Lecture 15:

Beam-Beam interaction & 1-D resonances

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Beyond first-order theory I know of no useful result from perturbation theory in celestial mechanics…

L.G. Taff

A) Round Beam-Beam interaction

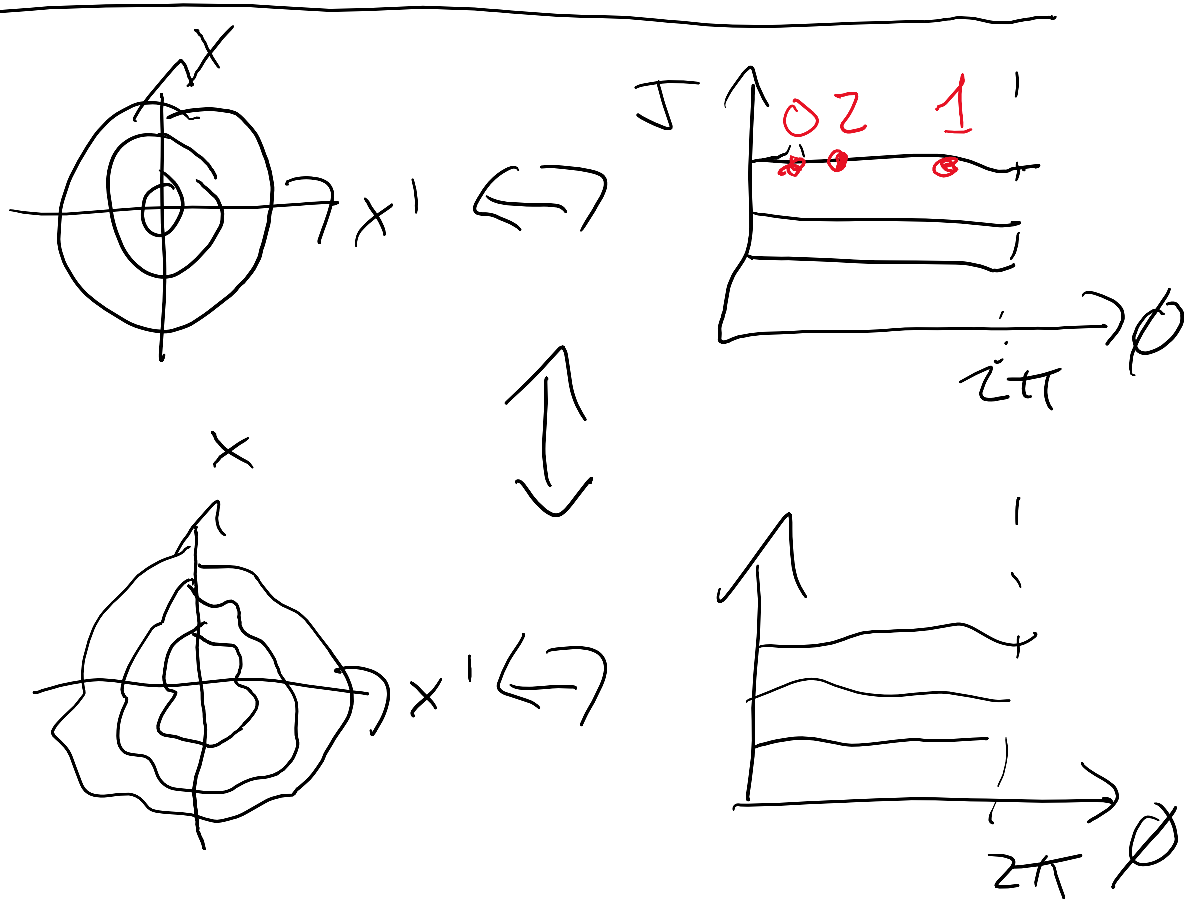
B) First order theory of 1D resonances

C) Resonance island tunes & widths

First order theory that works! Good example

Taxonomy:

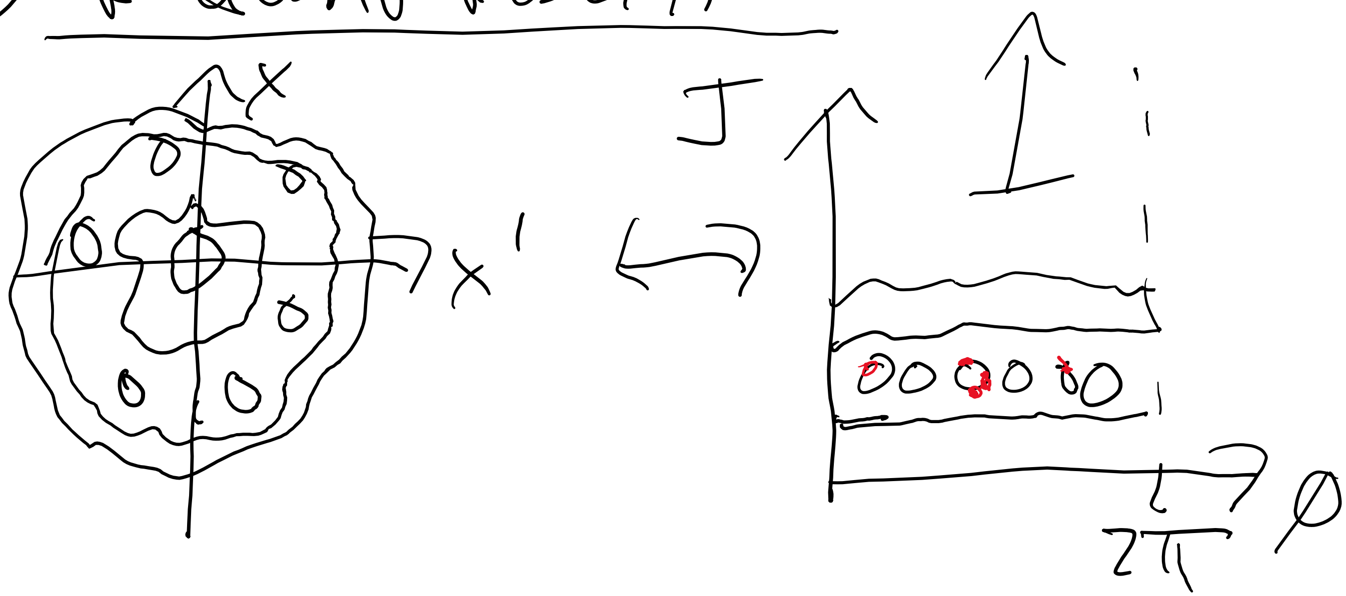
① REGULAR NON-RESONANT



- Roughly constant J

- Given enough time, get arbitrarily close to any phase φ!!

