmatrix} x \\ x' \end{pmatrix}_3 - \begin{pmatrix} x \\ x' \end{pmatrix}_0 \approx$$

$$3\mu \begin{pmatrix} x' \\ x \end{pmatrix}_0 - g\left[\begin{pmatrix} S_3 \\ C_3 \end{pmatrix} x_0^2 + \begin{pmatrix} S_2 \\ C_2 \end{pmatrix} x_1^2 + \begin{pmatrix} S_1 \\ C_1 \end{pmatrix} x_2^2\right]$$

$\mu = 2\pi\,\delta Q$ strength $C_k = \cos\left(k\,\dfrac{2\pi}{3}\right)$, $S_k = \sin(\ )$

– MORE SUCCINCTLY: 3-turn KOBAYASHI HAMILTONIAN

$$\boxed{H_3 = \frac{\mu}{2}\left(x^2 + x'^2\right) + \frac{g}{9}\sum_{k=1}^{3}\left(C_k x + S_k x'\right)^3}$$

$(x, x') \Rightarrow$ NORMALIZED space

where $\Delta x \simeq \dfrac{\partial H_3}{\partial x'} \cdot \Delta t$ with $\Delta t = 3$

$\Delta x' \simeq -\dfrac{\partial H_3}{\partial x} \cdot \Delta t$

— H_3 is APPROXIMATELY constant along a trajectory

— Factorizing H_3 gives

$$\boxed{H_3 = \frac{g}{3}\left(\frac{2\mu}{g}\right)^3 + \frac{g}{12}\left(x + \sqrt{3}x' + \frac{4\mu}{g}\right)\left(x - \sqrt{3}x' + \frac{4\mu}{g}\right)\left(x - \frac{2\mu}{g}\right)}$$

$$0 = \quad + \underbrace{\left(x + \sqrt{3}x' + \frac{4\mu}{g}\right)}_{0}\ \text{OR}\ \underbrace{\left(x - \sqrt{3}x' + \frac{4\mu}{g}\right)}_{0}\ \text{OR}\ \underbrace{\left(x - \frac{2\mu}{g}\right)}_{0}$$

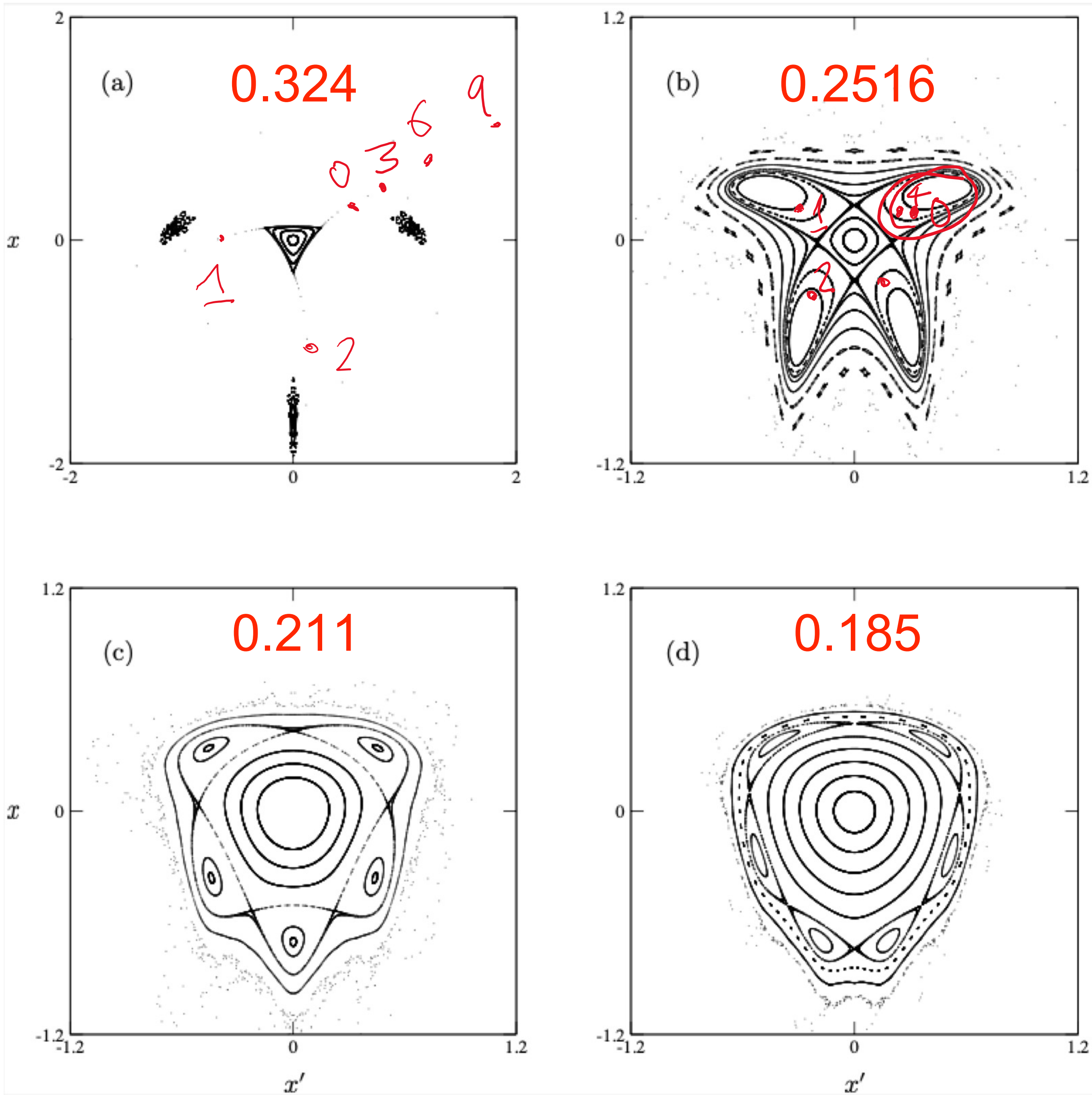
3 STRAIGHT LINES!!

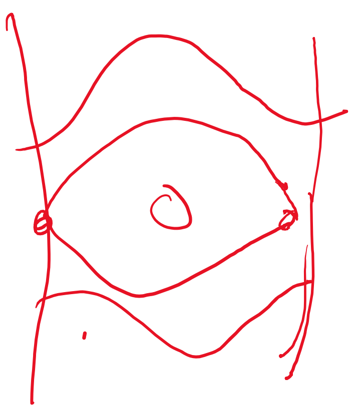
— EG2:

$$H_3 = \frac{g}{3}\left(\frac{2\mu}{g}\right)^3 : \qquad 0 = (\quad)(\quad)(\quad)$$

& $Q = 0.324$

9.3 Henon map in normalized phase space at different tunes Q





$\boxed{\text{EG} \quad Q = 0.324}$

- 3 STRAIGHT LINES !!.
- Regular circular motion near the origin
- RAPID divergent motion along 3 arms (slow ~~ex~~traction)
- 3 outer "islands" unpredicted by any perturbation theory

$\boxed{\text{EG} \quad Q = 0.2516} \approx 1/4$

- 4 RESONANCE ISLANDS ! Phase locking
- Island hopping
- Chains of many small islands appear

$\boxed{\text{EG } 0.211}$ $\approx 1/5$

- CHAOTIC "FLY SPECK" DOTS are visually distinctive
- 5 islands are more rotationally symmetric

$\boxed{\text{EG } Q = 0.185}$ $\approx \frac{1}{6}$

- 6 smaller islands, closer to a DYNAMIC APERTURE

D) Taxonomy of 1-D motion

HÉNONS TAXONOMY of typical behavior

① REGULAR MOTION

- Roughly circular motion (normalized phase space)
- Enough turns (dots) make "continuous" lines

② REGULAR DIVERGENT (e.g. Q = 0.324)

- Amplitude increases rapidly & regularly
- Exploit for resonant "slow" extraction

③ REGULAR RESONANT

- Island hopping : ONLY SOME PHASES ARE ACCESIBLE !!

④ CHAOS : Fly specks visually distinctive!!

LYAPUNOV EXPONENT

- Start 2 particles VERY close together
  (e.g. one digit in double precision)

32 bit 10^{-13}
64 bit 10^{-28}

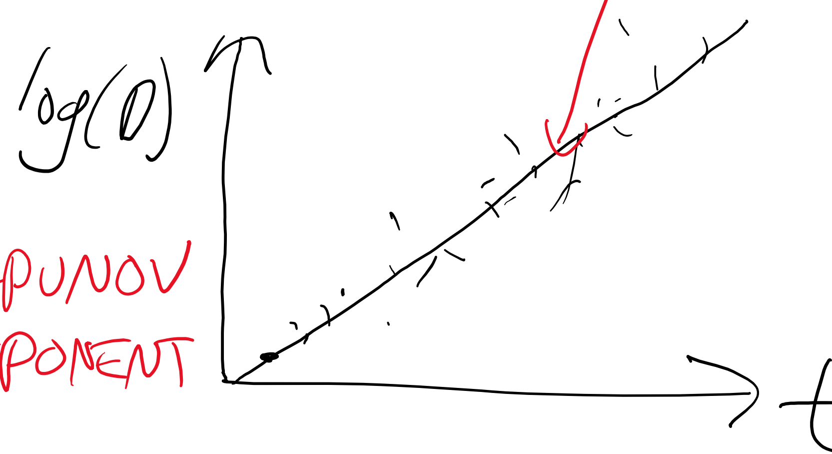
Q: How do they diverge in time?