② REGULAR RESONANT



- Island chain

- Some particles skip from island to island
  - ⇒ Only some phases are accessible: phase locked
  - ⇒ Tune (long term average) is EXACTLY $Q = p/n$
- Note resemblance to STANDARD MAP

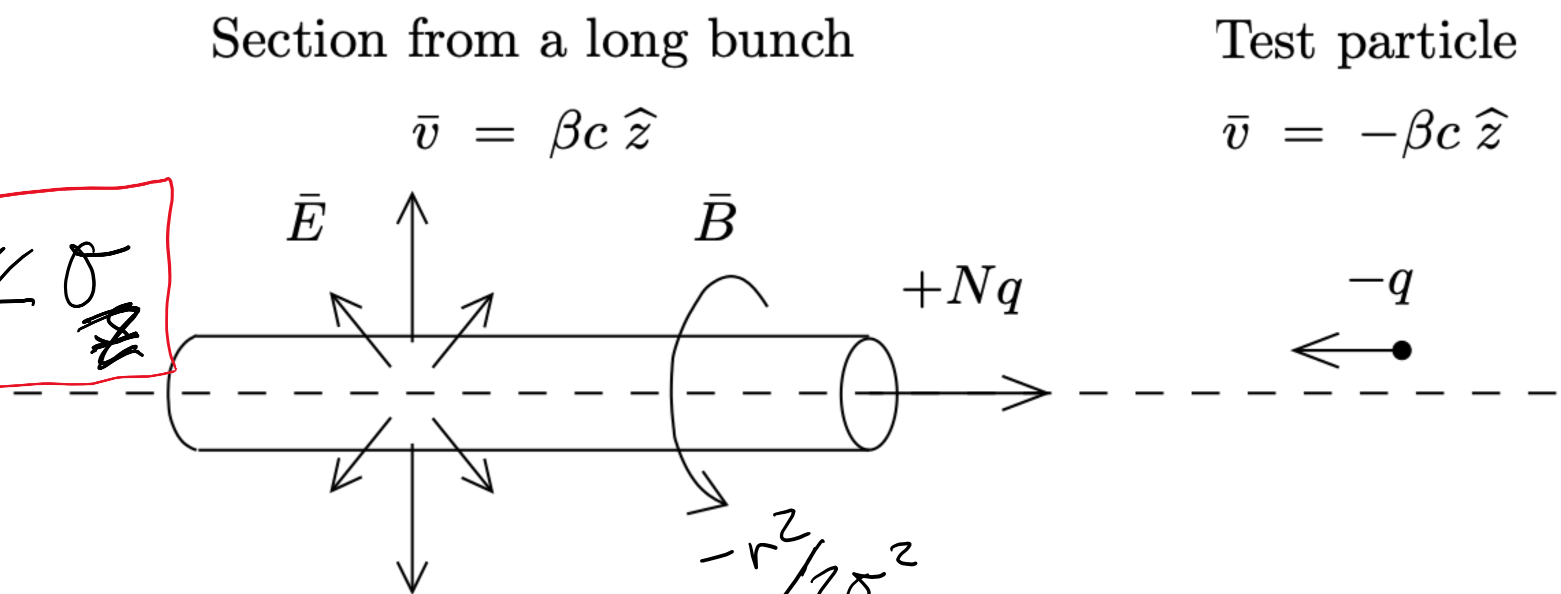
③ RAPIDLY DIVERGENT

④ CHAOS! (See chapter 16)

A) Round Beam-Beam interaction

15.1 Test particle enters a long Gaussian cylindrical bunch

$\boxed{\sigma \ll \sigma_z}$



$$\rho = \rho(z) \cdot \frac{e^{-r^2/2\sigma^2}}{2\pi\sigma^2}$$

$$\bar{F}_\perp = q(\bar{E} + \bar{v} \times \bar{B})$$

Gauss' Law:

$$\boxed{F_E = \frac{q^2}{2\pi\epsilon_0} \cdot \rho(z) \cdot \frac{1}{r}\left[1 - e^{-r^2/2\sigma^2}\right]}$$

ALSO $F_B = \beta^2 F_E$

BEAM-BEAM $\quad F_\perp = (1+\beta^2) F_E \simeq 2 F_E$

[SPACE CHARGE $\quad F_\perp = (1-\beta^2) F_E = \dfrac{F_E}{\gamma^2}$ $\quad$ LOW ENERGY!]

INTEGRATE ONE COLLISION

$$\boxed{\Delta x' = -\frac{4\pi}{\beta^*} \xi_0 \frac{2\sigma^2}{x} \left(1 - e^{-x^2/2\sigma^2}\right)}$$

Assume $\beta^* \gg \sigma_z$

1-D: $y = y' = 0$

where

(A) $\quad \boxed{\xi = \dfrac{N r_0}{4\pi \epsilon_N}}$

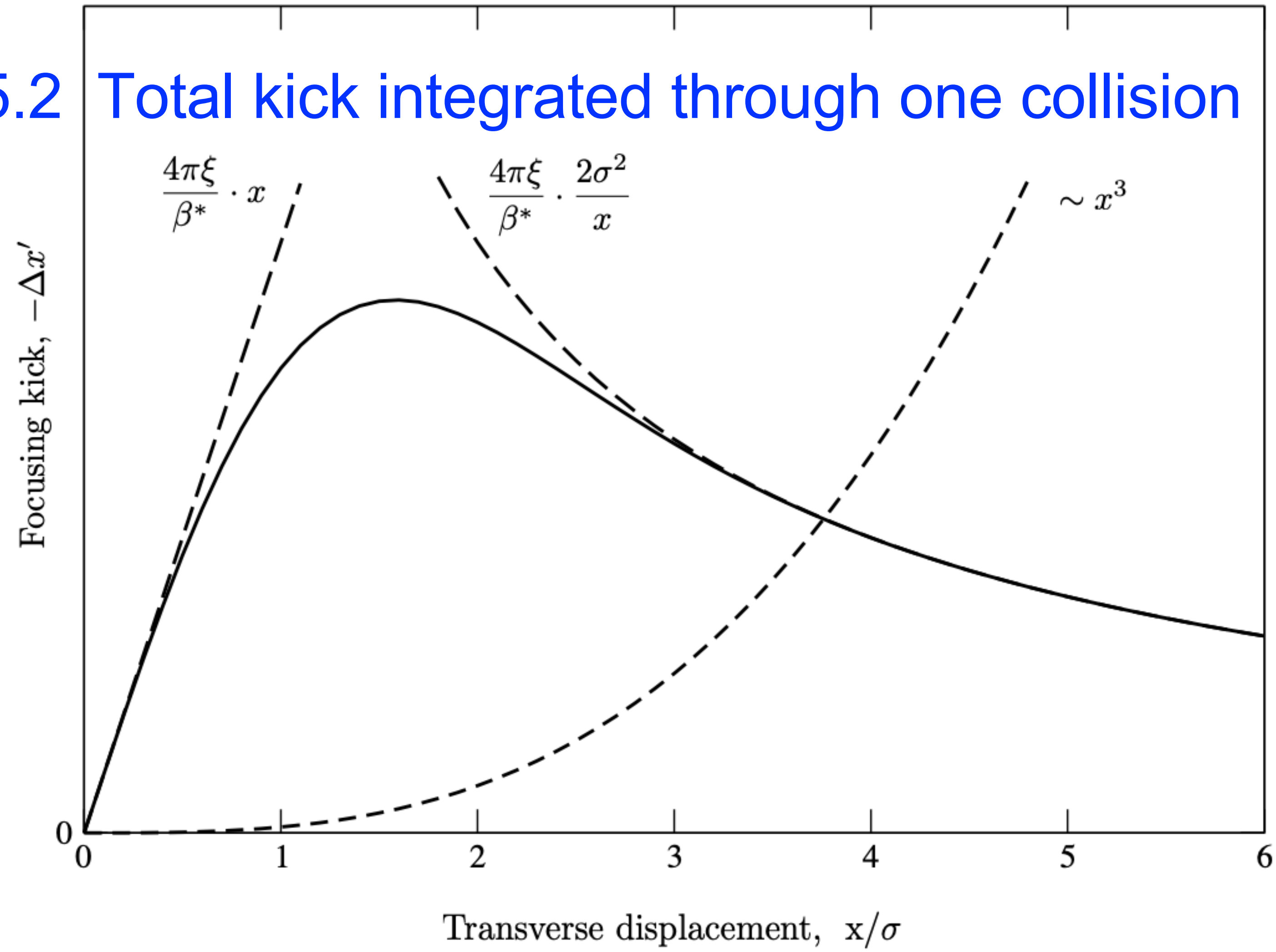
Bunch population

Classical radius

RMS normalised emittance

"BEAM-BEAM PARAMETER"

15.2 Total kick integrated through one collision



- SMALL $x \ll \sigma$: $\Delta x' \simeq -\frac{4\pi\xi}{\beta^*}\cdot x$ QUADRUPOLAR
- LARGE x : $\Delta x' \sim 1/x$ WELL-BEHAVED
- MAGNET : "BADLY" BEHAVED AT LARGE x Why??

SMALL X TUNE SHIFT

$$\boxed{\Delta Q = \frac{\beta^*}{4\pi} \cdot \frac{1}{f} = \xi}$$

ξ sometimes called "tune shift parameter"

– Independent of β^*! And of γ!! (see Ⓐ)

LUMINOSITY

$$\boxed{L = f_{REV}\, M \, \frac{(\beta\gamma)}{r_0} \, \frac{N\xi}{\beta^*}}$$

revolution f — # of bunches

Maximise L by increasing ξ & $N (\sim \xi)$, decrease β^*

ELECTRON/BEAMS have damping, so ξ_H & ξ_V are larger:

$e^+ - e^-$: $\xi_{MAX} \sim 0.1$

$h - h$: $\xi_{MAX} \sim 0.01$

B) First order theory of 1D resonances

EQUATIONS OF MOTION

Kobayashi Hamiltonian H_n with $Q \approx p/n$

$$\boxed{H_n = 2\pi\left(Q_0 - \frac{p}{n}\right) J + 2\pi\xi\, U(J) - 2\pi\xi\, V_n(J) \cos(n\phi)}$$

Detuning function   Resonance function

$$U'(J) = \frac{2}{J}\left[1 - e^{-J/2}\, I_0(J/2)\right]$$

$$V_n'(J) = (-1)^{n/2}\left(\frac{4}{J}\right) e^{-J/2}\, I_{n/2}(J/2)$$

NOTE

1) $\boxed{a = \sqrt{2J}\,\cdot\sigma}$!! dimensionless J

2) Prime denotes differentiation w.r.t. J

3) I_m is a modified Bessel function

Net (small) motion over $\underline{\underline{n}}$ turns:

$$\Delta\phi = n \cdot \frac{\partial H_n}{\partial J}, \quad \Delta J = -n \cdot \frac{\partial H_n}{\partial \phi}$$

HOW DOES THIS SOLUTION BEHAVE?