$$\text{e.g. } D = (x_A - x_B)^2 + (x'_A + x'_B)^2$$

$e^{\lambda t}$

A1: LINEAR ⇒ Regular

A2: EXPONENTIAL ⇒ Chaos



$$\boxed{D(t) \sim D_0 e^{\lambda t}}$$

LYAPUNOV EXPONENT

E) Dynamic Aperture (DA)

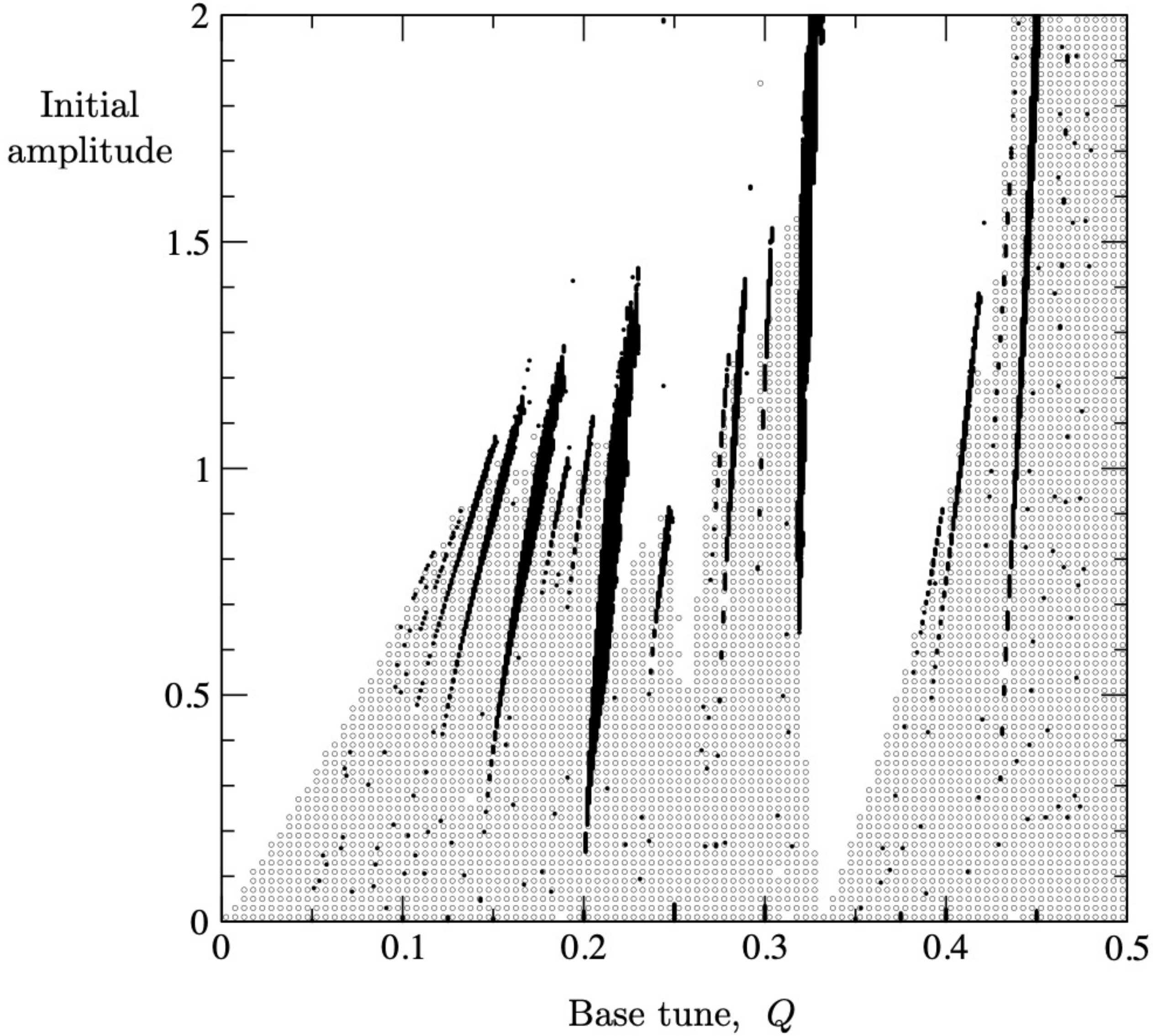
- IN ~~SI~~MULATION there is (usually) a clear cut maximum amplitude beyond which particles rapidly get lost – hit the beampipe
- IN REALITY there is noise/excitation/diffusion that drives particles across DA
  $\Rightarrow$ beam current decreases with time
- EVEN WITH JUST 1 SEXTUPOLE IN 1-D this is a complex story!!

Q1: Which tunes give better DAs?

Q2: Why is this naive? What is missing?

9.4 Dynamic Aperture vs tune under the Henon map.

Even one sextupole drives ALL resonances $Q = l/N$



IN THE END …

… it is hard to disagree with Taff:

"*Beyond first-order theory I know of no useful result from perturbation theory in celestial mechanics … Frequently the second approximation produces nonsensical results.*"