① $\underline{\xi = 0}$ $\quad H_n = 2\pi(Q_0 - p/n)J$

Phase advance over n turns

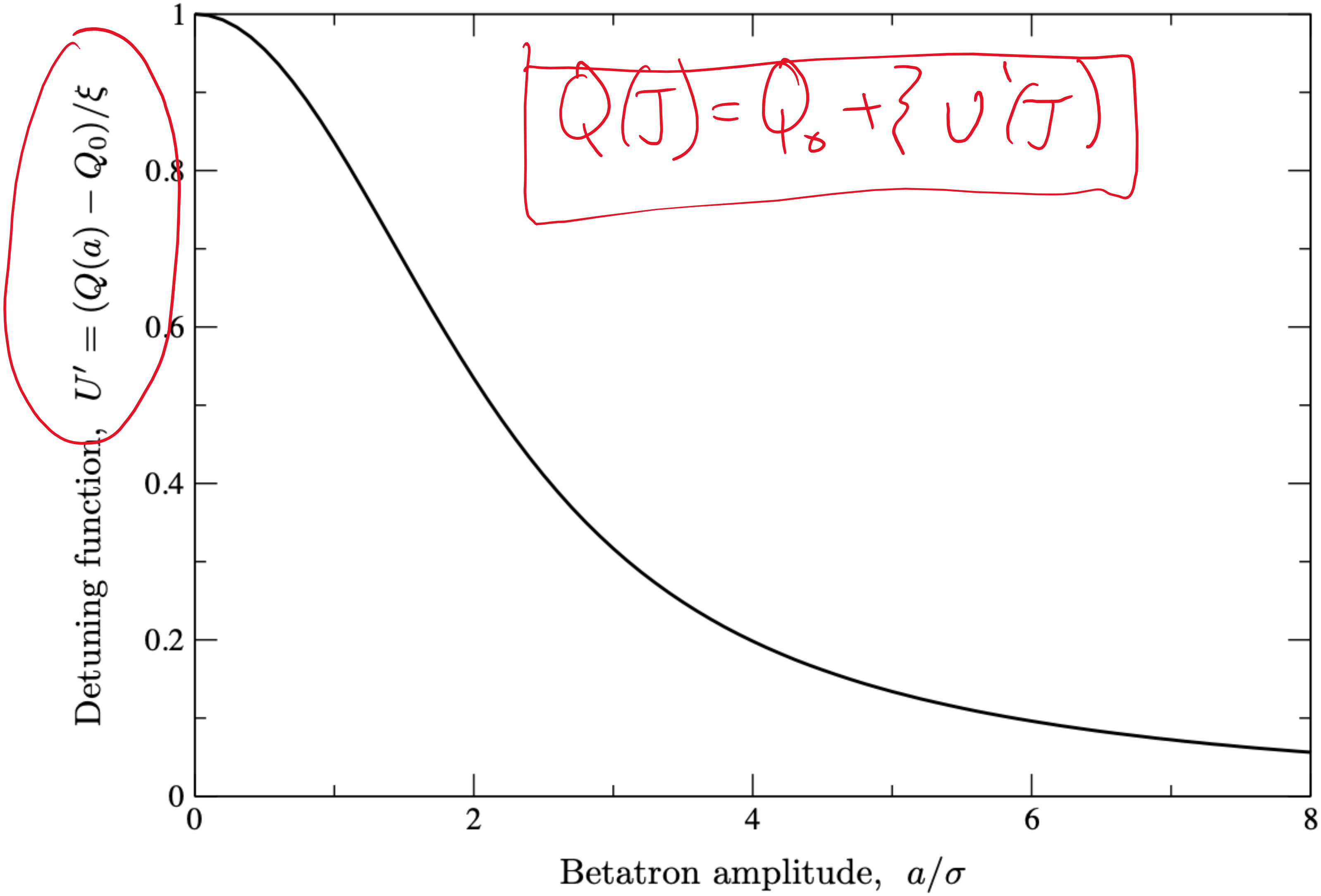
$$\phi_{t+n} - \phi_t = 2\pi(Q_0 - p/n) \quad \text{SMALL}$$

② $\xi \neq 0$: TUNE average rate of phase advance divided by 2π

$$\boxed{Q(J) = Q_0 + \xi U'(J)}$$

IF $\langle \cos(n\phi) \rangle = 0$

15.3 Detuning function *U'* for a single round collision



③ Resonant action J_R (a constant)

DEFINE by $\quad Q = Q_0 + \xi U'(J_R) \equiv \frac{p}{n} = \frac{2}{6}$

EXPAND equations of motion around

J_R

$\Rightarrow \quad I = J - J_R$

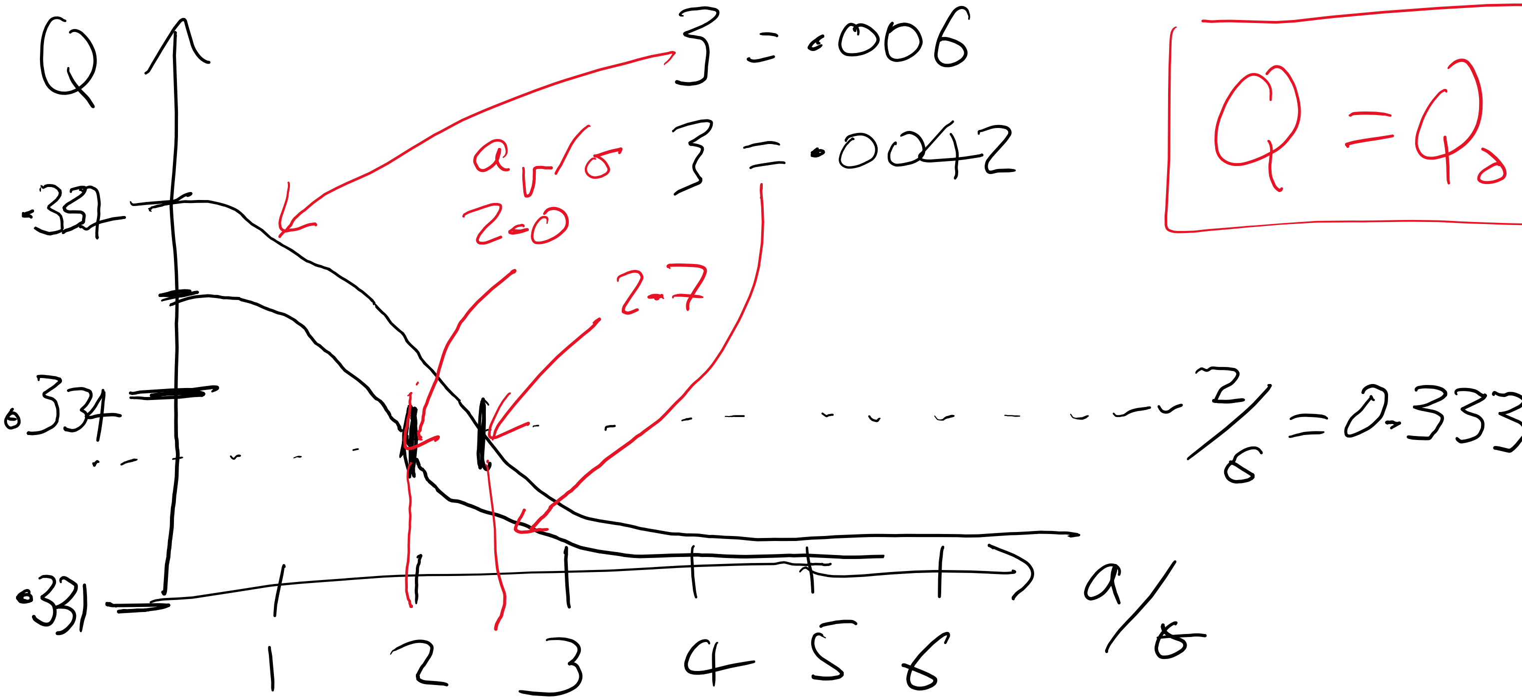
WHAT does $V_n(J)$ do ?

(BREAK)

WHERE ARE THE ISLANDS?

E.G. $Q_0 = 0.331$, near $Q = \frac{p}{n} \to \frac{2}{6}$

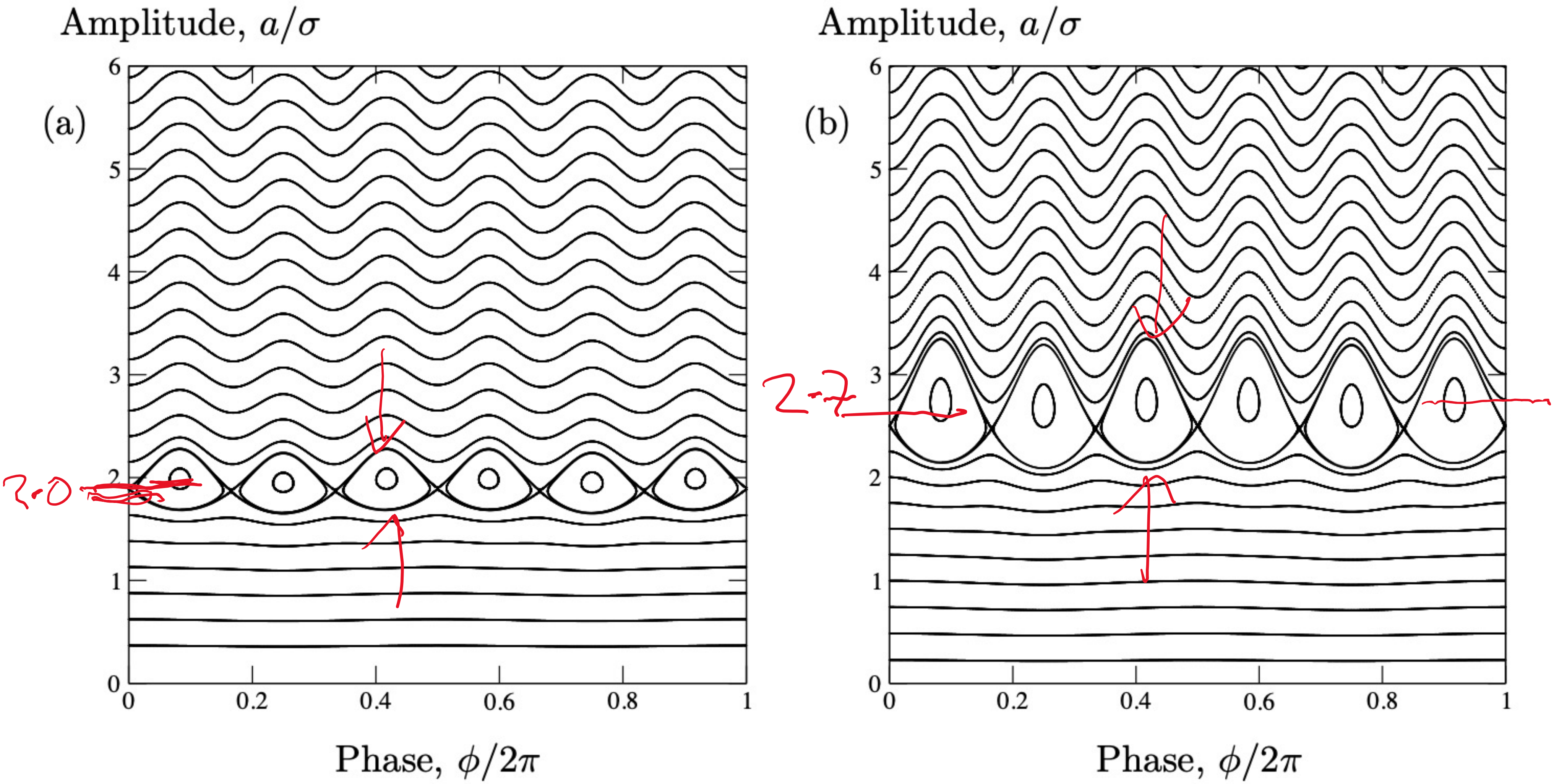
$$\boxed{Q = Q_0 + \xi\, U'(J)}$$



$$\frac{a_R}{\sigma} = 2.0 \qquad [.0042]$$

$$= 2.7 \qquad [.006]$$

15.4 Six-island chains with ξ = 0.0042 & 0.006



$Q_0 = 0.331$

- PREDICTED island locations are accurate!!
- ISLANDS get bigger at larger amplitudes!!

RESONANCE HAMILTONIAN

EXPAND $U(J) = U(J_R) + U'(J_R) I + \frac{1}{2} U''(J) I^2 + \dots$

$$V_n(J) = V_n(J_R) + \dots$$

(B)

$$\boxed{H_{Rn}(\phi, I) = 2\pi \left[\frac{1}{2} (\xi U''_R) I^2 - (\xi V_{Rn}) \cos(n\phi) \right]}$$

Compare to standard map (pendulum) Hamiltonian

$$\boxed{H(\theta, \theta') = \frac{1}{2} \theta'^2 - \cos(\theta)}$$

K.E. P.E.

$\Rightarrow$ the fundamental difference is "n"

NOTE: $V_n(J) = 0$ for ODD n $\Rightarrow$ NO odd resonances

C) Resonance island tunes & widths

HOW WIDE ARE ISLANDS?

Motion near a resonance island is described by H_{Rn}

$$\phi_{t+n} - \phi_t = n \cdot \frac{\partial H_{Rn}}{\partial I} \, , \quad I_{t+n} - I_t = -n \, \frac{\partial H_{Rn}}{\partial \phi} \qquad \text{(B)}$$

Or, in a continuous time approximation

$$\boxed{\begin{aligned} \frac{d\phi}{dt} &= \quad 2\pi \, \xi U_R'' \cdot I \\ \frac{dI}{dt} &= -2\pi \, n \, \xi V_{Rn} \cdot \sin(n\phi) \end{aligned}}$$

t is time in turn numbers

(C)

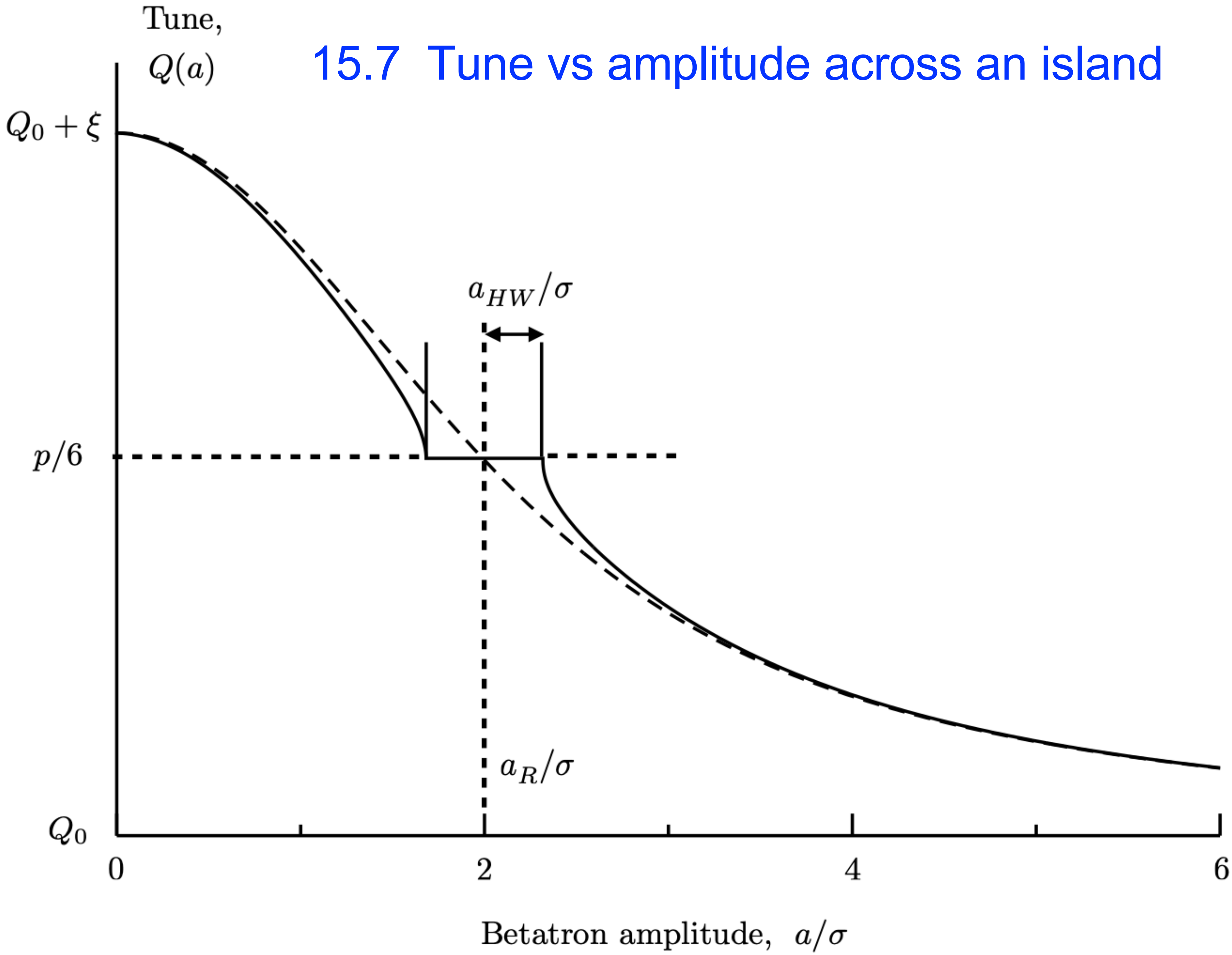
Fixed points at

$$I_{fp} = 0 \, , \quad \cos(n\phi_{fp}) = \pm 1$$

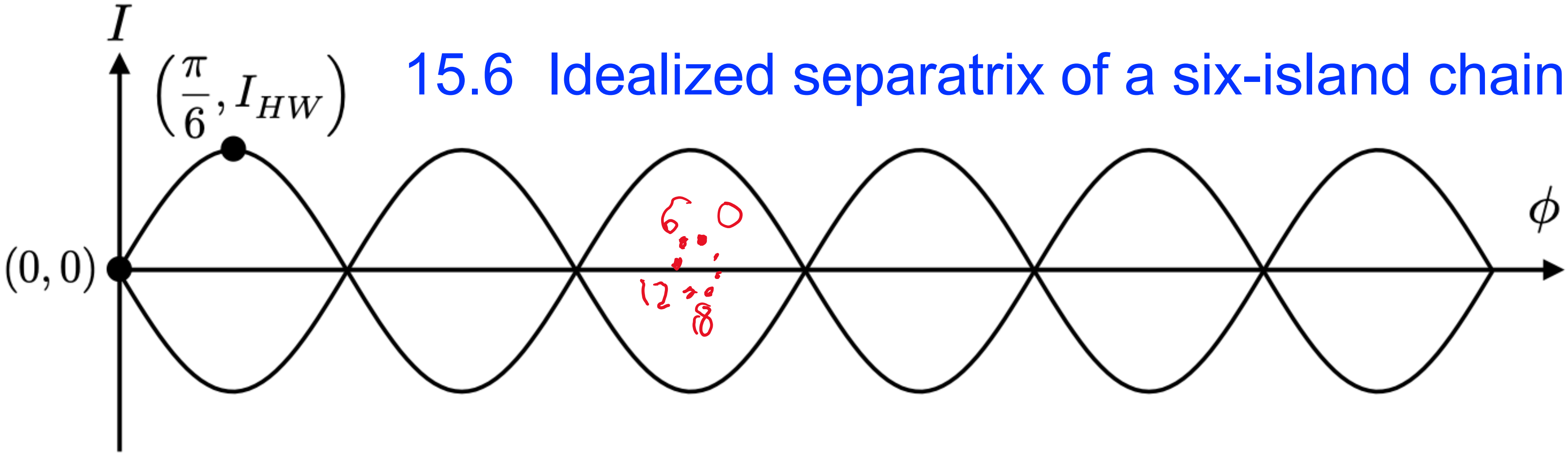
at (1) ISLAND CENTER, and (2) at SADDLE points

STABLE! UNSTABLE!

15.7 Tune vs amplitude across an island



15.6 Idealized separatrix of a six-island chain



H_{Rn} is CONSTANT along the separatrix

$$H_{Rn}(0,0) = H_{Rn}\left(\frac{\pi}{6}, I_{HW}\right)$$

and island width is

$$\boxed{I_{HW} \approx 2\left(\frac{V_{Rn}}{U''_{Rn}}\right)^{\frac{1}{2}}}$$

Independent of ξ !!

PREDICTED half-widths are accurate!!

E.G. $a_{HW} = 0.317\,\sigma$ when $Q_0 = .331$ & $\xi = .0042$

ISLAND TUNES

Near an island center: with $|\sin(n\phi)| \ll 1$

$$ⓒ \Rightarrow \quad I = a_I \cos(2\pi Q_I t)$$

$$\phi = \phi_{FP} + a_\phi \sin(2\pi Q_I t)$$

where island tune

$$\boxed{Q_I = n\xi\left(-\frac{1}{Rn}U_R''\right)^{1/2}}$$

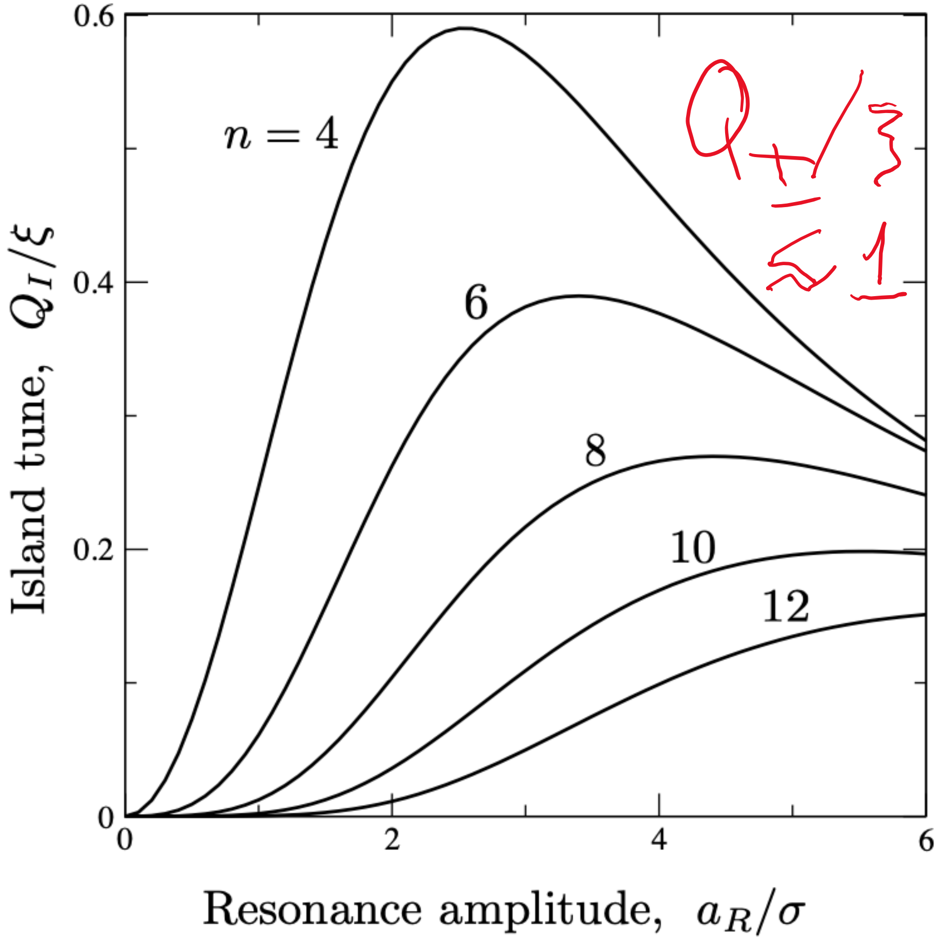
Measures the strength of the island conveniently

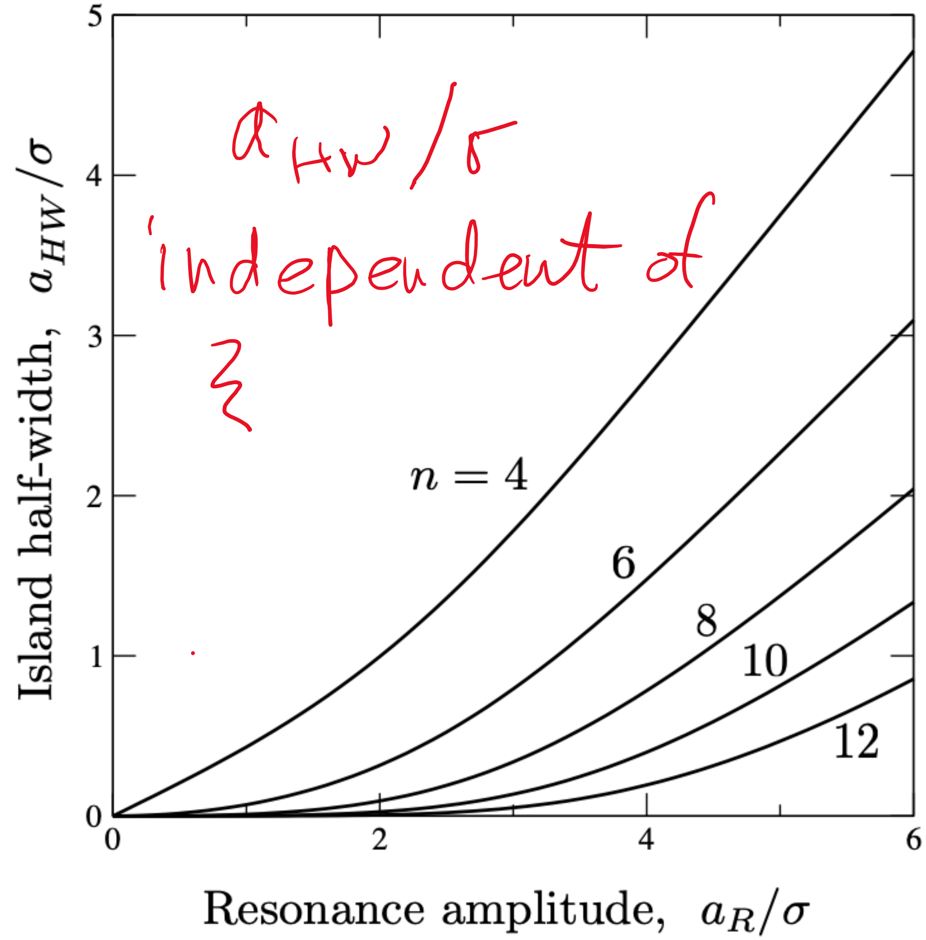
NOTE:

① $Q_I \sim \xi$ increases with beam-beam strength

② Q_I/ξ is of order 1 !!

15.5 Island tune & half-width vs resonance amplitude





ACCURATE PREDICTIONS !!

Q_I (and so also ξ) is a "control parameter" in considering external modulation effects, e.g. CHROMATIC TUNE MODULATION at Q_s

e.g., if $\delta = a_\delta \cos(2\pi Q_s \cdot t)$

then $Q_0 = Q_{00} + \chi a_\delta \cdot \cos(2\pi Q_s t)$

chromaticity $\simeq 2$ $\qquad$ 2×10^{-3}

tune modulation depth $\sim .004$ SIMILAR TO ξ values: DANGER! (See chapter 16)

ELECTRONS – FLAT BEAMS

- Electrons have both DAMPING (good!) & quantum EXCITATION (can be bad, e.g. phase space diffusion)
- Usually e^+-e^- collisions are FLAT:

$$\epsilon_H \gg \epsilon_V, \ \beta^*_H \gg \beta^*_V, \ \sigma^*_H \gg \sigma^*_V$$

- Nonetheless with roughly commensurate & beam-beam parameters since

$$\boxed{\xi_{H,V} = \frac{N r_0}{2\pi(\beta\gamma)} \cdot \frac{\beta^*_{H,V}}{\sigma_{H,V}(\sigma_H + \sigma_V)}}$$

- So what? What limits ξ_{MAX}?

- We have seen NO unstable motion (or diffusion) so far

- Where does CHAOS come from?

To be continued....